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SOME GEOMETRICAL COMMENTS ON VISION AND NEUROBIOLOGY

SEEING GAUSS AND GABOR WALKING BY, WHEN LOOKING THROUGH THE WINDOW OF THE PARMA AT LEUVEN IN THE COMPANY OF CASORATI

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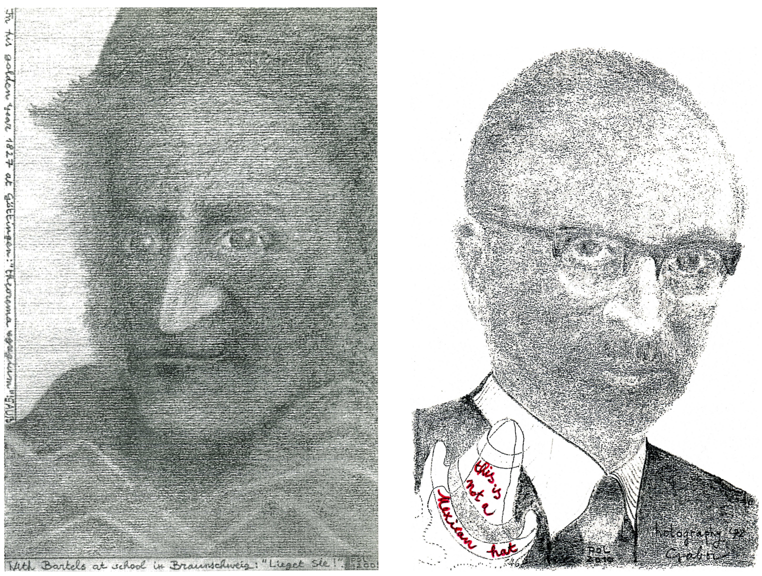

Gauss and Gabor

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In the 19 fifties, Kuffler and Hubel and Wiesel e.a. started to do neurobiological experiments concerning the activities of retinal and cortical neurons occurring in the process of vision. In relation with thus obtained findings, some could not resist to moreover start to phantasise in pretty unscientifical ways about how these factual activities would actually explain this process. Obviously, any large number N of such experiments could and can be further experimented and any multiple $m(> 1)$ of N such phantasies could and can be further phantasised. And ever since and up till now and like almost everywhere indeed, many eagerly have been involved in precisely such experiments and phantasies; (see, a.o., [1][2][3][4][5]). In the following, these things will be put at their proper place in the science of vision. In particular, it will be shown how "the Mexican ganglion hats" and "the V1 orientation detectors" appear as well determined compository aspects of the visual perception of contours in images via the extrinsic curvatures of the visual sensation of these images; (cfr. [6][7][8][9][10]).

In order to do so, from [7], we first recall the definition of *human visual sensation* $F(x, y)$ corresponding to a *luminosity function* $I(x, y)$ of a given image in an (x, y)–plane R^2; the function $I(x, y)$ is usually graphically extended in a z–direction R as a scalar field on the plane R^2, thus yielding a relief surface $z = I(x, y)$ of the image in the $3D$ $(x, y; z)$–space $R^3 = R^2 \times R$. In accordance with *"the horizontal (x)–vertical (y) effect"* which is rather significant indeed in human vision, the *scale space* of the given image was identified with the family of the smoothings $I(x, y; a_h, a_v) = I(x, y) * G_0(x, y; a_h, a_v)$ of the function $I(x, y)$ which result from *convolutions with elliptical* $2D$ Gaussians $G_0(x, y; a_h, a_v) = e^{-[(x/a_h)^2+(y/a_v)^2]}/2\pi a_h a_v$ of axes $a_h < a_v$; this constitutes a natural anisotropical retouch to the studies on the nature of observation as these were done under the hypothesis of isotropy of the visual field by Koenderink and van Doorn [11][12], (and, eventually, whenever such degree of sophistication would be needed, one could in addition also deal with the inhomogeneity of the visual field, e.g. in a way based on experiments as the one discussed in [13]). Then, by the *Law of Fechner*, for appropriate apertures (a_h, a_v), the human visual sensation of the given image is defined as $F(x, y) = S(x, y; a_h, a_v) = k. \ln I(x, y; a_h, a_v)$, whereby k is a suitable constant; thus, graphically, the visual sensation implied by the given image essentially is *the surface* M^2 *with Cartesian equation* $z = F(x, y)$ in R^3. By way of examples, from [7], in Figures 2, 3 and 4 are reproduced the graphs of the functions $I(x, y), I(x, y; a_h, a_v)$ and $F(x, y)$, respectively, corresponding to *the* *brightness illusion of Helmholtz* which is reproduced in Figure 1, (for more precise informations on concrete numerical values, see [7]).

