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SIGNED TOTAL k-DOMATIC NUMBERS OF DIGRAPHS

MARYAM ATAPOUR [1], SEYED MAHMOUD SHEIKHOLESLAMI [1]
AND LUTZ VOLKMANN [2]

Abstract. Let D be a finite and simple digraph with vertex set $V(D)$, and let $f : V(D) \to \{-1, 1\}$ be a two-valued function. If $k \geq 1$ is an integer and $\sum_{x \in N^-(v)} f(x) \geq k$ for each $v \in V(D)$, where $N^-(v)$ consists of all vertices of D from which arcs go into v, then f is a signed total k-dominating function on D. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total k-dominating functions of D with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$, for each $v \in V(D)$, is called a *signed total k-dominating family* (of functions) of D. The maximum number of functions in a signed total k-dominating family of D is the *signed total k-domatic number* of D, denoted by $d^t_{kS}(D)$. In this note we initiate the study of the signed total k-domatic numbers of digraphs and present some sharp upper bounds for this parameter.

1. Introduction

In this paper, D is a finite and simple digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. Its underlying graph is denoted $G(D)$. We write $\deg^+_D(v) = \deg^+(v)$ for the *outdegree* of a vertex v and $\deg^-_D(v) = \deg^-(v)$ for its *indegree*. The *minimum* and *maximum indegree* are $\delta^-(D)$ and $\Delta^-(D)$, respectively. The sets $N^+(v) = \{x \mid (v, x) \in A(D)\}$ and $N^-(v) = \{x \mid (x, v) \in A(D)\}$ are called the *outset* and *inset* of the vertex v. Likewise, $N^+[v] = N^+(v) \cup \{v\}$ and $N^-[v] = N^-(v) \cup \{v\}$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. For an arc $(x, y) \in A(D)$, the vertex y is an *outer neighbor* of x and x is an *inner neighbor* of y. Note that for any digraph D with

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m arcs,

$$\sum_{u\in V(D)} \deg^-(u) = \sum_{u\in V(D)} \deg^+(u) = m. \tag{1.1}$$

Consult [3] and [4] for notation and terminology which are not defined here.

For a real-valued function $f : V(D) \longrightarrow \mathbf{R}$ the weight of f is $w(f) = \sum_{v\in V(D)} f(v)$, and for $S \subseteq V(D)$, we define $f(S) = \sum_{v\in S} f(v)$, so $w(f) = f(V(D))$. If $k \geq 1$ is an integer, then the *signed total k-dominating function* is defined as a function $f : V(D) \longrightarrow \{-1, 1\}$ such that $f(N^-(v)) = \sum_{x\in N^-(v)} f(x) \geq k$ for every $v \in V(D)$. The *signed total k-domination number* for a digraph D is

$$\gamma^t_{kS}(D) = \min\{w(f) \mid f \text{ is a signed total } k\text{-dominating function of } D\}.$$

A $\gamma^t_{kS}(D)$-function is a signed total k-dominating function on D of weight $\gamma^t_{kS}(D)$. As the assumption $\delta^-(D) \geq k$ is necessary, we always assume that when we discuss $\gamma^t_{kS}(D)$, all digraphs involved satisfy $\delta^-(D) \geq k$ and thus $n(D) \geq k + 1$.

The signed total k-domination number of digraphs was introduced by Sheikholeslami and Volkmann [7]. When $k = 1$, the signed total k-domination number $\gamma^t_{kS}(D)$ is the usual *signed total domination number* $\gamma^t_S(D)$, which was introduced by Sheikholeslami in [6].

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total k-dominating functions on D with the property that $\sum_{i=1}^d f_i(v) \leq 1$ for each $v \in V(D)$, is called a *signed total k-dominating family* on D. The maximum number of functions in a signed total k-dominating family on D is the *signed total k-domatic number* of D, denoted by $d^t_{kS}(D)$. The signed total k-domatic number is well-defined and $d^t_{kS}(D) \geq 1$ for all digraphs D in which $d^-_D(v) \geq k$ for all $v \in V$, since the set consisting of any one STkD function forms a STkD family of D. A $d^t_{kS}(D)$*-family* of a digraph D is a STkD family containing $d^t_{kS}(D)$ STkD functions. When $k = 1$, the signed total k-domatic number of a digraph D is the usual *signed total domatic number* $d_{st}(D)$, which was introduced by Favaron and Sheikholeslami [1].

