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ON SOME GEOMETRIC PROPERTIES OF h-HOMOGENEOUS PRODUCTION FUNCTIONS IN MICROECONOMICS

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Abstract. Almost all economic theories presuppose a production function, either on the firm level or the aggregate level. In this sense the production function is one of the key concepts of mainstream neoclassical theories. There is a very important class of production functions that are often analyzed in microeconomics; namely, h-homogeneous production functions. This class of production functions includes many important production functions in microeconomics; in particular, the well-known generalized Cobb-Douglas production function and the ACMS production function.

In this paper we study geometric properties of h-homogeneous production functions via production hypersurfaces. As consequences, we obtain some characterizations for an h-homogeneous production function to have constant return to scale or to be a perfect substitute. Some applications to generalized Cobb-Douglas and ACMS production functions are also given.

1. Introduction

In microeconomics, a production function is a non-constant positive function that specifies the output of a firm, an industry, or an entire economy for all combinations of inputs. Almost all economic theories presuppose a production function, either on the firm level or the aggregate level. In this sense, the production function is one of the key concepts of mainstream neoclassical theories. By assuming that the maximum output technologically possible from a given set of inputs is achieved, economists using

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a production function in analysis are abstracting from the engineering and managerial problems inherently associated with a particular production process.

The primary purpose of the production function is to address allocative efficiency in the use of factor inputs in production and the resulting distribution of income to those factors. The engineering and managerial problems of technical efficiency are assumed to be solved, so that analysis can focus on the problems of allocative efficiency. Under certain assumptions, the production function can be used to derive a marginal product for each factor, which implies an ideal division of the income generated from output into an income due to each input factor of production (cf. [8],[9],[12]).

In microeconomics, there is an important class of production functions that are often analyzed; namely, h-homogeneous production functions. This class of production functions includes many important production functions in microeconomics; in particular, the well-known generalized Cobb-Douglas production function and the ACMS production function.

A production function $Q = f(x_1, \cdots, x_n)$ is said to be *h-homogeneous* or *homogeneous of degree h*, if given any positive constant t,

$$f(tx_1, \dots, tx_n) = t^h f(x_1, \dots, x_n) \tag{1.1}$$

for some constant h. If $h > 1$, the function exhibits increasing return to scale, and it exhibits decreasing return to scale if $h < 1$. If it is homogeneous of degree 1, it exhibits constant return to scale. Sometimes, a homogeneous function of degree 1 is called *linearly homogeneous.*

The presence of increasing returns means that a one percent increase in the usage levels of all inputs would result in a greater than one percent increase in output; the presence of decreasing returns means that it would result in a less than one percent increase in output. Constant returns to scale is the in-between case.

Each production function $f(x_1, \dots, x_n)$ can be identified with the non-parametric hypersurface of the Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$ given by

$$L(x_1, \dots, x_n) = (x_1, \dots, x_n, f(x_1, \dots, x_n)), \tag{1.2}$$

which is known as the *production hypersurface.*

In this paper we study some geometric properties of h-homogeneous production functions via their corresponding production hypersurfaces. As consequences, we obtain some characterizations for an h-homogeneous production function to have constant return to scale or to be a perfect substitute. Some applications to generalized Cobb-Douglas production function and the ACMS production function are also given.

2. Geometry of production hypersurfaces

For general references on the geometry of hypersurfaces, we refer to [3], [4], [10].

Let us denote the partial derivatives $\frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j}, \cdots, etc.$ by $f_i, f_{ij}, \cdots, etc.$ The Hessian $H(f)$ of f is the symmetric matrix (f_{ij}).

We put

$$w = \sqrt{1 + \sum_{i=1}^{n} f_i^2}. \tag{2.1}$$

2.1. **Basic definitions for geometry of hypersurfaces in $\mathbb{E}^{n+1}$.** Let M be a hypersurface of a Euclidean $(n+1)$-space. The *Gauss map* $\nu : M \to S^{n+1}$ maps M to the unit hypersphere S^n of $\mathbb{E}^{n+1}$. The Gauss map is a continuous map such that $\nu(p)$ is a unit normal vector $\xi(p)$ of M at p. The Gauss map can always be defined locally, i.e., on a small piece of the hypersurface. It can be defined globally if the hypersurface is orientable.