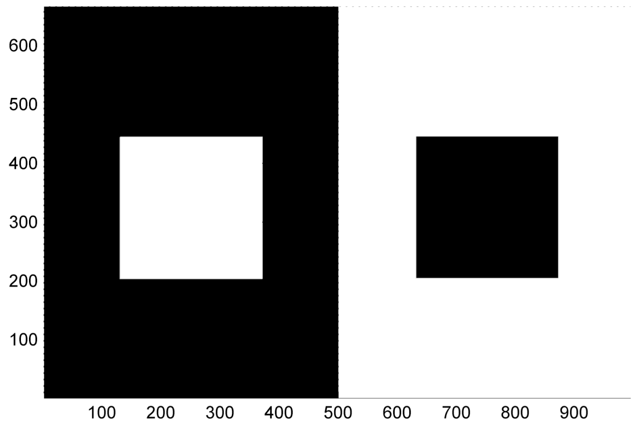

FIGURE 1. The illusion of Helmholtz, (the numbers on the axes represent distances in some arbitrary unit).

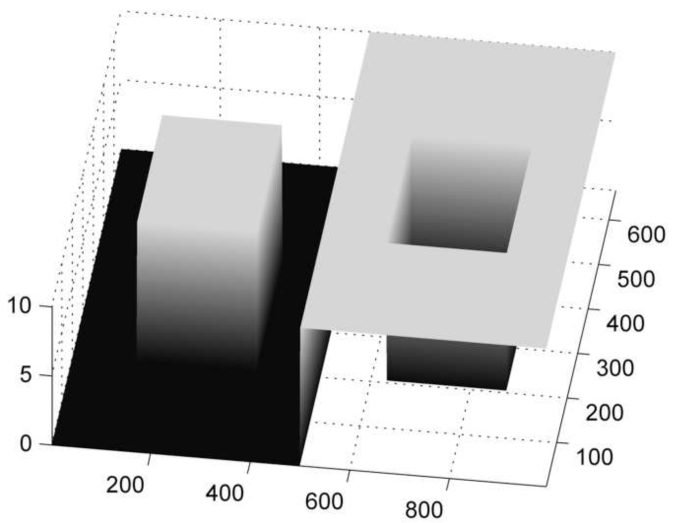

FIGURE 2. The relief of the function $I(x, y)$ for the illusion of Helmholtz.

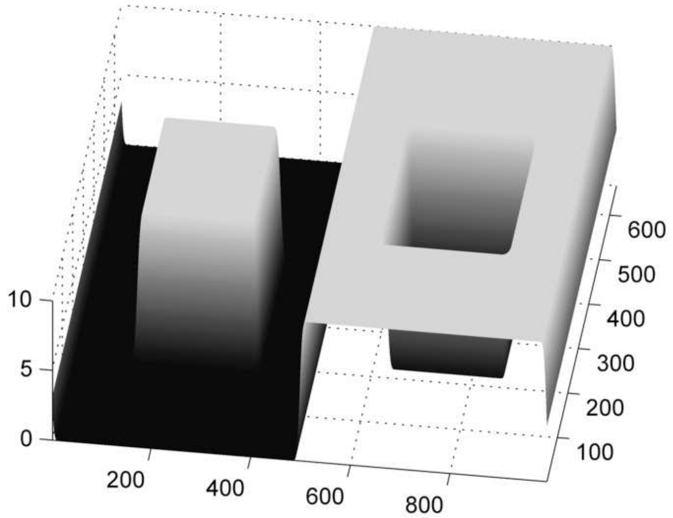

FIGURE 3. The relief of the function $I(x, y; a_h, a_v)$ for the illusion of Helmholtz.

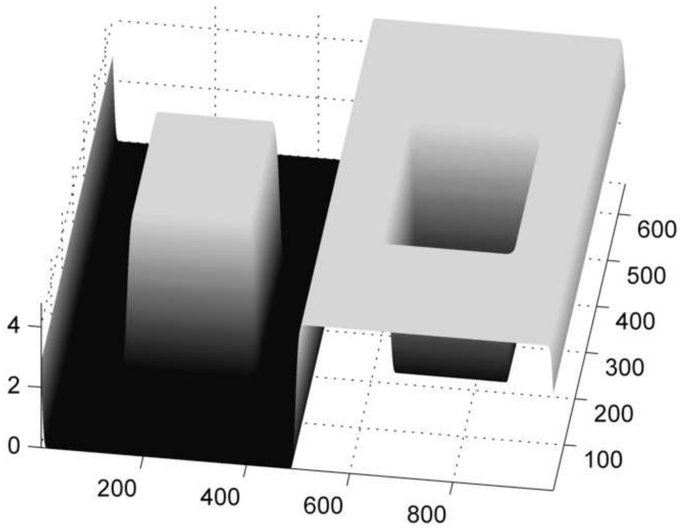

FIGURE 4. The relief of the function $F(x, y)$ for the illusion of Helmholtz.

