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ON HYPERSURFACES IN SPACE FORMS SATISFYING PARTICULAR CURVATURE CONDITIONS OF TACHIBANA TYPE

R. DESZCZ [1], M. GŁOGOWSKA [2], M. PLAUE [3], K. SAWICZ [4], AND M. SCHERFNER [5]

Abstract. We investigate hypersurfaces in space forms satisfying particular curvature conditions which are strongly related to pseudosymmetry. Expressing certain products of curvature tensors as linear combinations of Tachibana tensors we deduce several pseudosymmetry-type results.

1. Introduction

A semi-Riemannian manifold (M, g), $\dim M = n \geq 3$, is said to be locally symmetric if its curvature tensor R is parallel with respect to the Levi–Civita connection ∇, i. e., $\nabla R = 0$ holds on M. The last equation leads to the integrability condition

$$R \cdot R = 0 \tag{1.1}$$

and a semi-Riemannian manifold (M, g), $n \geq 3$, is called semisymmetric if (1.1) holds on M. We refer to Section 2 for precise definitions of the symbols used. Semisymmetric Riemannian manifolds were classified by Z. I. Szabó, locally, in [29] and there are several important results concerning such manifolds. So for example K. Nomizu conjectured in [22] that all complete irreducible semisymmetric Riemannian manifolds of dimension $n \geq 3$ are locally symmetric. This was answered in the negative by H. Takagi for $n = 3$ and by K. Sekigawa for $n \geq 3$ ([28]).

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Pseudosymmetric manifolds form an essential extension of the class of semisymmetric manifolds. We present a result that is related to this statement: Hypersurfaces M of dimension ≥ 3 and of type number two which are isometrically immersed in a Euclidean space (or more generally, in a semi-Euclidean space) are semisymmetric. This is not true if the ambient space is a space of non-zero constant curvature. Namely, on hypersurfaces M of type number two that are isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$ with signature $(s, n+1-s)$, $n \geq 3$, we have ([4])

$$R \cdot R \;=\; \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, R). \tag{1.2}$$

Here, $c = \frac{\widetilde{\kappa}}{n(n+1)}$ and $\widetilde{\kappa}$ are the sectional and scalar curvature of the ambient space, respectively, and $Q(g, R)$ is the Tachibana tensor of g and R. We note that hypersurfaces M in Riemannian spaces of constant curvature $N^{n+1}(c)$, $n \geq 3$, that have at most two distinct principal curvatures at every point also satisfy a condition of this kind (see Remark 5.1 (i) of the present paper). More generally, a semi-Riemannian manifold (M, g), $n \geq 3$, is said to be pseudosymmetric (see e.g. [10]) if the condition

$$R \cdot R \;=\; L_R\, Q(g, R) \tag{1.3}$$

holds on M, or more precisely, if (1.3) is satisfied on the set U_R of all points of M at which the curvature tensor R is not proportional to the Kulkarni–Nomizu tensor $g \wedge g$.

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be Ricci-pseudosymmetric if $R \cdot S = L_S\, Q(g, S)$ holds on M, i. e., if this condition is satisfied on the set U_S of all points of M at which the Ricci tensor S is not proportional to the metric tensor g. Every pseudosymmetric manifold is Ricci-pseudosymmetric. The converse statement is not true: For instance, every Cartan hypersurface of dimension $n = 6, 12, 24$ is a Ricci-pseudosymmetric but non-pseudosymmetric manifold that satisfies (see e.g. [15])

$$R \cdot S \;=\; \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, S). \tag{1.4}$$

A 3-dimensional Cartan hypersurface satisfies (1.3) with $L_R = \frac{\widetilde{\kappa}}{12}$. Proposition 3.2 and Theorem 4.2 of [4] imply that every hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 5$, that has principal curvatures $\lambda > 0$, $-\lambda$ and 0 at every point with the multiplicities p, p and $n - 2p$, $p \geq 1$, is a Ricci-pseudosymmetric but non-pseudosymmetric manifold satisfying (1.4). This has a direct geometrical meaning, if we regard the so called austerity. It can be formulated as an algebraic condition on the second fundamental form of a hypersurface and mainly asserts that the eigenvalues of its second fundamental form, when measured in any normal direction, occur in oppositely signed pairs [2]. Thus, we can state that several austere hypersurfaces satisfy (1.4).

Pseudosymmetry and Ricci-pseudosymmetry are certain special conditions of pseudosymmetry type. We may consider other conditions of this kind like the following particular one on hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$: the tensor $R \cdot R$, $R \cdot C$, $C \cdot R$ or $R \cdot C - C \cdot R$ may be written as a linear combination of the Tachibana tensors $Q(S, R)$, $Q(g, R)$, $Q(g, g \wedge S)$ and $Q(S, g \wedge S)$. For instance,

$$\begin{aligned} R \cdot C &= \alpha_1 Q(S,R) + \alpha_2 Q(g,R) \\ &\quad + \alpha_3 Q(g, g \wedge S) + \alpha_4 Q(S, g \wedge S) \end{aligned} \tag{1.5}$$

where $\alpha_1, \ldots \alpha_4$, are functions and C denotes the Weyl conformal curvature tensor. In Section 4, we investigate hypersurfaces M in $N^{n+1}(c)$, $n \geq 4$ that satisfy (1.5), which we call hypersurfaces of Tachibana type. It is obvious that (1.3) implies

$$R \cdot C = L_C \, Q(g, C). \tag{1.6}$$

For hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, the converse statement is also true (Remark 5.1 (ii)). We note that from (1.6), by making use of (2.2), it follows that every pseudosymmetric manifold of dimension ≥ 4 satisfies (1.5). More generally, every Ricci-pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfies (1.5) and is therefore of Tachibana type (cf. [15], Proposition 5.1 (iv)).

We prove (Theorem 4.1) that on hypersurfaces in space forms satisfying (1.5) we have

$$R \cdot R = Q(g, B) \tag{1.7}$$

where B is a generalized curvature tensor. We call hypersurfaces satisfying (1.7) of special Tachibana type.

In the last section, we will further investigate such hypersurfaces, and also other special conditions of Tachibana type, namely hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, on which the tensors $R \cdot R$, $R \cdot C$, $C \cdot R$ or $R \cdot C - C \cdot R$ may be expressed by the Tachibana tensor $Q(g, B)$, where B is a generalized curvature tensor.

We prove (see Theorems 5.1–5.3) that in every case the tensor B may be written as a linear combination of R and the tensors $g \wedge g$, $g \wedge S$, $g \wedge S^2$ and $S \wedge S$. We also determine the coefficients of the decomposition (1.5).

2. Preliminaries

Throughout this paper, all manifolds are assumed to be connected paracompact manifolds of class C^∞. Let (M,g) be an n-dimensional, $n \geq 3$, semi-Riemannian manifold, and let ∇ be its Levi–Civita connection and $\Xi(M)$ the Lie algebra of vector fields on M. We define on M the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X,Y)$ of $\Xi(M)$, respectively, by

$$\begin{aligned} (X \wedge_A Y)Z &= A(Y,Z)X - A(X,Z)Y\,, \\ \mathcal{R}(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \end{aligned}$$

where A is a symmetric $(0,2)$-tensor on M and $X,Y,Z \in \Xi(M)$. The Ricci tensor S, the Ricci operator $\mathcal{S}$ and the scalar curvature κ of (M,g) are defined by $S(X,Y) = tr\{Z \to \mathcal{R}(Z,X)Y\}$, $g(\mathcal{S}X,Y) = S(X,Y)$ and $\kappa = tr\,\mathcal{S}$, respectively. The endomorphism $\mathcal{C}(X,Y)$ is defined by

$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y)Z.$$

Now the $(0,4)$-tensor G, the Riemann–Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M,g) are defined by

$$\begin{aligned} G(X_1,X_2,X_3,X_4) &= g((X_1 \wedge_g X_2)X_3, X_4), \\ R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3, X_4), \\ C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3, X_4), \end{aligned}$$

respectively, where $X_1, X_2, \ldots \in \Xi(M)$. We define the following subsets of M: $U_R = \{x \in M \,|\, R - \frac{\kappa}{(n-1)n} G \neq 0 \text{ at } x\}$, $U_S = \{x \in M \,|\, S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$ and $U_C = \{x \in M \,|\, C \neq 0 \text{ at } x\}$. We note that $U_S \cap U_C \subset U_R$.

Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let B be a $(0,4)$-tensor associated with $\mathcal{B}(X,Y)$ by

$$B(X_1,X_2,X_3,X_4) = g(\mathcal{B}(X_1,X_2)X_3,X_4)\,. \tag{2.1}$$

The tensor B is said to be a generalized curvature tensor ([23]) if the following conditions are fulfilled: $B(X_1,X_2,X_3,X_4) = B(X_3,X_4,X_1,X_2)$ and

$$B(X_1,X_2,X_3,X_4) + B(X_3,X_1,X_2,X_4) + B(X_2,X_3,X_1,X_4) = 0.$$

Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$, and let B be the tensor defined by (2.1). We extend the endomorphism $\mathcal{B}(X,Y)$ to a derivation $\mathcal{B}(X,Y)\cdot$ of the algebra of tensor fields on M, assuming that it commutes with contractions and $\mathcal{B}(X,Y)\cdot f = 0$ for any smooth function f on M. Now for a $(0,k)$-tensor field T, $k \geq 1$, we can define the $(0,k+2)$-tensor $B\cdot T$ by

$$\begin{aligned} & (B\cdot T)(X_1,\dots,X_k,X,Y) = (\mathcal{B}(X,Y)\cdot T)(X_1,\dots,X_k) \\ = \; & -T(\mathcal{B}(X,Y)X_1,X_2,\dots,X_k) - \cdots - T(X_1,\dots,X_{k-1},\mathcal{B}(X,Y)X_k)\,. \end{aligned}$$

In addition, if A is a symmetric $(0,2)$-tensor, we define the $(0,k+2)$-tensor $Q(A,T)$ by

$$\begin{aligned} & Q(A,T)(X_1,\dots,X_k,X,Y) = (X\wedge_A Y\cdot T)(X_1,\dots,X_k) \\ = \; & -T((X\wedge_A Y)X_1,X_2,\dots,X_k) - \cdots - T(X_1,\dots,X_{k-1},(X\wedge_A Y)X_k)\,. \end{aligned}$$

In this manner we obtain the $(0,6)$-tensors $B\cdot B$ and $Q(A,B)$. Substituting $\mathcal{B} = \mathcal{R}$ or $\mathcal{B} = \mathcal{C}$, $T = R$ or $T = C$ or $T = S$, $A = g$ or $A = S$ in the above formulas, we get the tensors $R\cdot R$, $R\cdot C$, $C\cdot R$, $R\cdot S$, $Q(g,R)$, $Q(S,R)$, $Q(g,C)$ and $Q(g,S)$.

For a symmetric $(0,2)$-tensor E and a $(0,k)$-tensor T, $k \geq 2$, we define their Kulkarni–Nomizu product $E\wedge T$ by ([7])

$$\begin{aligned} & (E\wedge T)(X_1,X_2,X_3,X_4;Y_3,\dots,Y_k) \\ = \; & E(X_1,X_4)T(X_2,X_3,Y_3,\dots,Y_k) + E(X_2,X_3)T(X_1,X_4,Y_3,\dots,Y_k) \\ & - E(X_1,X_3)T(X_2,X_4,Y_3,\dots,Y_k) - E(X_2,X_4)T(X_1,X_3,Y_3,\dots,Y_k). \end{aligned}$$

The tensor $E\wedge T$ will be called the Kulkarni–Nomizu tensor of E and T. The following tensors are generalized curvature tensors: R, C and $E\wedge F$, where E and F are symmetric $(0,2)$-tensors. We have $G = \frac{1}{2}\, g\wedge g$ and

$$C = R - \frac{1}{n-2}\, g\wedge S + \frac{\kappa}{(n-2)(n-1)}\, G. \tag{2.2}$$

For symmetric $(0,2)$-tensors E and F we have (see e.g. [8], Section 3)

$$Q(E, E\wedge F) = -Q(F,\overline{E}). \tag{2.3}$$

We also have (cf. [7], eq. (3))

$$E \wedge Q(E,F) \;=\; -Q(F,\overline{E}). \tag{2.4}$$

For a symmetric $(0,2)$-tensor A we denote by $\mathcal{A}$ the endomorphism related to A by $g(\mathcal{A}X,Y) = A(X,Y)$. Now we define the tensor A^p, $p \geq 2$, by $A^p(X,Y) = A^{p-1}(\mathcal{A}X,Y)$.

Let A be a symmetric $(0,2)$-tensor A and T a $(0,p)$-tensor, $p \geq 2$. Following [18], we will call the tensor $Q(A,T)$ the Tachibana tensor of A and T, or the Tachibana tensor for short. We like to point out that in some papers, $Q(g,R)$ is called the Tachibana tensor (see e.g. [19], [20], [21], [24] and [30]). By an application of (2.3) we obtain on M the identities

$$Q(g, g \wedge S) \;=\; -Q(S,G) \quad \text{and} \quad Q(S, g \wedge S) \;=\; -\frac{1}{2}\, Q(g, S \wedge S).$$

From the tensors g, R and S we define the following $(0,6)$-Tachibana tensors: $Q(S,R)$, $Q(g,R)$, $Q(g, g \wedge S)$ and $Q(S, g \wedge S)$. Using (2.3) we can check that other $(0,6)$-Tachibana tensors that are constructed from g, R and S may be expressed by the four Tachibana tensors above or vanish identically on M.

Let B_{hijk}, T_{hijk} and A_{ij} be the local components of the generalized curvature tensors B and T and a symmetric $(0,2)$-tensor A on M, respectively, where Latin indices range from 1 to n. The local components $(B\cdot T)_{hijklm}$ and $Q(A,T)_{hijklm}$ of the tensors $B \cdot T$ and $Q(A,T)$ are the following:

$$(B \cdot T)_{hijklm} \;=\; g^{rs}(T_{rijk}B_{shlm} + T_{hrjk}B_{silm} + T_{hirk}B_{sjlm} + T_{hijr}B_{sklm}),$$

$$\begin{aligned} Q(A,T)_{hijklm} \;=\;& A_{hl}T_{mijk} + A_{il}T_{hmjk} + A_{jl}T_{himk} + A_{kl}T_{hijm} \\ & - A_{hm}T_{lijk} - A_{im}T_{hljk} - A_{jm}T_{hilk} - A_{km}T_{hijl}. \end{aligned}$$

If we contract the last equation with g^{ij} and g^{hm}, we obtain

$$\begin{aligned} g^{rs}Q(A,T)_{hrsklm} \;=\;& A^s_l T_{skhm} - A^s_l T_{shmk} - A^s_m T_{skhl} + A^s_m T_{shlk} \\ & + Q(A, Ric(T))_{hklm}, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} g^{rs}Q(A,T)_{rijkls} \;=\;& -A^s_i T_{sljk} + A^s_l T_{sijk} + A^s_j T_{sikl} + A^s_k T_{silj} \\ & + A_{lk} Ric(T)_{ij} - A_{jl} Ric(T)_{ik} - g^{rs}A_{rs}T_{lijk}. \end{aligned} \tag{2.6}$$

Lemma 2.1. *Let (M, g), $n \geq 3$, be a semi-Riemannian manifold. Suppose that the following equation is satisfied at a point of M:*

$$S^s_h R_{sklm} + S^s_l R_{skmh} + S^s_m R_{skhl} \;=\; S^s_k R_{shml} + S^s_l R_{shkm} + S^s_m R_{shlk}, \tag{2.7}$$