The differential $d\nu$ of the Gauss map ν can be used to define a type of extrinsic quantity, known as the *shape operator* or Weingarten map. Since at each point $p \in M$, the tangent space T_pM is an inner product space, the shape operator S_p can be defined as a linear operator on this space by the formula:

$$g(S_p v, w) = g(d\nu(v), w) \tag{2.2}$$

for $v, w \in T_pM$, where g is the metric tensor on M induced from the Euclidean metric on $\mathbb{E}^{n+1}$. The second fundamental form σ is related with the shape operator S by

$$g(\sigma(v, w), \xi(p)) = g(S_p(v), w) \tag{2.3}$$

for tangent vectors v, w of M at p. The eigenvalues of the shape operator S_p are called the principal curvatures. The determinant of the shape operator S_p is called the *Gauss-Kronecker curvature*, which is denoted by $G(p)$. Thus the Gauss-Kronecker curvature $G(p)$ is nothing but the product of the principal curvature at p. When $n = 2$, the Gauss-Kronecker curvature is simply called the *Gauss curvature*, which is intrinsic due to Gauss' theorema egregium.

The *mean curvature* is the trace of the shape operator divided by the dimension of the hypersurface. Contrast to the Gauss curvature, the mean curvature is extrinsic, which depends on the immersion of the hypersurface. A hypersurface is called *minimal* if its mean curvature vanishes identically.

Curves on a Riemannian manifold N which minimize length between the endpoints are called geodesics; they are the shape that an elastic band stretched between the two points would take. Mathematically, they are described using partial differential equations from the calculus of variations. For a given unit tangent vector $u \in T_pN$, there exists a unique unit speed geodesic $\gamma_u(t)$ in N through p such that $\gamma_u'(0) = u$. For a given 2-plane section π of the tangent space T_pN, all of geodesics through p and tangent to π form a surface in some neighborhood of p. The Gauss curvature of this surface at p is called the *sectional curvature* of π.

In differential geometry, the Riemann curvature tensor, or Riemann-Christoffel tensor is the most standard way to express curvature of Riemannian manifolds. It associates a tensor to each point of a Riemannian manifold that measures the extent to which the metric tensor is not locally isometric to the metric of a Euclidean space.

On a Riemannian manifold N there exists a unique affine connection ∇, called the *Levi-Civita connection* which preserves the metric, i.e, $\nabla g = 0$, and it is torsion-free, i.e., for any vector fields X and Y, we have $\nabla_X Y - \nabla_Y X = [X, Y]$, where $[\ ,\ ]$ denotes the Lie bracket of vector fields.

The *Riemann curvature tensor* R is given in terms of the Levi-Civita connection ∇ by the following formula:

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w. \tag{2.4}$$

For each pair of tangent vectors u, v, $R(u, v)$ is a linear transformation of the tangent space of the manifold. It is linear in u and v, and so defines a tensor.

If $u = \frac{\partial}{\partial x_i}$ and $v = \frac{\partial}{\partial x_j}$ are coordinate vector fields, then $[u, v] = 0$ and therefore the formula (2.4) simplifies to

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w. \tag{2.5}$$

The curvature tensor measures non-commutativity of the covariant derivative, and as such is the integrability obstruction for the existence of an isometry with Euclidean space. In this context, a Riemannian manifold is called *flat* if its Riemann curvature tensor vanishes identically.

The *Ricci tensor* of a Riemannian manifold N at a point $p \in N$ is defined to be the trace of the linear map $T_pN \to T_pN$ given by

$$w \mapsto R(w, u)v.$$

A Riemannian manifold is called an Einstein space if its Ricci tensor is proportional to its metric tensor. And it is called *Ricci-flat* if its Ricci tensor vanishes identically.