Next, from [8][9][10][6][7], we recall that *the contours that we perceive* in early vision when looking at a given image, basically, are *lines of extremal values of the Casorati curvature of the sensation surface* M^2 of this image in R^3. The Casorati curvature C of a surface M^2 in R^3 may well be considered as, for our senses, *the most intuitive*, and, for mathematics, *the most simple curvature* among all extrinsic curvatures of surfaces M^2 in R^3, since it vanishes identically, i.e. it is zero at all points of such a surface, if and only if this surface is a plane, i.e. if and only if M^2 is the only kind of surface in R^3 which is not curved at all in the sense of our kind's natural experiences, and, since, in terms of the so–called *principal curvatures* k_1 and k_2 of a surface M^2 in R^3, essentially C is their first elementary symmetric function which has this property, namely $k_1^2+k_2^2$; (as general references about principal and Casorati curvatures and about the notion of curvature in general, see [14][15][16][17][18]). By way of examples, from [7], in Figure 6 are shown the sides of the "squares" of Figure 1 as they are determined as such by extremal values of this curvature, the curvature surfaces with Cartesian equation $z = C(x, y)$ in R^3 themselves being shown in Figure 5, (for more precise informations on the numerical values involved as well as for some comments on the fact that the white and black squares of Figure 1 actually are not perceived as squares but as proper rectangles, the one "standing up" and the other one "lying down", respectively, see [7][6]).

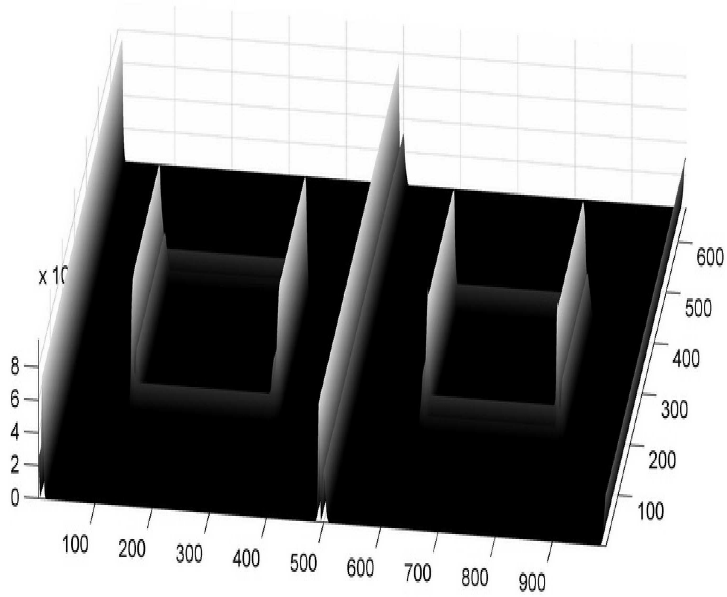

FIGURE 5. The Helmholtz brightness illusion: the Casorati curvature surfaces $z = F_{xx}^2 + F_{xy}^2 + F_{yx}^2 + F_{yy}^2$ of the Fechner surfaces $z = F(x, y)$ corresponding to the given white and black squares.

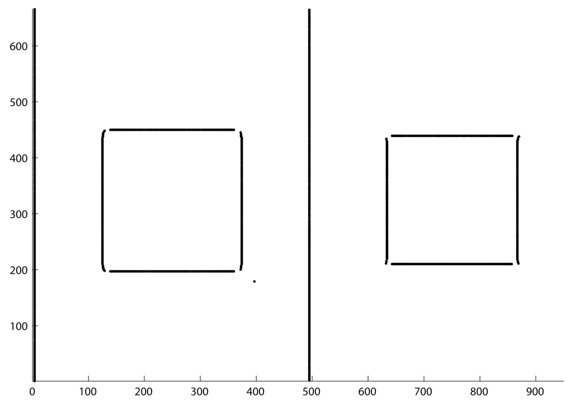

FIGURE 6. The Helmholtz brightness illusion: rectangles determined by the extrema of the Casorati curvature surfaces $z = F_{xx}^2 + F_{xy}^2 + F_{yx}^2 + F_{yy}^2$ of the Fechner surfaces $z = F(x, y)$ corresponding to the given white and black squares.