Then at this point we have

$$S^s_h R_{sklm} + S^s_l R_{skmh} + S^s_m R_{skhl} \;=\; 0. \tag{2.8}$$

Proof. Summing (2.7) cyclically in h, l, m we obtain

$$\begin{aligned} &3\,(S^s_h R_{sklm} + S^s_l R_{skmh} + S^s_m R_{skhl}) \\ &= S^s_h (R_{smkl} + R_{slmk}) + S^s_l (R_{shkm} + R_{smhk}) + S^s_m (R_{slkh} + R_{shlk}), \end{aligned} \tag{2.9}$$

which yields

$$\begin{aligned} &3\,(S^s_h R_{sklm} + S^s_l R_{skmh} + S^s_m R_{skhl}) \\ &= -S^s_h R_{sklm} - S^s_l R_{skmh} - S^s_m R_{skhl}, \end{aligned} \tag{2.10}$$

completing the proof. □

Proposition 2.1. *Let (M, g), $n \geq 4$, be a semi-Riemannian manifold that satisfies*

$$R \cdot R \;=\; Q(S, R) + L\, Q(g, C) \tag{2.11}$$

on $U_C \subset M$. Then the condition (2.8) *holds on M.*

Proof. From the equation

$$(R \cdot R)_{hijklm} \;=\; Q(S, R)_{hijklm} + L\, Q(g, C)_{hijklm}, \tag{2.12}$$

by contraction with g^{ij}, we obtain

$$S^s_h R_{sklm} + S^s_k R_{shlm} \;=\; S^s_l R_{skhm} + S^s_l R_{shkm} - S^s_m R_{skhl} - S^s_m R_{shkl}, \tag{2.13}$$

i. e., the equation (2.7). This, together with Lemma 2.1, implies (2.8) on U_C. Clearly, at every point of $M \setminus U_C$ we have

$$(R \cdot R)_{hijklm} \;=\; Q(S, R)_{hijklm}.$$

Contracting this with g^{ij}, we again obtain (2.7) which by Lemma 2.1 implies (2.8) on $M \setminus U_C$; this completes the proof. □

Remark 2.1. The last proposition also is true for every 3-dimensional semi-Riemannian manifold since on such manifolds we have the identity

$$R \cdot R \;=\; Q(S, R).$$

3. Curvature conditions

Let M be a hypersurface isometrically immersed in $N_s^{n+1}(c)$, $n \geq 4$. We denote by $U_H \subset M$ the set of all points at which the tensor H^2 is not a linear combination of the metric tensor g and the second fundamental tensor H. We have $U_H \subset U_C \cap U_S \subset M$.

Hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, such that at every point of U_H the tensor $R \cdot C$ is a linear combination of the Tachibana tensors $Q(S, R)$, $Q(g, R)$ and $Q(g, g \wedge S)$ were investigated in [15]. This condition means that

$$R \cdot C = \alpha_1 Q(S, R) + \alpha_2 Q(g, R) + \alpha_3 Q(g, g \wedge S) \tag{3.1}$$

holds on $U_H \subset M$, where α_1, α_2 and α_3 are functions on this set. In this paper we will investigate hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, for which at every point of $U_H \subset M$ the tensor $R \cdot C$ may be expressed as a linear combination of the tensors $Q(S, R)$, $Q(g, R)$, $Q(g, g \wedge S)$ and $Q(S, g \wedge S)$, i.e., (1.5) holds on U_H, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are functions on this set and α_4 is non-zero. According to [9] (Corollary 4.1), for a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, if at every point of $U_H \subset M$ one of the tensors $R \cdot C$, $C \cdot R$ or $R \cdot C - C \cdot R$ is a linear combination of the tensor $R \cdot R$ and a finite sum of the Tachibana tensors of the form $Q(A, B)$, where A is a symmetric $(0, 2)$-tensor and B a generalized curvature tensor, then

$$H^3 = tr(H)\, H^2 + \psi\, H + \rho\, g \tag{3.2}$$

holds on U_H, where ψ and ρ are functions. In particular, if (1.5) is satisfied on U_H then (3.2) holds on this set. Conversely, if (3.2) holds on $U_H \subset M$ for a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, then on this set we have among other results (see e.g. [26], Theorem 5.1):

$$\begin{aligned} R \cdot C &= -\frac{\rho}{n-2} Q(g, g \wedge H) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)} Q(g, R) \\ &\quad + Q(S, R) + \frac{(n-3)\widetilde{\kappa}}{(n-2)n(n+1)} Q(g, g \wedge S), \end{aligned} \tag{3.3}$$

$$\begin{aligned} C \cdot R &= \frac{1}{n-2}\left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)}\right) Q(g, R) \\ &\quad + \frac{n-3}{n-2} Q(S, R) + \frac{(n-3)\widetilde{\kappa}}{(n-2)n(n+1)} Q(g, g \wedge S), \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
R\cdot C - C\cdot R &= -\frac{\rho}{n-2}\, Q(g, g\wedge H) + \frac{1}{n-2}\, Q(S,R)\\
&\quad -\frac{1}{n-2}\Big(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\Big)\, Q(g,R),
\end{aligned}
\tag{3.5}
$$

and ([25], eqs. (3.7), (3.9))

$$
\rho\, H = S^2 + \Big(\varepsilon\psi - \frac{2(n-1)\widetilde{\kappa}}{n(n+1)}\Big)\, S + \lambda_1\, g,
\tag{3.6}
$$

$$
R\cdot S = \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g,S) + \rho\, Q(g,H),
\tag{3.7}
$$

respectively, where κ is the scalar curvature of M, $\varepsilon = \pm 1$ and

$$
\lambda_1 = \Big(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \varepsilon\psi\Big)\frac{(n-1)\widetilde{\kappa}}{n(n+1)} + \rho\, tr(H).
$$

If (3.2) and $\rho = 0$ hold at a point of U_H, i. e. at this point we have

$$
H^3 = tr(H)\, H^2 + \psi\, H,
\tag{3.8}
$$

then (3.3), (3.6) and (3.7) turn into (3.1),

$$
S^2 = \Big(\frac{2(n-1)\widetilde{\kappa}}{n(n+1)} - \varepsilon\psi\Big)\, S - \lambda_1\, g
\tag{3.9}
$$

and (1.4), respectively.

Let $U_\rho \subset U_H$ be the set of all points at which (3.2) with $\rho \neq 0$ holds. Examples of hypersurfaces in Euclidean spaces with three distinct principal curvatures that satisfy (3.2) on U_ρ are given in [27]. The curvature tensor R of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, for which (3.2) holds on U_ρ, is expressed by ([25], Theorem 3.2)

$$
\begin{aligned}
&2\varepsilon\rho^2\Big(R - \frac{\widetilde{\kappa}}{n(n+1)}\, G\Big)\\
&= \Big(S^2 - \Big(\frac{2(n-1)\widetilde{\kappa}}{n(n+1)} - \varepsilon\psi\Big)\, S + \Big(\Big(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \varepsilon\psi\Big)\frac{(n-1)\widetilde{\kappa}}{n(n+1)} + \rho\, tr(H)\Big)\, g\Big)\\
&\quad \wedge\Big(S^2 - \Big(\frac{2(n-1)\widetilde{\kappa}}{n(n+1)} - \varepsilon\psi\Big)\, S + \Big(\Big(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \varepsilon\psi\Big)\frac{(n-1)\widetilde{\kappa}}{n(n+1)} + \rho\, tr(H)\Big)\, g\Big),
\end{aligned}
\tag{3.10}
$$

i. e., R may be written on U_ρ as a linear combination of the Kulkarni–Nomizu tensors constructed from the tensors g, S and S^2. If the curvature tensor R of a semi-Riemannian manifold (M, g), $n \geq 4$, is given on $U_C \cap U_S$ as a linear combination of the Kulkarni–Nomizu tensors $g\wedge g$, $g\wedge S$ and $S\wedge S$, then such a manifold is called a Roter-type manifold. Such manifolds were recently investigated in [12] and [13]. We also refer to [17] for a survey on Roter-type manifolds, as well as on Roter-type hypersurfaces. We like to note that the curvature tensor of generalized (κ, μ)-space forms splits into six terms (see e.g. [3]).

