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GROWTH AND OSCILLATION OF SOME CLASS OF DIFFERENTIAL POLYNOMIALS IN THE UNIT DISC

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Abstract. In this paper, we study the growth and the oscillation of complex differential equations $f^{''}+A_1(z)f^{'}+A_0(z)f=0$ and $f^{''}+A_1(z)f^{'}+A_0(z)f=F$, where $A_0\not\equiv 0$, A_1 and F are analytic functions in the unit disc $\Delta=\{z:|z|<1\}$ with finite iterated $p-$order. We obtain some results on the iterated $p-$order and the iterated exponent of convergence of zero-points in Δ of the differential polynomials $g_f=d_1f^{'}+d_0f$ and $g_f=d_1f^{'}+d_0f+b$, where d_1,d_0,b are analytic functions such that at least one of $d_0(z),d_1(z)$ does not vanish identically with $\rho_p(d_j)<\infty$ $(j=0,1)\,,\rho_p(b)<\infty$.

1. Introduction

Throughout this paper, we assume that the reader is familiar with the fundamental results and standard notations of the Nevanlinna theory ([15], [16], [20], [24]) in the unit disc $\Delta=\{z\in\mathbb{C}:|z|<1\}$. Many important results have been obtained on the complex oscillation theory of differential polynomials generated by solutions of differential equations in $\mathbb{C}$, refer to see [2], [3], [13], [21], [27] and others. Recently, there has been an increasing interest in studying the growth of analytic solutions of linear differential equations in the unit disc by making use of Nevanlinna theory. The analysis of slowly growing solutions have been studied in [12], [16], [23]. Fast growth of solutions are considered by [4], [7], [8], [9], [10], [11], [14], [16], [17]. This development makes it possible to start investigating unit disc analogous of existing plane results on

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value distribution of differential polynomials. Thus a natural question is: What can be said about the problem of complex oscillation theory of differential polynomials generated by solutions of differential equations in the unit disc $\Delta = \{z : |z| < 1\}$?

Recently in [9], Cao and Yi obtained some results on analytic solutions of the differential equation

$$f'' + Af = 0,$$

where $A(z)$ is an analytic function in Δ. In [7], [8], [10], [17], [26] some results on the growth and complex oscillation theory of analytic solutions of higher order linear differential equations in Δ where obtained.

In this paper, we continue to consider this subject and investigate the complex oscillation theory of differential polynomials generated by analytic solutions of second order linear differential equations in Δ.

We need to give some definitions and discussions. Firstly, let us give two definitions about the degree of small growth order of functions in Δ as polynomials on the complex plane $\mathbb{C}$. There are many types of definitions of small growth order of functions in Δ (i.e., see [11], [12]).

Definition 1.1. Let f be a meromorphic function in Δ, and

$$D(f) = \limsup_{r \to 1^-} \frac{T(r,\ f)}{-\log\ (1-r)} = b.$$

If $b < \infty$, we say that f is of finite b degree (or is non-admissible); if $b = \infty$, we say that f is of infinite degree (or is admissible).

Definition 1.2. Let f be an analytic function in Δ, and

$$D_M(f) = \limsup_{r \to 1^-} \frac{\log^+ M(r,\ f)}{-\log\ (1-r)} = a < \infty \ \ (\text{or } a = \infty),$$

then we say that f is a function of finite a degree (or of infinite degree), where $M(r,\ f) = \max_{|z|=r} |f(z)|$.

Now we give the definitions of iterated order and growth index to classify generally the functions of fast growth in Δ as those in $\mathbb{C}$ (see [6], [18], [19]). Let us define inductively, for $r \in [0,1)$, $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large in $(0,1)$, $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.3. [7], [8] Let f be a meromorphic function in Δ. Then the iterated p-order of f is defined by

$$\rho_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ T(r,\ f)}{-\log\ (1-r)} \quad (p \geqslant 1\ \text{ is an integer}),$$

where $\log_1^+ x = \log^+ x = \max\{\log x, 0\}$, $\log_{p+1}^+ x = \log^+ \log_p^+ x$. For $p = 1$, this notation is called order and for $p = 2$ hyper-order ([16], [22]).