In this paper we initiate the study of the signed total k-domatic number of digraphs, and we present different bounds on $d^t_{kS}(D)$. Some of our results are extensions of well-known properties of the signed total domatic number $d_{st}(D) = d^t_{1S}(D)$ of digraphs (see for example [1]) as well as the signed total k-domatic number of graphs G (see for example [2, 5]).

We make use of the following results and observations in this paper.

Observation 1.1. *Let D be a digraph of order n. Then $\gamma^t_{kS}(D) = n$ if and only if $k \le \delta^-(D) \le k+1$ and for each $v \in V(D)$ there exists a vertex $u \in N^+(v)$ such that* $\deg^-(u) = k$ *or* $\deg^-(u) = k+1$.

Proof. If $k \le \delta^-(D) \le k+1$ and for each $v \in V(D)$ there exists a vertex $u \in N^+(v)$ such that $\deg^-(u) = k$ or $\deg^-(u) = k+1$, then trivially $\gamma^t_{kS}(D) = n$.

Conversely, assume that $\gamma^t_{kS}(D) = n$. By assumption $k \le \delta^-(D)$. Let, to the contrary, $\delta^-(D) > k+1$ or there exists a vertex $v \in V(D)$ such that $\deg^-(u) \ge k+2$ for each $u \in N^+(v)$. If $\delta^-(D) > k+1$, define $f : V(D) \to \{-1, 1\}$ by $f(v) = -1$ for some fixed v and $f(x) = 1$ for $x \in V(D) \setminus \{v\}$. Obviously, f is a signed total k-dominating function of D with weight less than n, a contradiction. Thus $k \le \delta^-(D) \le k+1$. Now let $v \in V(D)$ and $\deg^-(u) \ge k+2$ for each $u \in N^+(v)$. Define $f : V(D) \to \{-1, 1\}$ by $f(v) = -1$ and $f(x) = 1$ for $x \in V(D) \setminus \{v\}$. Again, f is a signed total k-dominating function of D, a contradiction. This completes the proof. $\square$

Observation 1.2. *Let $k \ge 1$ be an integer, and let D be a digraph with $\delta^-(D) \ge k$. If for every vertex $v \in V(D)$ the set $N^+(v)$ contains a vertex x such that* $\deg^-(x) \le k+1$, *then $d^t_{kS}(D) = 1$.*

Proof. Assume that $N^+(v)$ contains a vertex x_v such that $\deg^-(x_v) \le k+1$ for every vertex $v \in V(D)$, and let f be a signed total k-dominating function on D. Since $\deg^-(x_v) \le k+1$, we deduce that $f(v) = 1$. Hence $f(v) = 1$ for each $v \in V(D)$ and thus $d^t_{kS}(D) = 1$. $\square$

Observation 1.3. *The signed total k-domatic number of a digraph is an odd integer.*

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed total k-dominating family on D such that $d = d^t_{kS}(D)$. Suppose to the contrary that $d^t_{kS}(D)$ is an even integer. If $x \in V(D)$ is an arbitrary vertex, then $\sum_{i=1}^{d} f_i(x) \le 1$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number and we obtain $\sum_{i=1}^{d} f_i(x) \le 0$ for each $x \in V(G)$. If v is an arbitrary vertex, then it follows that

$$d \cdot k = \sum_{i=1}^{d} k \le \sum_{i=1}^{d} \sum_{x \in N^-(v)} f_i(x) = \sum_{x \in N^-(v)} \sum_{i=1}^{d} f_i(x) \le 0.$$

which is a contradiction, and the proof is complete. $\square$

2. Properties and upper bounds

In this section we present basic properties of the signed total k-domatic number, and we find some sharp upper bounds for this parameter.