For a hypersurface M of $\mathbb{E}^{n+1}$, the *equation of Gauss* is given by

$$g(R(u, v)w, x) = g(\sigma(u, x), \sigma(v, w)) - g(\sigma(u, w), \sigma(v, x)). \tag{2.6}$$

2.2. **Basic geometric results for production hypersurfaces.** The following basic geometric results are well-known.

Proposition 2.1. *For the production hypersurface of* $\mathbb{E}^{n+1}$ *defined by*

$$L(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f(x_1, \ldots, x_n)), \tag{2.7}$$

and w *given by* (2.1) *we have:*

(1) *The unit normal* ξ *is*

$$\xi = \frac{-1}{w}(f_1, \ldots, f_n, -1). \tag{2.8}$$

(2) *The coefficient* $g_{ij} = g(\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j})$ *of the metric tensor is*

$$g_{ij} = \delta_{ij} + f_i f_j, \quad \delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.9}$$

(3) *The volume element is*

$$dV = \sqrt{g_{ij}}\, dx_1 \wedge \cdots \wedge dx_n = w\, dx_1 \wedge \cdots \wedge dx_n. \tag{2.10}$$

(4) *The inverse matrix* (g^{ij}) *of* (g_{ij}) *is*

$$g^{ij} = \delta_{ij} - \frac{f_i f_j}{w^2}. \tag{2.11}$$

(5) *The matrix of the second fundamental form* σ *is*

$$\sigma_{ij} = \frac{f_{ij}}{w}. \tag{2.12}$$

(6) *The matrix of the shape operator* S *is*

$$a_i^j = \sum_k \sigma_{ik} g^{kj} = \frac{f_{ij}}{w} - \sum_k \frac{f_{ik} f_k f_j}{w^3}. \tag{2.13}$$

(7) *The mean curvature H is*

$$H=\frac{1}{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\Big(\frac{f_j}{w}\Big). \tag{2.14}$$

(8) *The Gauss-Kronecker curvature G is*

$$G=\frac{\det(\sigma_{ij})}{\det(g_{ij})}=\frac{\det(f_{ij})}{w^{n+2}}. \tag{2.15}$$

(9) *The sectional curvature K_{ij} of the plane section spanned by $\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}$ is*

$$K_{ij}=\frac{f_{ii}f_{jj}-f_{ij}^2}{w^2(1+f_i^2+f_j^2)}. \tag{2.16}$$

(10) *The Riemann curvature tensor R satisfies*

$$g\Big(R\Big(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\Big)\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_\ell}\Big)=\frac{f_{i\ell}f_{jk}-f_{ik}f_{j\ell}}{w^4}. \tag{2.17}$$

3. h-Homogeneous production functions

A production function is called a *perfect substitute* if it is linearly homogeneous, which takes the form

$$Q(x_1,\ldots,x_n)=\sum_{i=1}^{n}a_ix_i$$

for some nonzero constants $a_1,\ldots,a_n$.

A perfect substitute with inputs capital and labour has the properties that the marginal and average physical products of both capital and labour can be expressed as functions of the capital-labour ratio alone. Moreover, in this case if each input is paid at a rate equal to its marginal product, the firm's revenues will be exactly exhausted and there will be no excess economic profit.

Let us put

$$\mathbf{R}_+=\{r\in\mathbf{R}:r>0\}\ \text{ and }\ \mathbf{R}^n_+=\{(x_1,\ldots,x_n):x_1,\ldots,x_n>0\}.$$

The purpose of this section is to prove the following geometric characterizations for an h-homogeneous production function to have constant return to scale or to be a perfect substitute.