Remark 1.1. If f is analytic in Δ, then the iterated p-order of f is defined by

$$\rho_{M,p}(f) = \limsup_{r \to 1^-} \frac{\log_{p+1}^+ M(r,\ f)}{-\log\ (1-r)} \quad (p \geqslant 1 \text{ is an integer}).$$

Remark 1.2. It follows by M. Tsuji ([24], p. 205) that if f is an analytic function in Δ, then we have the inequalities

$$\rho_1(f) \leqslant \rho_{M,1}(f) \leqslant \rho_1(f) + 1,$$

which are the best possible in the sense that there are analytic functions g and h such that $\rho_{M,1}(g) = \rho_1(g)$ and $\rho_{M,1}(h) = \rho_1(h) + 1$, see [12]. However, it follows by Proposition 2.2.2 in [19] that $\rho_{M,p}(f) = \rho_p(f)$ for $p \geqslant 2$.

Definition 1.4. [7] The growth index of the iterated order of a meromorphic function $f(z)$ in Δ is defined by

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible;} \\ \min\{j \in \mathbb{N} : \rho_j(f) < +\infty\}, & \text{if } f \text{ is admissible;} \\ +\infty, & \text{if } \rho_j(f) = +\infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

For an analytic function f in Δ, we also define

$$i_M(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible,} \\ \min\{j \in \mathbb{N} : \rho_{M,j}(f) < +\infty\}, & \text{if } f \text{ is admissible,} \\ +\infty, & \text{if } \rho_{M,j}(f) = +\infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

Remark 1.3. If $\rho_p(f) < \infty$ or $i(f) \leqslant p$, then we say that f is of finite iterated $p-$order; if $\rho_p(f) = \infty$ or $i(f) > p$, then we say that f is of infinite iterated $p-$order. In particular, we say that f is of finite order if $\rho(f) < \infty$ or $i(f) \leqslant 1$; f is of infinite order if $\rho(f) = \infty$ or $i(f) > 1$.

Definition 1.5. [5] Let f be a meromorphic function in Δ. Then the iterated p-type of f, with iterated p-order $0 < \rho_p(f) < \infty$ is defined by

$$\sigma_p(f) = \limsup_{r\to 1^-} (1-r)^{\rho_p(f)} \log_{p-1} T(r,\ f) \quad (p \geqslant 1 \ \text{ is an integer}).$$

If f is an analytic function in Δ, then the iterated p-type of f, with iterated p-order $0 < \rho_{M,p}(f) < \infty$ is defined by [17]

$$\sigma_{M,p}(f) = \limsup_{r\to 1^-} (1-r)^{\rho_{M,p}(f)} \log_p M(r,\ f) \quad (p \geqslant 1 \ \text{ is an integer}).$$

Definition 1.6. [8] Let f be a meromorphic function in Δ. Then the iterated exponent of convergence of the sequence of zeros of $f(z)$ is defined by

$$\lambda_p(f) = \limsup_{r\to 1^-} \frac{\log_p^+ N\left(r, \frac{1}{f}\right)}{-\log\ (1-r)} \quad (p \geqslant 1 \ \text{ is an integer}),$$

where $N\left(r, \frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z : |z| < r\}$. For $p = 1$, this notation is called exponent of convergence of the sequence of zeros and for $p = 2$ hyper-exponent of convergence of the sequence of zeros.

Similarly, the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \limsup_{r\to 1^-} \frac{\log_p^+ \overline{N}\left(r, \frac{1}{f}\right)}{-\log\ (1-r)} \quad (p \geqslant 1 \ \text{ is an integer}),$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| < r\}$. For $p = 1$, this notation is called exponent of convergence of the sequence of distinct zeros and for $p = 2$ hyper-exponent of convergence of the sequence of distinct zeros.