Theorem 2.1. *Let D be a digraph and $v \in V(D)$. Then*

$$d_{kS}^t(D) \leq \begin{cases} \dfrac{\deg^-(v)}{k} & \text{if } \deg^-(v) \equiv k \pmod 2 \\ \dfrac{\deg^-(v)}{k+1} & \text{if } \deg^-(v) \equiv k+1 \pmod 2\,. \end{cases}$$

Moreover, if the equality holds, then for each function f_i of a STkD family $\{f_1, f_2, \ldots, f_d\}$ and for every $u \in N^-(v)$, $\sum_{u \in N^-(v)} f_i(u) = k$ if $\deg^-(v) \equiv k \,(\mathrm{mod}\ 2)$, $\sum_{u \in N^-(v)} f_i(u) = k+1$ if $\deg^-(v) \equiv k+1 \,(\mathrm{mod}\ 2)$ and $\sum_{i=1}^{d} f_i(u) = 1$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a STkD family of D such that $d = d_{kS}^t(D)$. If $\deg^-(v) \equiv k \,(\mathrm{mod}\ 2)$, then

$$\begin{aligned} d &= \sum_{i=1}^{d} 1 \leq \sum_{i=1}^{d} \frac{1}{k} \sum_{u \in N^-(v)} f_i(u) \\ &= \frac{1}{k} \sum_{u \in N^-(v)} \sum_{i=1}^{d} f_i(u) \leq \frac{1}{k} \sum_{u \in N^-(v)} 1 \\ &= \frac{\deg^-(v)}{k}. \end{aligned}$$

Similarly, if $\deg^-(v) \equiv k+1 \,(\mathrm{mod}\ 2)$, then

$$\begin{aligned} d &= \sum_{i=1}^{d} 1 \leq \sum_{i=1}^{d} \frac{1}{k+1} \sum_{u \in N^-(v)} f_i(u) \\ &= \frac{1}{k+1} \sum_{u \in N^-(v)} \sum_{i=1}^{d} f_i(u) \leq \frac{1}{k+1} \sum_{u \in N^-(v)} 1 \\ &= \frac{\deg^-(v)}{k+1}. \end{aligned}$$

If $d_{kS}^t(D) = \frac{\deg^-(v)}{k}$ when $\deg^-(v) \equiv k \,(\mathrm{mod}\ 2)$ or $d_{kS}^t(D) = \frac{\deg^-(v)}{k+1}$ when $\deg^-(v) \equiv k+1 \,(\mathrm{mod}\ 2)$, then the two inequalities occurring in the proof of each corresponding case become equalities, which gives the properties given in the statement. $\square$

Corollary 2.1. *Let D be a digraph and $1 \le k \le \delta^-(D)$. Then*

$$d^t_{kS}(D) \le \begin{cases} \dfrac{\delta^-(D)}{k}, & \text{if } \delta^-(D) \equiv k \pmod 2, \\ \dfrac{\delta^-(D)}{k+1}, & \text{if } \delta^-(D) \equiv k+1 \pmod 2. \end{cases}$$

Corollary 2.2. *Let $k \ge 1$ be an integer, and let D be a $(k+2)$-inregular digraph of order n. If $k \ge 2$ or $k = 1$ and $n \not\equiv 0 \pmod 3$, then $d^t_{kS}(D) = 1$.*

Proof. By Corollary 2.1, $d^t_{kS}(D) \le \frac{k+2}{k}$. If $k \ge 2$, then it follows from Observation 1.3 that $d^t_{kS}(D) = 1$. Now let $k = 1$. Then $d^t_{kS}(D) = 1$ or $d^t_{kS}(D) = 3$ by Observation 1.3. Suppose to the contrary that $d^t_{kS}(D) = 3$. Let f belong to a signed total k-dominating family on D of order 3. By Theorem 2.1, we have $\sum_{x \in N^-(v)} f(x) = 1$ for every $v \in V(D)$. This implies that

$$n = \sum_{v \in V(D)} \sum_{x \in N^-(v)} f(x) = \sum_{x \in N^-(v)} \sum_{v \in V(D)} f(x) = 3w(f).$$

Since $w(f)$ is an integer, 3 is a divisor of n which contradicts the hypotheses $n \not\equiv 0 \pmod 3$, and the proof is complete. □

Corollary 2.3. *Let $k \ge 1$ be an integer, and let D be a $(k+3)$-inregular digraph of order n. Then $d^t_{kS}(D) = 1$.*