Actually, as far as we can see, all optical visual illusions, the static as well as the dynamic ones, can accurately be well described as such. Besides, several kinds of similarly "strange" phenomena which manifest themselves in the visual arts and "unrealistic" psychophysical experiences of luminance intensities in certain images, can also accurately be described in equally natural and elementary ways associated with the just recalled general geometrical descriptions of visual sensations and perceptions as will be reported on in some of our subsequent papers, partly also in co–operation with A. Kaplarević–Mališić.

From the above, we repeat that, for a given image: (i) the corresponding *visual sensation* is given by $z = F(x, y) = k. \ln I(x, y; a_h, a_v)$ whereby $I(x, y; a_h, a_v)$ stands for the convolution $I(x, y) * G_0(x, y; a_h, a_v)$ of the *luminosity* $I = I(x, y)$ *of the image* with a *"horizontal–vertical" elliptical Gaussian*, say $F = k. \ln(I * Gauss)$ for short, whereby Gauss $= G_0(x, y; a_h, a_v)$, and, (ii) *the contours that we perceive when looking at this image* are determined by the main geometrical characteristics of the surface M^2 which is the graph in R^3 of the observation $z = F(x, y)$, i.e. by lines of extremal values of the Casorati curvature function C of this surface M^2 in R^3. Now, at this stage, when properly and concretely dealing with *the extrinsic geometry of surfaces* M^2 *in* R^3, it is crucial to decide on which *geometrical structure* would be most appropriate to be put on R^3 in the situation under consideration. Obviously, the standard Euclidean structure which amounts to put on R^3 the Riemannian metric $ds^2 = dx^2 + dy^2 + dz^2$, which essentially means to perform the geometrical measurements in R^3 itself and also the geometrical measurements related to surfaces M^2 in R^3 in accordance with the *Theorem of Pythagoras*, for this will not do well, i.e. to study visual sensation surfaces M^2, which are the graphs in R^3 of functions $F : R^2 \to R$, in the Euclidean space $E^3 = (R^3, ds^2)$ is theoretically complete out of the question: the co–ordinate directions x, y on the one hand and the co–ordinate direction z on the other hand having completely different natures indeed. The most natural geometrical structure to put on R^3 for the study of the extrinsic geometry of the surfaces $M^2 : z = F(x, y)$ may very well be the *degenerate Riemannian metric* $ds_*^2 = dx^2 + dy^2$, as was clearly motivated by Koenderink and van Doorn in [19]. Hence, further on, we will study the extrinsic geometry of the surfaces M^2 under consideration in R^3 as surfaces in the $3D$ *1–fold isotropical space* $I^3 = (R^3, ds_*^2)$; a basic reference for the geometry of surfaces M^2 in I^3 is [20]. For some comments on the comparison in practice of the geometries of E^3 and of I^3 in the present context, see [7]. In I^3, the Casorati curvature C of a surface M^2 which is the graph of a function $F : R^2 \to R : (x, y) \mapsto z = F(x, y)$ is given by $C = F_{xx}^2 + F_{xy}^2 + F_{yx}^2 + F_{yy}^2$, whereby the indices x and y refer to the *partial differentiations* of F with respect to the co–ordinates in the image plane R^2. And, thus, finally one can readily appreciate to what extent that *elliptical Gaussian filterings* (cfr. the functioning of cortical orientation detecting neurons) and that *Gabor filterings* (cfr. the functioning of ON–OFF ganglion cells) of the function $I(x, y)$ associated with a given image, do actually may play some roles in the process of vision. Namely, in full generality, in view of the formula for the Casorati curvature for surfaces M^2 in the space I^3 and by the rules for differentiations of logarithmic and of exponential functions and of convolutions of functions and also by the approximations of goniometrical functions via Taylor–Maclaurin expansion, -the *"Gabors"* or Gabor functions involved here resulting from the products of the *"Gaussians"* with goniometrical functions- , indeed one can straightforwardly notice how various combinations of convolutions $I * Gauss$ and $I * Gabor$ of the luminance function $I(x, y)$ with Gaussians and Gabors in some non trivial ways actually do mix together to compose the curvature function C; (as a basic reference for studying scalings in vision, see [21]). And, of course, in the particular cases of some very "special" images, (say, whereby the influence of the Law of Fechner might be not so important, -so that the convolution of I with the Gaussian in F could then "directly" be differentiated itself, without having to go through the logarithm- , and, say, whereby the luminosity $I(x, y)$ essentially might only depend on one of the variables, -in case of dependance on x only, then identically having $I_{xy} = I_{yx} = I_{yy} = 0$ -), then the curvature C could turn out basically to become some very simple combination of smoothings $I * Gauss$ and $I * Gabor$, (say, more precisely for the just scetched special situation, basically as the square of a linear combination of one convolution $I * Gauss$ with one convolution $I * Gabor$, both to be considered as functions of x alone, since then $(I * Gauss)'' = I * (Gauss'') \approx c_1.(I * Gauss) + c_2.(I * Gabor)$, for some coefficients c_1 and c_2, whereby $''$ means second order differentiation with respect to x). In such cases, one could be tempted to substitute some "theories" on neural activities in place of the general description of images by curvatures. Yet, even by small perturbations of such special images, (say, for the cases alluded to before, by a random rotation of the given image, the corresponding new sensation function $I(x, y)$ would then depend essentially both on x and y and would then yield as curvature C of this new image a possibly rather complicated compilation of many convolutions of I with Gaussians and Gabors), in general, one could not help to realise that these neurobiological "theories" do not "work", simply by confronting the perceived images with what should be expected to be seen according to these "theories", whereas the image description as given by the curvature C proper remains effective indeed. We will present some concrete illustrations of these latter comments in our subsequent paper [22].