The condition (3.7), via (3.6), turns into

$$R \cdot S \;=\; Q(g, S^2 + (\varepsilon\psi - \frac{(2n-3)\widetilde{\kappa}}{n(n+1)})\, S). \tag{3.11}$$

Therefore, we can say that on U_ρ the tensors S and $A = S^2 + (\varepsilon\psi - \frac{(2n-3)\widetilde{\kappa}}{n(n+1)})\, S$ are pseudosymmetrically related ([6]).

In Section 4 we investigate hypersurfaces of Tachibana type, i. e., satisfying (1.5) on U_ρ. The main result of that section (Theorem 4.1) states that in space forms, such hypersurfaces are special in the sense that $R \cdot R = Q(g, B)$ holds on U_ρ for a generalized curvature tensor B. We also give an explicit formula for B which shows that R and B are pseudosymmetrically related on U_ρ.

In the last section, we further investigate hypersurfaces that satisfy on U_H the equation $R \cdot R = Q(g, B)$ for a generalized curvature tensor B. We prove that the tensor B can be written as a linear combination of R, $S \wedge S$, $g \wedge S^2$, $g \wedge S$ and G (Theorem 5.1).

Moreover, in that section we consider hypersurfaces such that the tensor $R \cdot C$, resp. the tensor $C \cdot R$ or the tensor $R \cdot C - C \cdot R$, is equal to the Tachibana tensor $Q(g, B)$, where B is a generalized curvature tensor. We prove (Theorem 5.2) that in every case, the tensor B is a certain linear combination of the curvature tensor R and the Kulkarni–Nomizu tensors $g \wedge g$, $g \wedge S$, $g \wedge S^2$ and $S \wedge S$.

4. Hypersurfaces in space forms of Tachibana type

On every hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, we have ([14])

$$R \cdot R \;=\; Q(S, R) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, Q(g, C). \tag{4.1}$$

Proposition 2.1 together with (2.6) and (4.1) implies the following lemma:

Lemma 4.1. *The following identity holds on every hypersurface M in $N_s^{n+1}(c)$, $n \geq 3$:*

$$g^{rs} Q(S, R)_{rijkls} \;=\; -\kappa\, R_{lijk} - S_i^s R_{sljk} + S_{ij} S_{kl} - S_{ik} S_{jl}. \tag{4.2}$$

We also note that (4.1), i. e.,

$$(R \cdot R)_{hijklm} \;=\; Q(S, R)_{hijklm} - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, Q(g, C)_{hijklm}, \tag{4.3}$$

by contraction with g^{ij}, yields

$$(R \cdot S)_{hklm} \;=\; g^{rs} Q(S,R)_{hrsklm}. \tag{4.4}$$

Using results of [15], we can prove:

Lemma 4.2. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, which satisfies the Tachibana-type condition* (1.5) *on $U_\rho \subset U_H \subset M$. Then the function α_4 is non-zero at every point of this set.*

Proof. Suppose that α_4 vanishes at $x \in U_\rho$. Evidently, at x the condition (1.5) is equivalent to (3.1). In addition, if $\alpha_1 \neq 1$ at x then in view of Theorem 6.4 of [15], (1.2) holds at this point. This clearly yields (1.4). Now, (1.4) together with (3.7) give $\rho\, Q(g,H) = 0$, which implies $\rho = 0$ at x, a contradiction. If $\alpha_1 = 1$ at x, then, using Theorem 6.2 in [15], we have at this point:

$$R \cdot C \;=\; Q(S,R) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, Q(g,R) + \frac{(n-3)\widetilde{\kappa}}{(n-2)n(n+1)}\, Q(g, g \wedge S).$$

However, considering Theorem 6.1 of [15], this is equivalent to (1.4), which together with (3.7) gives $\rho\, Q(g,H) = 0$, and as a consequence: $\rho = 0$ at x, which is a contradiction. Thus α_4 is non-zero at every point of U_ρ. The last remark completes the proof. □

Lemma 4.3. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, which satisfies the Tachibana-type condition* (1.5) *on $U_\rho \subset U_H \subset M$. Then:*

(i) *On U_ρ we have*

$$\alpha_4 \;=\; -\alpha_1, \tag{4.5}$$

$$\alpha_3 = -\frac{1}{n-2}\Big(\alpha_1\big(\kappa + \varepsilon\psi - \frac{(2n-3)\widetilde{\kappa}}{n(n+1)}\big) + \alpha_2\Big), \tag{4.6}$$

$$\begin{aligned} &(\alpha_1 - 1)\, Q(S,R) + \Big(\alpha_2 + \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\Big)\, Q(g,R) \\ &+ \frac{1}{n-2}\Big(\varepsilon\psi - \frac{(3n-5)\widetilde{\kappa}}{n(n+1)} - \alpha_1\big(\kappa + \varepsilon\psi - \frac{(2n-3)\widetilde{\kappa}}{n(n+1)}\big) - \alpha_2\Big)\, Q(g, g \wedge S) \\ &- \alpha_1\, Q(S, g \wedge S) + \frac{1}{n-2}\, Q(g, g \wedge S^2) \;=\; 0. \end{aligned} \tag{4.7}$$

(ii) *At every point of* U_ρ *we have* $\alpha_1 \neq 1$. *Moreover,*

$$Q(S,R) \;=\; Q(g,T) \tag{4.8}$$

holds on U_ρ, *where the* $(0,4)$*-tensor* T *is defined by*

$$\begin{aligned} T \;=\; & (1-\alpha_1)^{-1}\,((\alpha_2 + \frac{(n-2)\widetilde{\kappa}}{n(n+1)})\,R + \frac{\alpha_1}{2}\,S\wedge S + \frac{1}{n-2}\,g\wedge S^2 \\ & + \frac{1}{n-2}(\varepsilon\psi - \frac{(3n-5)\widetilde{\kappa}}{n(n+1)} - \alpha_1(\kappa + \varepsilon\psi - \frac{(2n-3)\widetilde{\kappa}}{n(n+1)}) - \alpha_2)\,g\wedge S). \end{aligned} \tag{4.9}$$

Proof. (i) Contracting (1.5), i. e.,

$$\begin{aligned} (R\cdot C)_{hijklm} \;=\; & \alpha_1\,Q(S,R)_{hijklm} + \alpha_2\,Q(g,R)_{hijklm} \\ & + \alpha_3\,Q(g,g\wedge S)_{hijklm} + \alpha_4\,Q(S,g\wedge S)_{hijklm}, \end{aligned} \tag{4.10}$$

with g^{ij} and using (2.5) and (4.4), we obtain

$$-\alpha_1\,(R\cdot S) \;=\; Q(g,\alpha_4\,S^2 + (\alpha_2 + (n-2)\alpha_3 - \kappa\alpha_4)\,S). \tag{4.11}$$

By applying (3.7) to this result we get

$$\begin{aligned} & -\alpha_1\rho\,Q(g,H) \\ =\; & Q(g,\alpha_4\,S^2 + (\frac{\alpha_1\widetilde{\kappa}}{n(n+1)} + \alpha_2 + (n-2)\alpha_3 - \kappa\alpha_4)\,S). \end{aligned} \tag{4.12}$$