Proof. By Corollary 2.1, $d^t_{kS}(D) \le \frac{k+3}{k+1}$. Therefore Observation 1.3 implies that $d^t_{kS}(D) = 1$. □

Theorem 2.2. *Let $k \ge 1$ be an integer, and let D be an r-inregular digraph of order n such that $r \ge k$. If $r < 3k$, then $d^t_{kS}(D) = 1$, and if $r \ge 3k$ and $(n, r) = 1$, then*

$$d^t_{kS}(D) < \begin{cases} \dfrac{r}{k}, & \text{if } r \equiv k \pmod 2, \\ \dfrac{r}{k+1}, & \text{if } r \equiv k+1 \pmod 2. \end{cases}$$

Proof. If $r < 3k$, then it follows from Corollary 2.1 that $d^t_{kS}(D) \le \frac{r}{k} < 3$. Therefore Observation 1.3 implies that $d^t_{kS}(D) = 1$.

Now assume that $r \ge 3k$ and $(n, r) = 1$. First let $r = \delta^-(D) \equiv k \pmod 2$ (if $\delta^-(D) \equiv k+1 \pmod 2$, then the proof is similar). Suppose to the contrary that $d^t_{kS}(D) \ge \frac{\delta^-(D)}{k}$. Then by Corollary 2.1, $d^t_{kS}(D) = \frac{\delta^-(D)}{k}$. Let f belong to a signed total k-dominating family on D of order $\frac{\delta^-(D)}{k}$. By Theorem 2.1, we have

$\sum_{x\in N^-(v)} f(x) = k$ for every $v \in V(D)$. This implies that

$$nk = \sum_{v\in V(D)} \sum_{x\in N^-(v)} f(x) = \sum_{x\in N^-(v)} \sum_{v\in V(D)} f(x) = rw(f).$$

Since $w(f)$ is an integer and $(n, r) = 1$, the number r is a divisor of k. It follows from $k \le \delta^-(D) = r$ that $k = r$, a contradiction to the hypothesis that $r \ge 3k$. $\square$

Theorem 2.3. *Let D be a digraph with $\delta^-(D) \ge k$, and let $\Delta = \Delta(G(D))$ be the maximum degree of $G(D)$. Then*

$$d^t_{kS}(D) \le \begin{cases} \dfrac{\Delta}{2k}, & \text{if } \delta^-(D) \equiv k \pmod 2, \\ \dfrac{\Delta}{2(k+1)}, & \text{if } \delta^-(D) \equiv k+1 \pmod 2. \end{cases}$$

Proof. First of all, we show that $\delta^-(D) \le \Delta/2$. Suppose to the contrary that $\delta^-(D) > \Delta/2$. Then $\Delta^+(D) \le \Delta - \delta^-(D) < \Delta/2$, and (1.1) leads to the contradiction

$$\frac{\Delta \cdot |V(D)|}{2} < \sum_{u\in V(D)} \deg^-(u) = \sum_{u\in V(D)} \deg^+(u) < \frac{\Delta \cdot |V(D)|}{2}.$$

Applying Corollary 2.1, we deduce the desired result. $\square$

Let D be a digraph. By D^{-1} we denote the digraph obtained by reversing all the arcs of D. A digraph without directed cycles of length 2 is called an *oriented graph*. An oriented graph D is a *tournament* when either $(x, y) \in A(D)$ or $(y, x) \in A(D)$ for each pair of distinct vertices $x, y \in V(D)$.

Theorem 2.4. *For every oriented graph D of order n and $1 \le k \le \min\{\delta^-(D), \delta^-(D^{-1})\}$,*

$$d^t_{kS}(D) + d^t_{kS}(D^{-1}) \le \frac{n-1}{k} \tag{2.1}$$

with equality if and only if D is an r-regular tournament of order $n = 2r + 1$ and $k = r$.