Theorem 3.1. *An h-homogeneous production function has constant return to scale if and only if the production hypersurface has null Gauss-Kronecker curvature.*

Proof. Assume that $Q = f(x_1, \dots, x_n)$ is a homogeneous production function of degree h. Then

$$f(tx_1, \dots, tx_n) = t^h f(x_1, \dots, x_n) \tag{3.1}$$

for any $t \in \mathbf{R}_+$ and any $(x_1, \dots, x_n) \in \mathbf{R}_+^n$. Since f is assumed to be h-homogeneous, the Euler Homogeneous Function Theorem implies that

$$x_1 f_1 + x_2 f_2 + \cdots + x_n f_n = hf. \tag{3.2}$$

After taking the partial derivatives of (3.2) with respect to $x_1, \dots, x_n$, respectively, we obtain

$$\begin{aligned} x_1 f_{11} + x_2 f_{12} + \cdots + x_n f_{1n} &= (h-1) f_1, \\ x_1 f_{12} + x_2 f_{22} + \cdots + x_n f_{2n} &= (h-1) f_2, \\ &\vdots \\ x_1 f_{1n} + x_2 f_{2n} + \cdots + x_n f_{nn} &= (h-1) f_n. \end{aligned} \tag{3.3}$$

Now, let us assume that the production hypersurface has null Gauss-Kronecker curvature. Then statement (8) of Proposition 2.1 implies that $\det(f_{ij}) = 0$. Since system (3.3) admits positive solutions for $x_1, \dots, x_n$, it follows from (3.3) that either $h = 1$ or $f_1 = \cdots = f_n = 0$. But the latter case cannot occur since the production function is non-constant. Consequently, we must have $h = 1$. Therefore the production function must has constant return to scale.

Conversely, if the production function has constant return to scale, then $h = 1$. Thus system (3.3) reduces to

$$\begin{aligned} x_1 f_{11} + x_2 f_{12} + \cdots + x_n f_{1n} &= 0, \\ x_1 f_{12} + x_2 f_{22} + \cdots + x_n f_{2n} &= 0, \\ &\vdots \\ x_1 f_{1n} + x_2 f_{2n} + \cdots + x_n f_{nn} &= 0, \end{aligned} \tag{3.4}$$

which is impossible unless $\det(f_{ij}) = 0$, because there exist some solutions for $x_1, \dots, x_n$. Hence the Gauss-Kronecker curvature of the production hypersurface must vanishes according to statement (8) of Proposition 2.1. $\square$

Theorem 3.2. *An h-homogeneous production function with more than two factors is a perfect substitute if and only if the production hypersurface is flat.*

Proof. Let $Q = f(x_1, \dots, x_n)$ be an h-homogeneous production function with more than 2 factors. Then $n \geq 3$. Suppose that the production hypersurface is flat, then it follows from statement (5) of Proposition 2.1 and the equation of Gauss that

$$f_{i\ell} f_{jk} = f_{ij} f_{k\ell}, \quad 1 \leq i, j, k, \ell \leq n. \tag{3.5}$$

It is easy to verify that (3.5) implies that each cofactor H_{ij} of the Hessian $H(f)$ of the production function f vanishes identically. Therefore the production hypersurface must has null Gauss-Kronecker curvature. Hence, according to Theorem 3.1, the production function has constant return to scale. Consequently, we have the homogeneous system (3.4), which can be rewritten as

$$\begin{aligned} x_1 f_{11} + x_2 f_{12} + \cdots + x_{n-1} f_{1n-1} &= -x_n f_{1n}, \\ x_1 f_{12} + x_2 f_{22} + \cdots + x_{n-1} f_{2n-1} &= -x_n f_{2n}, \\ &\vdots \\ x_1 f_{1n} + x_2 f_{2n} + \cdots + x_{n-1} f_{n-1n} &= -x_n f_{nn} \end{aligned} \tag{3.6}$$

Since the cofactors of the Hessian $H(f)$ satisfy $H_{1n} = H_{2n} = \cdots = H_{nn} = 0$, and the system (3.6) admits positive solutions for $x_1, \dots, x_{n-1}$, we obtain from (3.6) that $f_{1n} = \cdots = f_{nn} = 0$.

Similarly, we also have $f_{ij} = 0$ for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, n-1\}$. Consequently, the production function f is a linear function. Therefore, it is a perfect substitute.

The converse is trivial. ∎

Since Ricci-flat 3-manifolds are always flat, Theorem 3.2 implies the following.