References

[1] R. L. Gregory (Ed.), *The Oxford Companion to the Mind*, Oxford University Press, Oxford, 1998.

[2] R. Snowden, P. Thompson and T. Troscianko, *Basic Vision -An Introduction to Visual Perception-*, Oxford University Press, Oxford, 2006.

[3] M. W. Levine, *Levine and Shefner: Fundamentals of Sensation and Perception*, Oxford University Press, Oxford, 2002.

[4] H. R. Schiffman, *Sensation and Perception*, J. Wiley and Sons Inc, New York, 1995.

[5] D. M. Eagleman, *Visual Illusions and Neurobiology*, Nature Neuroscience Reviews **2** (2001), 920–926.

[6] B. Ons and P. Verstraelen, *A geometrical description of visual sensation*, Kragujevac J. Math. **33** (2010), 5–15.

[7] B. Ons and P. Verstraelen, *A geometrical description of visual sensation II*, Kragujevac J. Math. **34** (2010), 5–24.

[8] P. Verstraelen, *A remark on visual illusions and neuroscience: "basically, we only see what there is to be seen"*, Psychological Reports, Laboratory of Experimental Psychology, K. U. Leuven, 2003.

[9] L. Verstraelen, *The geometry of eye and brain*, Soochow J. Math. **30** (2004), 367-376.

[10] L. Verstraelen, *A geometrical descripition of visual perception*, Kragujevac J. Math. **28** (2005), 7–17.
[11] J. J. Koenderink *The structure of images*, Biol. Cybern. **50** (1984), 363–370.
[12] J. J. Koenderink and A. J. van Doorn, *The structure of relief*, Adv. Imag. Electr. Physics **103** (1998), 65-150.
[13] B. Ons, P. Verstraelen and J. Wagemans, *A Theory of Visual Anisotropy*, (preprint).
[14] W. Kühnel, *Differentialgeometrie (Kurven–Flächen–Mannigfaltigkeiten)*, Vieweg, Weisbaden, 2008; (there is also some English translation: A. M. S., Student Mathematical Library, volume 16).
[15] L. Verstraelen, *A concise mini History of Geometry*, Tangenta (Journal for Mathematics and Informatics of the Serbian Mathematical Society) **57(1)** (2009), 1–7, (in Serbian).
[16] F. Casorati, *Mesure de la courbure suivant l'idée commune*, Acta Math. **14** (1890), 95–110.
[17] S. Haesen, D. Kowalczyk and L. Verstraelen, *On the extrinsic principal directions of Riemannian submanifolds*, Note di Mathematica **29** (2008), 41–53.
[18] B. Y. Chen, *Pseudo–Riemannian Geometry, δ–invariants and Applications*, World. Sci. Publ. Co, Singapore, 2011.
[19] J. J. Koenderink and A. J. van Doorn, *Image Processing Done Right*, Lecture Notes in Computer Science, **2350** (2002), 158–172.
[20] H. Sachs, *Isotrope Geometrie des Raumes*, Vieweg, Weisbaden, 1990.
[21] B. M. ter Haar Romeny (Ed.), *Geometry - Driven Diffusion in Computer Vision*, Kluwer Acad. Publ., Dordrecht, 1994.
[22] B. Ons and P. Verstraelen, *Some comments on Hermann's grid*, (in preparation).

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