Definition 1.7. [4], [8] Let f be a meromorphic function in Δ. Then the iterated exponent of convergence of the sequence of fixed points of $f(z)$ is defined by

$$\tau_p(f) = \lambda_p(f-z) = \limsup_{r\to 1^-} \frac{\log_p^+ N\left(r, \frac{1}{f-z}\right)}{-\log\ (1-r)} \ (p \geqslant 1 \text{ is an integer}).$$

For $p = 1$, this notation is called exponent of convergence of the sequence of fixed points and for $p = 2$ hyper-exponent of convergence of the sequence of fixed points.

Similarly, the iterated exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}_p(f) = \overline{\lambda}_p(f-z) = \limsup_{r\to 1^-} \frac{\log_p^+ \overline{N}\left(r, \frac{1}{f-z}\right)}{-\log\ (1-r)} \ (p \geqslant 1 \ \text{ is an integer}).$$

For $p = 1$, this notation is called exponent of convergence of the sequence of distinct fixed points and for $p = 2$ hyper-exponent of convergence of the sequence of distinct fixed points. Thus $\overline{\tau}_p(f) = \overline{\lambda}_p(f - z)$ is an indication of oscillation of distinct fixed points of $f(z)$.

Consider the linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.1}$$

where $k \geqslant 1$ is an integer and $A_0, A_1, \ldots, A_{k-1}$ are analytic functions in Δ. In [7], Cao and Yi have investigated the growth of solutions of the above equation and have obtained the following result.

Theorem 1.1. [7] *Let $p \geqslant 1$ be an integer and $i(A_0) = p$, and let $A_0(z), \ldots, A_{k-1}(z)$ be the coefficients of* (1.1) *analytic in Δ. If*

$$\max\{i(A_j) : j = 1, \ldots, k-1\} < p$$

or

$$\max\{\rho_p(A_j) : j = 1, \ldots, k-1\} < \rho_p(A_0),$$

then $i(f) = p + 1$ and $\rho_p(A_0) \leqslant \rho_{p+1}(f) = \rho_{M,p+1}(f) \leqslant \alpha_M$ holds for all solutions $f \not\equiv 0$ of (1.1)*, where $\alpha_M = \max\{\rho_{M,p}(A_j) : j = 0, \ldots, k-1\}$.*

In [5], the author and El Farissi have generalized Theorem 1.1 for second order differential equations as follows.

Theorem 1.2. [5] *Let $A_0(z), A_1(z), d_0, d_1, d_2$ be analytic functions in Δ and $i(A_0) = p$ ($p \geqslant 1$ is an integer) such that $\max\{\rho_p(A_1), \rho_p(d_j)\ (j = 0, 1, 2)\} < \rho_p(A_0) = \rho$ ($0 < \rho < \infty$), $\sigma_p(A_0) = \sigma$ ($0 < \sigma < \infty$), and let $\varphi \not\equiv 0$ be an analytic function in Δ with $\rho_p(\varphi) < \infty$. If $f \not\equiv 0$ is a solution of the equation*

$$f'' + A_1(z) f' + A_0(z) f = 0,$$

then the differential polynomial $g_f = d_2 f'' + d_1 f' + d_0 f$ satisfies

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(g_f) = \rho_p(f) = \infty,$$
$$\rho_p(A_0) \leqslant \overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi)$$
$$= \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_{M,p+1}(f) \leqslant \alpha_M,$$

where $\alpha_M = \max\{\rho_{M,p}(A_j) : j = 0, 1\}$.

In [25], Tu and Yi obtained the following result in the complex plane.