From the last relation, in view of Lemma 2.4 in [14], it follows that

$$-\alpha_1\rho\,H \;=\; \alpha_4\,S^2 + (\frac{\alpha_1\widetilde{\kappa}}{n(n+1)} + \alpha_2 + (n-2)\alpha_3 - \kappa\alpha_4)\,S + \lambda_2\,g \tag{4.13}$$

holds on U_ρ, where λ_2 is a function. Now (4.13), together with (3.6), gives

$$\begin{aligned} & -(\alpha_1+\alpha_4)\,S^2 \\ =\; & (\alpha_1(\varepsilon\psi - \frac{(2n-3)\widetilde{\kappa}}{n(n+1)}) + \alpha_2 + (n-2)\alpha_3 - \kappa\alpha_4)\,S + \lambda_3\,g \end{aligned} \tag{4.14}$$

where $\lambda_3 = \alpha_1 + \lambda_2$. Suppose that $\alpha_1 \neq -\alpha_4$ at x. Thus, at x the tensor S^2 is a linear combination of the tensors S and g , i. e.,

$$S^2 \;=\; \beta_1\,S + \beta_2\,g, \qquad \beta_1,\beta_2 \in I\!R,$$

holds at x. Therefore, (3.10) reduces at x to

$$R \;=\; \frac{\beta_3}{2}\,S\wedge S + \beta_4\,g\wedge S + \beta_5\,G, \qquad \beta_3,\beta_4,\beta_5 \in I\!R. \tag{4.15}$$

We note that $\beta_3 \neq 0$. In fact, if we had $\beta_3 = 0$, then – in a standard way – we would obtain $C = 0$ from (4.15), a contradiction. Equation (4.15) implies (see e.g. Section 3

of [17]): $R\cdot R = \beta_6\, Q(g,R)$, hence $R\cdot S = \beta_6\, Q(g,S)$. Since $x \in U_H$ the last condition yields $\beta_6 = \frac{\widetilde{\kappa}}{n(n+1)}$ ([4], Proposition 3.2 and Theorem 3.1; see also the introduction of this paper). Now from (3.7) it follows that $\rho = 0$ at x, a contradiction. Thus we proved that (4.5) holds at x. Now if we apply (4.5) to (4.14), we obtain (4.6). Finally (1.5) and (3.3), via (3.6), (4.5) and (4.6), lead to (4.7) – completing the proof of (i).
(ii) Suppose that $\alpha_1 = 1$ holds at $x \in U_\rho$. Then, (4.7) is equivalent to

$$\begin{aligned} & Q(g, (\alpha_2 + \frac{(n-2)\widetilde{\kappa}}{n(n+1)})\, R + \frac{1}{2}\, S\wedge S \\ & + \frac{1}{n-2}\, g\wedge (S^2 - (\frac{(n-2)\widetilde{\kappa}}{n(n+1)} + \kappa)\, S)) \;=\; 0, \end{aligned} \tag{4.16}$$

which in view of Lemma 1.1 (iii) in [5] yields

$$(\alpha_2 + \frac{(n-2)\widetilde{\kappa}}{n(n+1)})\, R + \frac{1}{2}\, S\wedge S + g\wedge B \;=\; 0 \tag{4.17}$$

where B is the $(0,2)$-tensor defined by

$$B \;=\; \frac{1}{n-2}\,(S^2 - (\frac{(n-2)\widetilde{\kappa}}{n(n+1)} + \kappa\alpha_2)\, S + \lambda_4\, g), \qquad \lambda_4 \in I\!R. \tag{4.18}$$

Contracting (4.17), i. e.,

$$\begin{aligned} & (\alpha_2 + \frac{(n-2)\widetilde{\kappa}}{n(n+1)})\, R_{hijk} + S_{hk}S_{ij} - S_{hj}S_{ik} \\ & + g_{hk}B_{ij} + g_{ij}B_{hk} - g_{hj}B_{ik} - g_{ik}B_{hj} \;=\; 0, \end{aligned} \tag{4.19}$$

with S_l^h we obtain

$$\begin{aligned} & (\alpha_2 + \frac{(n-2)\widetilde{\kappa}}{n(n+1)})\, S_l^r R_{rijk} + S^2_{lk}S_{ij} - S^2_{lj}S_{ik} \\ & + S_{lk}B_{ij} - S_{lj}B_{ik} + g_{ij}D_{lk} - g_{ik}D_{lj} \;=\; 0 \end{aligned} \tag{4.20}$$

where D is the $(0,2)$-tensor defined by $D_{ij} = B_{ir}S^r_j$. If we symmetrize (4.20) in l, i, we get

$$\begin{aligned} & (\alpha_2 + \frac{(n-2)\widetilde{\kappa}}{n(n+1)})\, (R\cdot S)_{lijk} + Q(S,S^2)_{lijk} \\ & - Q(S,B)_{lijk} + Q(g,D)_{lijk} \;=\; 0. \end{aligned} \tag{4.21}$$

We already noted in the introduction that if (1.5) is satisfied on U_H, (3.2) holds on this set. Now, using Proposition 3.2 in [25] (eq. (3.10)), we see that the tensor D is a linear combination of the tensors g, S and S^2. Similarly, we note that from (3.11) it follows that the tensor $R \cdot S$ is a linear combination of the tensors $Q(g, S)$ and $Q(g, S^2)$. Using these facts together with (4.18) and (4.21), we can deduce that

$$Q(S, S^2) + \beta_1 Q(g, S) + \beta_2 Q(g, S^2) = 0, \quad \beta_1, \beta_2 \in \mathbb{R}, \tag{4.22}$$

holds at x. From the last equation, applying Lemma 2.4 (ii) in [14], it follows that at x the tensor S^2 is a linear combination of the tensors g and S. Therefore, (3.10) turns into (4.15). But this, in the same way as in the proof of (i), leads to $\rho = 0$, a contradiction. Thus $\alpha_1 \neq 1$ holds at every point of U_ρ. Now (4.8) is an immediate consequence of (4.7) and the proposition is proved. □

Lemma 4.4. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, that satisfies the Tachibana-type condition* (1.5) *on $U_\rho \subset U_H \subset M$. Then, on U_ρ we have*

$$\alpha_1 = -\alpha_4 = -\frac{1}{n-2}, \tag{4.23}$$

$$\alpha_2 = \frac{1}{n-2}(\kappa + \varepsilon\psi - \frac{(n^2 - 3n + 3)\widetilde{\kappa}}{n(n+1)}), \tag{4.24}$$

$$\alpha_3 = \frac{(n-3)\widetilde{\kappa}}{(n-2)n(n+1)}. \tag{4.25}$$

Proof. Contracting (4.8), i. e.,

$$Q(S, R)_{hijklm} = Q(g, T)_{hijklm}, \tag{4.26}$$

with g^{ij} and g^{hm}, and using (2.6), (4.2) and (4.4), we obtain

$$(R \cdot S)_{hklm} = Q(g, Ric(T))_{hklm}, \tag{4.27}$$

$$\begin{aligned} &-\kappa R_{lijk} - S_i^s R_{sljk} + S_{ij} S_{kl} - S_{ik} S_{jl} \\ &= -(n-1) T_{lijk} + g_{kl} Ric(T)_{ij} - g_{jl} Ric(T)_{ik}, \end{aligned} \tag{4.28}$$

respectively. Furthermore, (4.27), via (3.7), turns into

$$Q(g, Ric(T) - \frac{\widetilde{\kappa}}{n(n+1)} S - \rho H) = 0 \tag{4.29}$$

which, in view of Lemma 2.4 (i) in [14] and (3.6), implies

$$Ric(T) = \frac{\widetilde{\kappa}}{n(n+1)} S + \rho H + \beta_1 g. \tag{4.30}$$