Theorem 3.3. *A three-factor h-homogeneous production function is a perfect substitute if and only if the production hypersurface is Ricci-flat.*

For two-factor h-homogeneous production functions, we also have the following.

Theorem 3.4. *A two-factor h-homogeneous production function is a perfect substitute if and only if the production surface is a minimal surface.*

Proof. Assume that $Q = f(x_1, x_2)$ is a homogeneous production function of degree h. Then

$$f(tx_1, tx_2) = t^h f(x_1, x_2) \tag{3.7}$$

for any $t \in \mathbf{R}_+$ and any $(x_1, x_2) \in \mathbf{R}^n_+$. Suppose that the corresponding production surface is minimal, i.e., $H = 0$ identically. Then it follows from statement (7) of Proposition 2.1 that

$$(f_{11} + f_{22})(1 + f_1^2 + f_2^2) = \sum_{j,k=1}^{2} f_j f_k f_{jk}. \tag{3.8}$$

Since the production surface $L(x_1, x_2) = (x_1, x_2, f(x_1, x_2))$ is minimal, the following surface defined by

$$\tilde{L}(\lambda x_1, \lambda x_2) = (\lambda x_1, \lambda x_2, f(\lambda x_1, \lambda x_2)), \quad \lambda \in \mathbf{R}_+,$$

is also minimal. Hence, in view of (3.7), we conclude that the surface given by

$$\hat{L}(x_1, x_2) = (x_1, x_2, \lambda^{h-1} f(x_1, x_2))$$

is also minimal. Consequently, after applying statement (7) of Proposition 2.1 once more, we also have

$$(f_{11} + f_{22})(\lambda^{2-2h} + (f_1^2 + f_2^2)) = \sum_{j,k=1}^{2} f_j f_k f_{jk}. \tag{3.9}$$

After comparing equations (3.8) and (3.9), either we obtain $h = 1$ or we have the following two equations:

$$f_{11} + f_{22} = 0, \tag{3.10}$$

$$f_1^2 f_{11} + 2 f_1 f_2 f_{12} + f_2^2 f_{22} = 0. \tag{3.11}$$

If $h = 1$, then the production function is linearly homogeneous. Thus the Euler Homogeneous Function Theorem yields

$$x_1 f_1 + x_2 f_2 = f. \tag{3.12}$$

Therefore, after taking the partial derivatives of (3.11) with respect to x_1 and x_2, respectively, we obtain

$$\begin{aligned} x_1 f_{11} + x_2 f_{12} &= 0, \\ x_1 f_{12} + x_2 f_{22} &= 0. \end{aligned} \tag{3.13}$$

Since this system admits some positive solutions for x_1, x_2, it implies $\det(f_{ij}) = 0$. Hence the production surface is flat. Thus the minimal surface is totally geodesic in $\mathbb{E}^3$. (cf. [3],[4]). Consequently, the production function is a perfect substitute.

Next, let us assume that $h \neq 1$. Then both (3.10) and (3.11) hold, from which we obtain

$$f_{22} = -f_{11}, \quad (f_1^2 - f_2^2)f_{11} + 2f_1f_2f_{12} = 0. \tag{3.14}$$

On the other hand, since f is h-homogeneous, the Euler Homogeneous Function Theorem gives

$$x_1f_1 + x_2f_2 = hf. \tag{3.15}$$

After taking the partial derivatives of (3.15) with respect to x_1, x_2 and applying the first equation in (3.14), we find

$$x_1f_{11} + x_2f_{12} = (h-1)f_1, \quad x_1f_{12} - x_2f_{11} = (h-1)f_2 \tag{3.16}$$

Now, after solving (3.16) for x_1, x_2 and substituting them into the second equation in (3.14), we derive that

$$(f_{11}^2 + f_{12}^2)(f_{11}(x_1^2 - x_2^2) + 2x_1x_2f_{12}) = 0. \tag{3.17}$$

Thus either f is a linear function or we have

$$f_{11}(x_1^2 - x_2^2) + 2x_1x_2f_{12} = 0. \tag{3.18}$$

If f is a linear function, then $h = 1$, which is a contradiction.