Theorem 1.3. [25] *Let $A_j(z)$ $(j=0,\dots,k-1)$ be entire functions satisfying $\rho(A_0)=\rho$ $(0<\rho<\infty)$, $\sigma(A_0)=\sigma$ $(0<\sigma<\infty)$, and let $\rho(A_j)\leqslant\rho$, $\sigma(A_j)<\sigma$ if $\rho(A_j)=\rho$ $(j=1,\dots,k-1)$, then every solution $f\not\equiv 0$ of* (1.1) *satisfies $\rho_2(f)=\rho(A_0)$.*

In the same paper, they posed the following question "*Can we get the same result as Theorem 1.3 when all the coefficients in* (1.1) *are analytic in the unit disc* $\{z:\ |z|<1\}$?"

In [26], they answer to this question and obtained the following result.

Theorem 1.4. [26] *Let $p\geqslant 1$ be an integer and $i(A_0)=p$, and let $A_0(z),\dots,A_{k-1}(z)$ be the coefficients of* (1.1) *analytic in Δ satisfying*

$$\max\{\rho_{M,p}(A_j): j=1,\dots,k-1\}\leqslant\rho_{M,p}(A_0),\rho_{M,p}(A_0)>0$$

and $\rho_{M,p}(A_j)=\rho_{M,p}(A_0)$ if $\sigma_{M,p}(A_j)<\sigma_{M,p}(A_0)$ $(j=1,\dots,k-1)$. Then every solution $f\not\equiv 0$ of (1.1) *satisfies $i(f)=p+1$ and $\rho_{M,p+1}(f)=\rho_{M,p}(A_0)$.*

2. Homogeneous case

The first main purpose of this paper is to study the growth, the oscillation and the relation between small functions and differential polynomials generated by homogeneous second order linear differential equations when the solutions are of infinite iterated $p-$order and the coefficients having the same iterated $p-$order. Furthermore, we give answer to the question of Tu and Yi and obtain similar result to Theorem 1.4 for the iterated $p-$order of analytic solutions f and analytic coefficients defined by the characteristic function of Nevanlinna (see Lemma 4.10).

We consider the differential equation

$$f''+A_1(z)f'+A_0(z)f=0, \tag{2.1}$$

where A_0, A_1 are analytic functions in the unit disc $\Delta=\{z:|z|<1\}$. It is well-known that all solutions of equation (2.1) are analytic functions in Δ and that there are exactly two linearly independent solutions of (2.1) (see [16]) . We obtain the following results.

Theorem 2.1. *Let $A_0(z),A_1(z)$ be analytic functions in Δ and $i(A_0)=p$ ($p\geqslant 1$ is an integer) such that $\rho_p(A_0)=\rho$ $(0<\rho<\infty)$, $\sigma_p(A_0)=\sigma$ $(0<\sigma<\infty)$, and let $\rho_p(A_1)<\rho_p(A_0)$ and $\sigma_p(A_1)<\sigma_p(A_0)$ if $\rho_p(A_0)=\rho_p(A_1)$. Let $d_1(z),d_0(z)$ be analytic functions in Δ such that at least one of d_0,d_1 does not vanish identically* *with* $\max\{\rho_p(d_j)\ \ (j=0,1)\} < \rho_p(A_0)$*. If* $f \not\equiv 0$ *is a solution of* (2.1)*, then the differential polynomial*

$$g_f = d_1 f' + d_0 f \tag{2.2}$$

satisfies

$$\rho_p(g_f) = \rho_p(f) = \infty,$$
$$\rho_p(A_0) \leqslant \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_{M,p+1}(f) \leqslant \alpha_M,$$

where $\alpha_M = \max\{\rho_{M,p}(A_j) : j = 0,1\}$.