Considering (3.6), this condition turns into

$$Ric(T) \;=\; S^2 + (\varepsilon\psi - \frac{(2n-3)\widetilde{\kappa}}{n(n+1)})\, S + \beta_2\, g \tag{4.31}$$

where β_1 and β_2 are functions on U_ρ. Now (4.28), via (4.31) and equation (3.6) in [25], i. e.,

$$\begin{aligned} S_{ir} g^{rs} R_{sljk} \;&=\; (\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \varepsilon\psi)\,(R_{iljk} - \frac{\widetilde{\kappa}}{n(n+1)}\, G_{iljk}) \\ &\quad + \frac{\widetilde{\kappa}}{n(n+1)}\,(g_{jl}S_{ik} - g_{kl}S_{ij}) - \rho\,(g_{ik}H_{jl} - g_{ij}H_{kl}), \end{aligned} \tag{4.32}$$

yields

$$\begin{aligned} &(n-1)(\alpha_1 - 1)\, T_{lijk} \\ =\;& (\alpha_1 - 1)(\kappa + \varepsilon\psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)})\, R_{lijk} \\ &-(\alpha_1 - 1)\,(S_{ij}S_{kl} - S_{ik}S_{jl}) + \lambda_3\, G_{lijk} \\ &+(\alpha_1 - 1)\,(\varepsilon\psi - \frac{2(n-1)\widetilde{\kappa}}{n(n+1)})\,(g_{ij}S_{kl} + g_{kl}S_{ij} - g_{jl}S_{ik} - g_{ik}S_{jl}) \\ &+(\alpha_1 - 1)\,(g_{ij}S^2_{kl} + g_{kl}S^2_{ij} - g_{jl}S^2_{ik} - g_{ik}S^2_{jl}) \end{aligned} \tag{4.33}$$

where λ_3 is some function on U_ρ. Now (4.9) and (4.33) yield

$$\begin{aligned} &((\alpha_1 - 1)(\kappa + \varepsilon\psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}) + (n-1)(\alpha_2 + \frac{(n-2)\widetilde{\kappa}}{n(n+1)}))\, R_{lijk} \\ &+((n-2)\alpha_1 + 1)\,(S_{ij}S_{kl} - S_{ik}S_{lj}) + \lambda_3\, G_{lijk} \\ &+(\alpha_1 + \frac{1}{n-2})\,(g_{ij}S^2_{kl} + g_{kl}S^2_{ij} - g_{jl}S^2_{ik} - g_{ik}S^2_{jl}) \\ &+((\alpha_1 - 1)\,(\varepsilon\psi - \frac{2(n-1)\widetilde{\kappa}}{n(n+1)}) + \frac{n-1}{n-2}(\varepsilon\psi - \frac{(3n-5)\widetilde{\kappa}}{n(n+1)} - \alpha_2 \\ &-\alpha_1(\kappa + \varepsilon\psi - \frac{(2n-3)\widetilde{\kappa}}{n(n+1)})))\,(g_{ij}S_{kl} + g_{kl}S_{ij} - g_{jl}S_{ik} - g_{ik}S_{jl}) \;=\; 0. \end{aligned} \tag{4.34}$$

By contracting (4.34) with S^l_m, using the fact that the tensor S^3 is a linear combination of the tensors S^2, S and g ([25], eq. (3.10)), and by symmetrizing the resulting equation in i, m, we get

$$(\alpha_1 + \frac{1}{n-2})\, Q(S, S^2) + \alpha_5\, Q(g, S) + \alpha_6\, Q(g, S^2) \;=\; 0 \tag{4.35}$$

where α_5 and α_6 are functions on U_ρ. If $\alpha_1 \neq -\frac{1}{n-2}$ at a point $x \in U_\rho$, then (4.22) holds at x. But this, in the same way as in the proof of Proposition 3.1 (i), leads to a contradiction. Therefore, (4.23) holds on U_ρ. Now (4.34), by making use of (4.23), reduces to

$$\begin{aligned}
&(n-1)(\alpha_2 - \frac{1}{n-2}(\kappa+\varepsilon\psi) + \frac{(n^2-3n+3)\widetilde{\kappa}}{(n-2)n(n+1)})\, R_{lijk} + \lambda_3\, G_{lijk}\\
&-\frac{n-1}{n-2}\,(\alpha_2 - \frac{n-1}{n-2}(\kappa+\varepsilon\psi) + \frac{(n^2-3n+3)\widetilde{\kappa}}{(n-2)n(n+1)})\,(g_{ij}S_{kl} + g_{kl}S_{ij}\\
&-g_{jl}S_{ik} - g_{ik}S_{jl}) \;=\; 0.
\end{aligned} \tag{4.36}$$

Since at every point of U_ρ the Weyl tensor C is non-zero and (4.36) leads to (4.24). But this, together with (4.6), yields (4.25), which completes the proof. □

In summary, Lemmas 4.3 and 4.4, together with (4.1), imply:

Theorem 4.1. *Let M be a hypersurface of Tachibana type in $N_s^{n+1}(c)$, $n \geq 4$, i.e. on $U_\rho \subset U_H \subset M$*

$$R\cdot C = \alpha_1 Q(S,R) + \alpha_2 Q(g,R) + \alpha_3 Q(g, g\wedge S) + \alpha_4 Q(S, g\wedge S)$$

holds for some functions $\alpha_1,\ldots,\alpha_4$. Then we have on U_ρ:

$$\begin{aligned}
\alpha_1 &= -\alpha_4 = -\frac{1}{n-2},\\
\alpha_2 &= \frac{1}{n-2}\left(\kappa+\varepsilon\psi - \frac{n^2-3n+3}{n(n+1)}\tilde{\kappa}\right),\\
\alpha_3 &= \frac{n-3}{(n-2)n(n+1)}\tilde{\kappa}.
\end{aligned}$$

Furthermore, on U_ρ,

$$R\cdot R = Q(g,B)$$

with

$$\begin{aligned}
B &= \left(\frac{1}{n-1}(\kappa+\varepsilon\psi) - \frac{1}{n(n+1)}\tilde{\kappa}\right) R - \frac{(n-2)}{n(n+1)}\tilde{\kappa}C\\
&\quad -\frac{1}{2(n-1)}S\wedge S + \frac{1}{n-1}g\wedge S^2 + \left(\frac{1}{n-1}\varepsilon\psi - \frac{2}{n(n+1)}\tilde{\kappa}\right) g\wedge S,
\end{aligned}$$

i.e. in space forms, every hypersurface of Tachibana type is special.

5. Hypersurfaces in space forms satisfying special conditions of Tachibana type

In the following, we further investigate hypersurfaces that satisfy the equation $R \cdot R = Q(g, B)$ on $U_H \subset M$ for a generalized curvature tensor B. In addition, we investigate hypersurfaces satisfying on $U_H \subset M$ the similar conditions

$$\begin{aligned} R \cdot C &= Q(g, B_1) \\ C \cdot R &= Q(g, B_2) \\ R \cdot C - C \cdot R &= Q(g, B_3) \end{aligned}$$

for generalized curvature tensors B_1, B_2, B_3.

From (1.7) and (4.1) we obtain

$$Q(S, R) = Q(g, B + \frac{(n-2)\widetilde{\kappa}}{n(n+1)} R - \frac{\widetilde{\kappa}}{n(n+1)} g \wedge S). \tag{5.1}$$

Furthermore, (1.7) by a suitable contraction yields

$$R \cdot S = Q(g, Ric(B)). \tag{5.2}$$

Applying this to the identity

$$R \cdot C = R \cdot R - \frac{1}{n-2} g \wedge (R \cdot S)$$

we get

$$R \cdot C = R \cdot R - \frac{1}{n-2} g \wedge Q(g, Ric(B))$$

which by (2.4) turns into

$$R \cdot C = R \cdot R + \frac{1}{n-2} Q(Ric(B), G).$$

Considering (4.1) this yields

$$R \cdot C = Q(S, R) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)} Q(g, C) + \frac{1}{n-2} Q(Ric(B), G). \tag{5.3}$$

In addition, in view of Corollary 4.1 of [9], (3.2) holds on U_H. If we now compare the right-hand sides of (3.7) and (5.2), we get

$$Q(g, Ric(B) - \frac{\widetilde{\kappa}}{n(n+1)} S - \rho H) = 0 \tag{5.4}$$

which implies via Lemma 2.4 in [14]:

$$Ric(B) = \frac{\widetilde{\kappa}}{n(n+1)} S + \rho H + \beta_1 g \tag{5.5}$$

where β_1 is a function on U_H. Furthermore, by applying (3.6) we find

$$Ric(B) = S^2 + (\varepsilon\psi - \frac{(2n-3)\widetilde{\kappa}}{n(n+1)})\, S + \lambda_2\, g \tag{5.6}$$

where λ_2 is a function on U_H. If we contract (5.1), i. e.,

$$Q(S,R)_{hijklm} = Q(g, B + \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, R - \frac{\widetilde{\kappa}}{n(n+1)}\, g \wedge S)_{hijklm},$$

with g^{hm} and use (2.6), (4.2), (4.32) and (5.6), we finally obtain:

Theorem 5.1. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. If a generalized curvature tensor B satisfies*

$$R \cdot R = Q(g, B) \tag{5.7}$$

on $U_H \subset M$, then on this set we have

$$\begin{aligned} B &= \frac{1}{n-1}\, ((\kappa + \varepsilon\psi - \frac{(n-1)^2\widetilde{\kappa}}{n(n+1)})\, R - \frac{1}{2}\, S \wedge S + g \wedge S^2 \\ &\quad + (\varepsilon\psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)})\, g \wedge S + \lambda\, G) \end{aligned} \tag{5.8}$$

where λ is some function on U_H.