If f is a nonlinear, then we must have (3.18). Now, after solving the first equation of (3.16) for f_{12} we get

$$f_{12} = \frac{(h-1)f_1 - x_1f_{11}}{x_2}. \tag{3.19}$$

By substituting (3.19) into (3.18) we find

$$f_{11} = \frac{2(h-1)x_1}{x_1^2 + x_2^2}f_1. \tag{3.20}$$

After solving the differential equation (3.20) for f_1 we obtain

$$f_1 = p(x_2)(x_1^2 + x_2^2)^{h-1} \tag{3.21}$$

for some function $p(x_2)$. Substituting (3.21) into (3.18) yields

$$x_2p'(x_2) = (1-h)p(x_2).$$

Hence we have $p(x_2) = cx_2^{1-h}$ for some constant c. After combining this with (3.21) we obtain

$$f_1 = cx_2^{1-h}(x_1^2 + x_2^2)^{h-1}. \tag{3.22}$$

Now, by substituting (3.22) into (3.15) we derive that

$$f_2 = \frac{hf}{x_2} - \frac{cx_1(x_1^2 + x_2^2)^{h-1}}{x_2^h}. \tag{3.23}$$

Therefore, after applying (3.18), (3.22) and (3.23) we find $c = 0$, which gives $f_1 = 0$ by (3.21). Finally, after combining $f_1 = 0$ with (3.8) we conclude that f is linearly homogeneous, which is a contradiction.

The converse is easy to verify. □

4. Applications to Cobb-Douglas' and ACMS production functions

In 1928, Cobb and Douglas introduced in [5] a famous two-factor production function, nowadays called *Cobb-Douglas production function.* The Cobb-Douglas function is widely used in economics to represent the relationship of an output to inputs. Similar functions were originally used by Knut Wicksell (1851-1926).

The Cobb-Douglas production function was first developed in 1927, when Paul H. Douglas (1892-1976) seeking a functional form to relate estimates he had calculated for workers and capital. He spoke with mathematician and colleague Charles W. Cobb (1875-1949) who suggested a function of the form

$$Y = bL^k C^{1-k}, \tag{4.1}$$

where L represents the labor input, C the capital input, b the total factor productivity and Y is the total production. Later work in the 1940s prompted them to allow for the exponents on C and L vary, which resulting in estimates that subsequently proved to be very close to improved measure of productivity developed at that time (cf. [6],[7]).

The Cobb-Douglas production function is especially notable for being the first time an aggregate or economy-wide production function had been developed, estimated, and then presented to the profession for analysis. It gave a landmark change in how economists approached macroeconomics. The Cobb-Douglas function has also been applied to many other contexts besides production.

In its generalized form the Cobb-Douglas production function may be written as

$$Q = bx_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad (x_1, \ldots, x_n) \in \mathbf{R}_+^n, \tag{4.2}$$

where b is a positive constant and $\alpha_1, \ldots, \alpha_n$ are nonzero constants.

Since the function Q in (4.2) is homogeneous with degree $h = \sum_{j=1}^n \alpha_j$, it has constant return to scale if and only if $\sum_{j=1}^n \alpha_j = 1$.

Our Theorem 3.1 implies immediately the following recent result of [14].

Corollary 4.1. *The generalized Cobb-Douglas production function has constant return to scale if and only if the production hypersurface has null Gauss-Kronecker curvature.*

For the two-factor Cobb-Douglas production function, Corollary 4.1 becomes the following.

Corollary 4.2. *The two-factor Cobb-Douglas production function has constant return to scale if and only if the production surface is flat.*

However, Corollary 4.2 is false if the Cobb-Douglas production function has more than two factors. In fact, our Theorem 3.2 implies the following.

Theorem 4.1. *The production hypersurface of the generalized Cobb-Douglas production function with more than two factors is always non-flat.*

Proof. Consider the generalized Cobb-Douglas production function given by (4.2) with $n \geq 3$. If the production hypersurface is flat, then Theorem 3.2 implies that the production function is a linear function, which is impossible since $n \geq 3$. Consequently, the production hypersurface cannot be flat. □

Since Ricci-flat 3-manifolds are always flat, Theorem 4.1 implies the following.