Proof. Since $\delta^-(D) + \delta^-(D^{-1}) \le n - 1$, Corollary 2.1 implies that

$$d^t_{kS}(D) + d^t_{kS}(D^{-1}) \le \frac{\delta^-(D)}{k} + \frac{\delta^-(D^{-1})}{k} \le \frac{n-1}{k}.$$

If D is an r-regular tournament of order $n = 2r + 1$ and $k = r$, then D^{-1} is also an r-regular tournament, and it follows from Observation 1.2 that

$$d^t_{kS}(D) + d^t_{kS}(D^{-1}) = 2 = \frac{2r}{k} = \frac{n-1}{k}.$$

If D is not a tournament or D is a non-regular tournament, then $\delta^-(D)+\delta^-(D^{-1}) \leq n-2$ and hence we deduce from Corollary 2.1 that

$$d^t_{kS}(D) + d^t_{kS}(D^{-1}) \leq \frac{n-2}{k}.$$

If D is an r-regular tournament, then $n = 2r+1$. If $k < r < 3k$, then Theorem 2.2 leads to

$$2 = d^t_{kS}(D) + d^t_{kS}(D^{-1}) < \frac{n-1}{k}.$$

Finally, assume that $r \geq 3k$. We observe that $(n, r) = (2r+1, r) = 1$. Using Theorem 2.2, we deduce that

$$d^t_{kS}(D) + d^t_{kS}(D^{-1}) < \frac{\delta^-(D)}{k} + \frac{\delta^-(D^{-1})}{k} = \frac{n-1}{k},$$

and the proof is complete. $\square$

Theorem 2.5. *Let D be a digraph of order n and $\delta^-(D) \geq k > 0$. Then $\gamma^t_{kS}(D) \cdot d^t_{kS}(D) \leq n$. Moreover if $\gamma^t_{kS}(D) \cdot d^t_{kS}(D) = n$, then for each $d = d^t_{kS}(D)$-family $\{f_1, f_2, \ldots, f_d\}$ of D each function f_i is a $\gamma^t_{kS}(D)$-function and $\sum_{i=1}^d f_i(v) = 1$ for all $v \in V$.*

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a STkD family of D such that $d = d^t_{kS}(D)$ and let $v \in V$. Then

$$\begin{aligned}
d \cdot \gamma^t_{kS}(D) &= \sum_{i=1}^{d} \gamma^t_{kS}(D) \\
&\leq \sum_{i=1}^{d} \sum_{v \in V} f_i(v) \\
&= \sum_{v \in V} \sum_{i=1}^{d} f_i(v) \\
&\leq \sum_{v \in V} 1 \\
&= n.
\end{aligned}$$

If $\gamma^t_{kS}(D) \cdot d^t_{kS}(D) = n$, then the two inequalities occurring in the proof become equalities. Hence for the $d^t_{kS}(D)$-family $\{f_1, f_2, \ldots, f_d\}$ of D and for each i, $\sum_{v \in V} f_i(v) = \gamma^t_{kS}(D)$, thus each function f_i is a $\gamma^t_{kS}(D)$-function, and $\sum_{i=1}^d f_i(v) = 1$ for all v. $\square$

Corollary 2.4. *If D is a digraph of order n, then $\gamma^t_{kS}(D) + d^t_{kS}(D) \leq n + 1$.*

Proof. By Theorem 2.5,

$$\gamma^t_{kS}(D) + d^t_{kS}(D) \le d^t_{kS}(D) + \frac{n}{d^t_{kS}(D)}. \tag{2.2}$$

Using the fact that the function $g(x) = x + n/x$ is decreasing for $1 \le x \le \sqrt{n}$ and increasing for $\sqrt{n} \le x \le n$, this inequality leads to the desired bound immediately. □

Corollary 2.5. *Let D be a digraph of order $n \ge 3$. If $2 \le \gamma^t_{kS}(D) \le n-1$, then*

$$\gamma^t_{kS}(D) + d^t_{kS}(D) \le n.$$

Proof. Theorem 2.5 implies that

$$\gamma^t_{kS}(D) + d^t_{kS}(D) \le \gamma^t_{kS}(D) + \frac{n}{\gamma^t_{kS}(D)}. \tag{2.3}$$

If we define $x = \gamma^t_{kS}(D)$ and $g(x) = x + n/x$ for $x > 0$, then because $2 \le \gamma^t_{kS}(D) \le n-1$, we have to determine the maximum of the function g on the interval $I : 2 \le x \le n-1$. It is easy to see that

$$\begin{aligned}
\max_{x\in I}\{g(x)\} &= \max\{g(2), g(n-1)\} \\
&= \max\{2 + \frac{n}{2}, n - 1 + \frac{n}{n-1}\} \\
&= n - 1 + \frac{n}{n-1} < n+1,
\end{aligned}$$

and we obtain $\gamma^t_{kS}(D) + d^t_{kS}(D) \le n$. This completes the proof. □