Theorem 5.2. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$.*

(i) *If a generalized curvature tensor B_1 satisfies*

$$R \cdot C = Q(g, B_1) \tag{5.9}$$

on $U_H \subset M$, then on this set we have

$$\begin{aligned} B_1 &= \frac{1}{n-1}\, ((\kappa + \varepsilon\psi - \frac{(n-1)^2\widetilde{\kappa}}{n(n+1)})\, R - \frac{1}{n-2}\, g \wedge S^2 \\ &\quad - \frac{1}{2}\, S \wedge S - \frac{1}{n-2}\, (\varepsilon\psi - \frac{(n-1)^2\widetilde{\kappa}}{n(n+1)})\, g \wedge S + \lambda\, G) \end{aligned} \tag{5.10}$$

where λ is some function on U_H.

(ii) *If a generalized curvature tensor B_2 satisfies*

$$C \cdot R = Q(g, B_2) \tag{5.11}$$

on $U_H \subset M$, then on this set we have

$$\begin{aligned} B_2 &= (\frac{\kappa}{n-1} + \frac{2\varepsilon\psi}{n-1} - \frac{\widetilde{\kappa}}{n+1})\, R + \lambda\, G \\ &\quad + \frac{n-3}{(n-2)(n-1)}((\varepsilon\psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)})g \wedge S - \frac{1}{2}\, S \wedge S + g \wedge S^2) \end{aligned} \tag{5.12}$$

where λ is a function on U_H.

(iii) *If a generalized curvature tensor B_3 satisfies*

$$R \cdot C - C \cdot R \;=\; Q(g, B_3) \tag{5.13}$$

on $U_H \subset M$, then on this set we have

$$\begin{aligned} B_3 \;=\;& (-\frac{\varepsilon\psi}{n-1} + \frac{\widetilde{\kappa}}{n(n+1)})\, R + (-\frac{\varepsilon\psi}{n-1} + \frac{2\widetilde{\kappa}}{n(n+1)})\, g \wedge S \\ & -\frac{1}{n-1}\, g \wedge S^2 - \frac{1}{2(n-2)(n-1)}\, S \wedge S + \lambda\, G \end{aligned} \tag{5.14}$$

where λ is a function on U_H.

Proof. (i) From (5.9), applying Corollary 4.1 in [9], it follows that (3.2) holds on U_H. As a consequence, we also have (3.3) and (3.6). Thus,

$$R \cdot C \;=\; Q(S, R) + Q(g, B_4) \tag{5.15}$$

on this set, where

$$\begin{aligned} B_4 \;=\;& -\frac{1}{n-2}\, g \wedge (S^2 + (\varepsilon\psi - \frac{2(n-1)\widetilde{\kappa}}{n(n+1)})\, S + \lambda_1\, g) \\ & -\frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, R + \frac{(n-3)\widetilde{\kappa}}{(n-2)n(n+1)}\, g \wedge S. \end{aligned} \tag{5.16}$$

Furthermore, (5.9) and (5.16) yield $Q(S,R) \;=\; Q(g, B_1 - B_4)$. Applying this to (4.1) we get

$$R \cdot R \;=\; Q(g, B_1 - B_4 - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, C).$$

Now, taking Theorem 5.1 into account, we have

$$\begin{aligned} B_1 \;=\;& B_4 + \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, C + \frac{1}{n-1}\, ((\varepsilon\psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)})\, g \wedge S + \lambda\, G) \\ & + \frac{1}{n-1}\, ((\kappa + \varepsilon\psi - \frac{(n-1)^2\widetilde{\kappa}}{n(n+1)})\, R - \frac{1}{2}\, S \wedge S + g \wedge S^2) \end{aligned}$$

which completes the proof of (i).

(ii) From (5.11), using Corollary 4.1 in [9], it follows that (3.2) holds on U_H. As a consequence, we also have (3.4) and (3.6). Clearly, on U_H we can present (3.4) in the following form:

$$C \cdot R \;=\; \frac{n-3}{n-2}\, Q(S, R) + Q(g, B_5) \tag{5.17}$$

where

$$B_5 = \frac{1}{n-2}((\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)})\, R + \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\, g \wedge S). \tag{5.18}$$

Moreover, (5.11) and (5.17) yield $(n-3)\, Q(S,R) = (n-2)\, Q(g, B_2 - B_5)$. If we apply this to (4.1), we get

$$R \cdot R = Q(g, \frac{n-2}{n-3}\,(B_2 - B_5) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, C).$$

Now, using Theorem 5.1, we have

$$\begin{aligned} B_2 &= B_5 + \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\, C + \frac{n-3}{(n-2)(n-1)}\,((\varepsilon\psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)})\, g \wedge S + \lambda_1\, G) \\ &\quad + \frac{n-3}{(n-2)(n-1)}\,((\kappa + \varepsilon\psi - \frac{(n-1)^2\widetilde{\kappa}}{n(n+1)})\, R - \frac{1}{2}\, S \wedge S + g \wedge S^2)) \end{aligned}$$

which completes the proof of (ii).

(iii) From (5.13), applying Corollary 4.1 in [9], it follows that (3.2) holds on U_H. As a consequence, we also have (3.5) and (3.6). Clearly, on U_H we can present (3.5) in the following form:

$$(n-2)(R \cdot C - C \cdot R) = Q(S,R) + Q(g, B_6) \tag{5.19}$$

where

$$\begin{aligned} B_6 &= -(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)})\, R \\ &\quad - g \wedge (S^2 + (\varepsilon\psi - \frac{2(n-1)\widetilde{\kappa}}{n(n+1)})\, S + \lambda_1\, g). \end{aligned} \tag{5.20}$$

Now, (5.13) and (5.19) yield $Q(S,R) = Q(g, (n-2)B_3 - B_6))$. If we apply this to (4.1), we get

$$R \cdot R = Q(g, (n-2)B_3 - B_6 - \frac{(n-2)\widetilde{\kappa}}{n(n+1)})\, C).$$

Now, in view of Theorem 5.1, we have

$$\begin{aligned} (n-2)\, B_3 &= B_6 + \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, C + \frac{1}{n-1}((\varepsilon\psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)})\, g \wedge S + \lambda_1\, G) \\ &\quad + \frac{1}{n-1}((\kappa + \varepsilon\psi - \frac{(n-1)^2\widetilde{\kappa}}{n(n+1)})\, R - \frac{1}{2}\, S \wedge S + g \wedge S^2) \end{aligned}$$

which completes the proof of (iii). □

We finally consider hypersurfaces already studied in [11] (see Proposition 5.1 (iii) therein):

Theorem 5.3. *Let M be a hypersurface in $N_s^{n+1}(c)$, $c \neq 0$, $n \geq 4$, that satisfies on $U_H \subset M$:*

$$(a)\ \ rank(H) \ = \ 2, \qquad (b)\ \ rank(H^2 - tr(H)\, H) \ = \ 1. \tag{5.21}$$