Corollary 4.3. *The production hypersurface of the three-factor generalized Cobb-Douglas production function is non-Ricci-flat.*

The CES production function is a type of production function that displays constant elasticity of substitution. In other words, the production technology has a constant percentage change in factor (e.g. labour and capital) proportions due to a percentage change in marginal rate of technical substitution.

In 1961, Arrow, Chenery, Minhas and Solow [2] introduced a two-factor CES production function given by

$$Q = F \cdot (aK^r + (1-a)L^r)^{\frac{1}{r}}. \tag{4.3}$$

where Q is the output, F the factor productivity, a the share parameter, K, L the primary production factors (capital and labor), $r = 1 - 1/s$ and $s = 1/(1-r)$ is the elasticity of substitution.

The generalized form of (4.3) is:

$$Q = b\Big(\sum_{i=1}^{n} a_i^{\rho} x_i^{\rho}\Big)^{\frac{h}{\rho}}, \quad (x_1, \dots, x_n) \in \mathbf{R}_+^n, \tag{4.4}$$

where $b > 0, \rho < 1, \rho \neq 0, h > 0$ and $a_i > 0$ for all $i \in \{1, 2, \dots, n\}$. This is known as Arrow-Chenery-Minhas-Solow (ACMS) production function or the generalized CES production function.

The same functional form arises as a utility function in consumer theory. For example, if there exist n types of consumption goods c_i, then aggregate consumption C could be defined using the CES aggregator:

$$C = \left(\sum_{i=1}^{n} a_i^{\frac{1}{s}} c_i^{\frac{s-1}{s}}\right)^{\frac{s}{s-1}}, \tag{4.5}$$

where the coefficients a_i are share parameters, and s is the elasticity of substitution.

The ACMS production function (4.4) is also known as the Armington aggregator which was discussed in microeconomics (see, for instance [1],[11]).

Since the ACMS production function is h-homogeneous, Theorem 3.1 also implies immediately the following recent result of [13].

Corollary 4.4. *The ACMS production function has constant return to scale if and only if the production hypersurface has null Gauss-Kronecker curvature.*

Two other immediate applications of Theorems 3.2 and 3.3 are the following.

Corollary 4.5. *The ACMS production function with more than two factors is a perfect substitute if and only if the production hypersurface is flat.*

Corollary 4.6. *The three-factor ACMS production function is a perfect substitute if and only if the production hypersurface is Ricci-flat.*

Theorem 3.1 also implies immediately the following.

Corollary 4.7. *The production hypersurface of the utility function C defined by* (4.5) *has null Gauss-Kronecker curvature.*

Remark 4.1. It was proved in [13] and [14] that the ACMS production function and the generalized Cobb-Douglas production function have deceasing/increasing return to scale if and only if the product hypersurfaces have positive/negative Gauss-Kronecker curvature.

For a general h-homogeneous production function, the decreasing/increasing return to scale property cannot be determined by the Gauss-Kronecker curvature of production hypersurface; even for two-factor h-homogeneous production functions. This fact can be seen from the following simple example.

Example 4.1. Consider the following homogeneous production function of degree 3 defined by

$$f(x, y) = x^2 y + \gamma y^3, \tag{4.6}$$

where γ is a constant. Then the Gauss curvature of the production surface is

$$G = \frac{4(3\gamma y^2 - x^2)}{(1 + 4x^2y^2 + (x^2 + 3\gamma y^2)^2)^2}. \tag{4.7}$$

Since the homogeneity of this production function is 3, the production function has increasing return to scale. On the other hand, the Gauss curvature of the production surface at the point $(1, 1)$ is positive or negative depending on $\gamma > \frac{1}{3}$ or $\gamma < \frac{1}{3}$, respectively. Consequently, the decreasing/increasing return to scale property cannot be determined by the Gauss curvature of production surface.

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