Corollary 2.6. *Let D be a digraph of order n and let $k \ge 1$ be an integer. If $\min\{\gamma^t_{kS}(D), d^t_{kS}(D)\} \ge 2$, then*

$$\gamma^t_{kS}(D) + d^t_{kS}(D) \le \frac{n}{2} + 2.$$

Proof. Since $\min\{\gamma^t_{kS}(D), d^t_{kS}(D)\} \ge 2$, it follows by Theorem 2.5 that $2 \le d^t_{kS}(D) \le \frac{n}{2}$. By (2.2) and the fact that the maximum of $g(x) = x + n/x$ on the interval $2 \le x \le n/2$ is $g(2) = g(n/2)$, we see that

$$\gamma^t_{kS}(D) + d^t_{kS}(D) \le d^t_{kS}(D) + \frac{n}{d^t_{kS}(D)} \le \frac{n}{2} + 2.$$

□

Observation 1.2 shows that Corollary 2.6 is no longer true if $\min\{\gamma^t_{kS}(D), d^t_{kS}(D)\} = 1$.

3. Signed total k-domatic number of graphs

The *signed total k-dominating function* of a graph G is defined in [8] as a function $f : V(G) \longrightarrow \{-1, 1\}$ such that $\sum_{x \in N_G(v)} f(x) \geq k$ for all $v \in V(G)$. The sum $\sum_{x \in V(G)} f(x)$ is the weight $w(f)$ of f. The minimum of weights $w(f)$, taken over all signed total k-dominating functions f on G is called the *signed total k-domination number* of G, denoted by $\gamma^t_{kS}(G)$. The special case $k = 1$ was defined and investigated in [10].

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(G)$, is called a *signed total k-dominating family* on G. The maximum number of functions in a signed total k-dominating family on G is the *signed total k-domatic number* of G, denoted by $d^t_{kS}(G)$. This parameter was introduced by Khodkar and Sheikholeslami in [5]. In the case $k = 1$, we write $d_{st}(G)$ instead of $d^t_{1S}(G)$ which was introduced by Henning [2].

The *associated digraph* $D(G)$ of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}(v) = N_G(v)$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 3.1. *If $D(G)$ is the associated digraph of a graph G, then $\gamma^t_{kS}(D(G)) = \gamma^t_{kS}(G)$ and $d^t_{kS}(D(G)) = d^t_{kS}(D)$.*

There are a lot of interesting applications of Observation 3.1, as for example the following results. Using Observation 1.3, we obtain the first one.

Corollary 3.1. (*Henning* [2]) *The signed total domatic number $d_{st}(G)$ of a graph G is an odd integer.*

Since $\delta^-(D(G)) = \delta(G)$, the next result follows from Observation 3.1 and Corollary 2.1.

Corollary 3.2. (*Khodkar and Sheikholeslami* [5]) *If G is a graph with minimum degree $\delta(G) \geq k$, then*

$$d^t_{kS}(G) \leq \begin{cases} \dfrac{\delta(G)}{k}, & \textit{if} \quad \delta(G) \equiv k \pmod 2, \\ \dfrac{\delta(G)}{k+1}, & \textit{if} \quad \delta(G) \equiv k+1 \pmod 2. \end{cases}$$

The case $k = 1$ in Corollary 3.2 can be found in [2].

In view of Observation 3.1 and Corollary 2.4, we obtain the next result immediately.

Corollary 3.3. (*Khodkar and Sheikholeslami* [5]) *If G is a graph of order n, then*

$$\gamma_{kS}^t(G) + d_{kS}^t(G) \leq n + 1.$$

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[1] Department of Mathematics,
Azarbaijan University of Tarbiat Moallem,
I. R. Iran
E-mail address: s.m.sheikholeslami@azaruniv.edu

[2] Lehrstuhl II für Mathematik,
RWTH Aachen University,
Germany
E-mail address: volkm@math2.rwth-aachen.de