Then (5.7), (5.9), (5.11) *and* (5.13) *are satisfied. Precisely, we have on U_H:*

$$R \cdot R \ = \ Q(g, B) \ = \ \frac{\kappa}{(n-1)n}\, Q(g, R), \tag{5.22}$$

$$\begin{aligned} R \cdot C \ &= \ Q(g, B_1) \\ &= \ \frac{1}{n-1}\, Q(g, \frac{\kappa}{n}\, R - \frac{1}{2}\, S \wedge S + \frac{(n-3)\kappa}{(n-2)n}\, g \wedge S), \end{aligned} \tag{5.23}$$

$$C \cdot R \ = \ Q(g, B_2) \ = \ 0, \tag{5.24}$$

$$\begin{aligned} R \cdot C - C \cdot R \ &= \ Q(g, B_3) \\ &= \ \frac{1}{n-1}\, Q(g, \frac{\kappa}{n}\, R - \frac{1}{2}\, S \wedge S + \frac{(n-3)\kappa}{(n-2)n}\, g \wedge S). \end{aligned} \tag{5.25}$$

Proof. First of all, (5.21)(a) implies ([4], Theorem 4.2)

$$R \cdot R \ = \ \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, R). \tag{5.26}$$

Now in view of Proposition 5.1 (iii) of [11] on U_H we have (30), (41)(b) and (48) of [11], i.e.

$$\begin{aligned} &(a)\ \ \frac{\kappa}{n-1} \ = \ \frac{\widetilde{\kappa}}{n+1}, \qquad (b)\ \mathrm{rank}\,(S - \frac{\kappa}{n}\, g) \ = \ 1, \\ &(c)\ \ H^3 \ = \ tr(H)\, H^2, \ \psi \ = \ 0. \end{aligned} \tag{5.27}$$

Thus (5.26) by (5.27)(a) turns into

$$R \cdot R \ = \ \frac{\kappa}{(n-1)n}\, Q(g, R). \tag{5.28}$$

Further, we note that (5.27)(b) is equivalent to

$$0 \ = \ \frac{1}{2}\, (S - \frac{\kappa}{n}\, g) \wedge (S - \frac{\kappa}{n}\, g) \ = \ \frac{1}{2}\, S \wedge S - \frac{\kappa}{n}\, g \wedge S + (\frac{\kappa}{n})^2\, G,$$

which gives

$$\frac{1}{2}\, Q(g, S \wedge S) \ = \ \frac{\kappa}{n}\, Q(g, g \wedge S). \tag{5.29}$$

Let B be a generalized curvature tensor defined by (5.8). Using (3.9), (5.27)(a), (5.27)(c) and (5.29) we can easily check that

$$Q(g, B) \ = \ \frac{\kappa}{(n-1)n}\, Q(g, R).$$

Therefore (5.28) turns into (5.22). Using (1.4), (2.3), (2.4), (5.8), (5.27)(a)(c) and (5.10) we find

$$\begin{aligned} R\cdot C &= R\cdot R - \frac{1}{n-2}\, g\wedge (R\cdot S) \\ &= Q(g,B) - \frac{\widetilde{\kappa}}{(n-2)n(n+1)}\, g\wedge Q(g,S) \\ &= Q(g,B) + \frac{\widetilde{\kappa}}{(n-2)n(n+1)}\, Q(S,G) \\ &= Q(g, B - \frac{\kappa}{(n-2)(n-1)n}\, g\wedge S) \\ &= \frac{1}{n-1}\, Q(g, \frac{\kappa}{n} R - \frac{1}{2} S\wedge S + \frac{(n-3)\kappa}{(n-2)n}\, g\wedge S), \end{aligned} \tag{5.30}$$

and

$$Q(g,B_1) = \frac{1}{n-1}\, Q(g, \frac{\kappa}{n} R - \frac{1}{2} S\wedge S + \frac{(n-3)\kappa}{(n-2)n}\, g\wedge S).$$

Therefore (5.23) holds on U_H. Applying to (4.1) the relations: (2.2), (5.27)(a) and (5.28) we obtain

$$Q(S,R) = \frac{\kappa}{n}\, Q(g,R) - \frac{\kappa}{(n-1)n}\, Q(g, g\wedge S). \tag{5.31}$$

Now (3.4), by making use of (5.27)(a)(c) and (5.31), reduces to $C\cdot R = 0$. Further, using (3.9), (5.27)(a)(c) and (5.29), we can check that $Q(g,B_2) = 0$, where the tensor B_2 is defined by (5.12). Thus we see that (5.24) holds on U_H.
By an application of (3.9), (5.27)(a)(c), (5.29) and (5.30) we obtain

$$R\cdot C - C\cdot R = R\cdot C = \frac{1}{n-1}\, Q(g, \frac{\kappa}{n} R - \frac{\kappa}{(n-2)n}\, g\wedge S),$$

$$Q(g,B_3) = \frac{1}{n-1}\, Q(g, \frac{\kappa}{n} R - \frac{\kappa}{(n-2)n}\, g\wedge S),$$

where the tensor B_3 is defined by (5.14). Thus we see that (5.25) holds on U_H. □

We finish our paper with the following remarks.

Remark 5.1.

(i) Let M be a hypersurface in $N_s^{n+1}(c)$, $n\geq 4$. We recall that $U_H \subset U_C \cap U_S \subset M$. Let $U = U_C \cap U_S \setminus U_H$. Thus, on this set we have $H^2 = \alpha\, H + \beta\, g$ where α and β are functions on U. The last relation implies on U (see e.g. [16], eq. (17)):

$$R\cdot R = (\frac{\widetilde{\kappa}}{n(n+1)} - \varepsilon\beta)\, Q(g,R), \qquad \varepsilon = \pm 1. \tag{5.32}$$

Let in addition (1.7) be satisfied on U. Now (1.7) and (5.32) yield

$$Q(g, B - (\frac{\widetilde{\kappa}}{n(n+1)} - \varepsilon\beta)\, R) \;=\; 0.$$

This, using Lemma 1.1 (iii) of [5], implies $B = (\frac{\widetilde{\kappa}}{n(n+1)} - \varepsilon\beta)\, R + \lambda\, G$ where λ is a function on U.

(ii) Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. As we noted in the introduction, if (1.3) is satisfied on $U_C \subset M$, then (1.6) and $L_C = L_R$ hold on U_C. The converse statement, taking Theorem 3.1 of [1] into account, is also true.

(iii) In Example 5.1 of [11] a particular hypersurface M in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $c \neq 0$, $n \geq 4$, was defined. That hypersurface satisfies (5.21). Now from Theorem 5.3 it follows that all special Tachibana-type conditions (1.7), (5.9), (5.11) and (5.13) hold on M.

(iv) In [8] (see Examples 4.1 and 5.1) a particular hypersurface M in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 4$, was defined. On that hypersurface we have: (5.21), (5.27)(c), $\kappa = 0$, $\operatorname{rank} S = 1$ and $S^2 = 0$. Therefore, $R \cdot R$, $R \cdot C$, $C \cdot R$, $Q(g, B)$, $Q(g, B_1)$, $Q(g, B_2)$ and $Q(g, B_3)$ vanish.

(v) We can also prove that the non-quasi-Einstein hypersurfaces M of type number two in $N_s^{n+1}(c)$, $n \geq 4$, satisfy all special Tachibana-type conditions investigated in this paper. It will be shown in a subsequent paper of the authors.

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[1,2] Department of Mathematics,
Wrocław University of Environmental and Life Sciences,
Grunwaldzka 53, 50-357 Wrocław, Poland
E-mail address: [1] `ryszard.deszcz@up.wroc.pl`
[2] `malgorzata.glogowska@up.wroc.pl`

[3,5] Department of Mathematics,
Technische Universität Berlin,
Str. d. 17. Juni 136, 10623 Berlin, Germany
E-mail address: [3] `plaue@math.tu-berlin.de`
[5] `scherfner@math.tu-berlin.de`

[4] Department of Applied Mathematics,
Karol Adamiecki University of Economics in Katowice,
Pułaskiego 25, 40-276 Katowice, Poland
E-mail address: `katarzyna.sawicz@ae.katowice.pl`