Theorem 2.2. *Let* $A_0(z), A_1(z)$ *be analytic functions in* Δ *and* $i(A_0) = p$ $(p \geqslant 1$ *is an integer) such that* $\rho_p(A_0) = \rho$ $(0 < \rho < \infty)$, $\sigma_p(A_0) = \sigma$ $(0 < \sigma < \infty)$*, and let* $\rho_p(A_1) < \rho_p(A_0)$ *and* $\sigma_p(A_1) < \sigma_p(A_0)$ *if* $\rho_p(A_0) = \rho_p(A_1)$*. Let* $d_0(z), d_1(z)$ *be analytic functions in* Δ *such that at least one of* d_0, d_1 *does not vanish identically with* $\max\{\rho_p(d_j)\ \ (j=0,1)\} < \rho_p(A_0)$*, and let* $\varphi \not\equiv 0$ *be analytic function in* Δ *of finite iterated p-order* $\rho_p(\varphi) < +\infty$*. If* $f \not\equiv 0$ *is a solution of* (2.1)*, then the differential polynomial* $g_f = d_1 f' + d_0 f$ *satisfies*

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(g_f) = \rho_p(f) = \infty$$

and

$$\rho_p(A_0) \leqslant \overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi)$$
$$= \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_{M,p+1}(f) \leqslant \alpha_M,$$

where $\alpha_M = \max\{\rho_{M,p}(A_j) : j = 0,1\}$.

Using Theorem 2.2, we obtain by setting $\varphi(z) \equiv z$ the following corollary.

Corollary 2.1. *Let* $A_0(z), A_1(z)$ *be analytic functions in* Δ *and* $i(A_0) = p$ $(p \geqslant 1$ *is an integer) such that* $\rho_p(A_0) = \rho$ $(0 < \rho < \infty)$, $\sigma_p(A_0) = \sigma$ $(0 < \sigma < \infty)$*, and let* $\rho_p(A_1) < \rho_p(A_0)$ *and* $\sigma_p(A_0) < \sigma_p(A_0)$ *if* $\rho_p(A_0) = \rho_p(A_1)$*. Let* $d_0(z), d_1(z)$ *be analytic functions in* Δ *such that at least one of* d_0, d_1 *does not vanish identically with* $\max\{\rho_p(d_j)\ \ (j=0,1)\} < \rho_p(A_0)$*. If* $f \not\equiv 0$ *is a solution of* (2.1)*, then the differential polynomial* $g_f = d_1 f' + d_0 f$ *satisfies*

$$\overline{\tau}_p(g_f) = \tau_p(g_f) = \rho_p(g_f) = \rho_p(f) = \infty$$

and

$$\rho_p(A_0) \leqslant \overline{\tau}_{p+1}(g_f) = \tau_{p+1}(g_f) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_{M,p+1}(f) \leqslant \alpha_M,$$

where $\alpha_M = \max\{\rho_{M,p}(A_j) : j = 0,1\}$.

3. Non-homogeneous case

We consider the differential equation

$$f'' + A_1 f' + A_0 f = F, \tag{3.1}$$

where A_0, A_1 and F are analytic functions in the unit disc Δ.

The second main purpose of this paper is to study the growth, the oscillation and the relation between small functions and differential polynomials generated by non-homogeneous second order linear differential equation (3.1) when the solutions are of finite iterated p-order.

Before we state our results, we denote by

$$\beta_0 = d_0' - d_1 A_0, \;\; \beta_1 = d_1' + d_0 - d_1 A_1, \tag{3.2}$$

$$h = d_1 \beta_0 - d_0 \beta_1 \tag{3.3}$$

and

$$\eta(z) = \frac{d_1 \left(\varphi' - b' - d_1 F\right) - \left(d_1' + d_0 - d_1 A_1\right)(\varphi - b)}{h}, \tag{3.4}$$

where $A_1(z)$, $A_0(z) \not\equiv 0$, F, d_j $(j = 0, 1)$, b and φ are analytic functions in the unit disc Δ with finite iterated p-order. We obtain the following results.

Theorem 3.1. *Let $A_1(z)$, $A_0(z) \not\equiv 0$, F be analytic functions of finite iterated p-order in Δ, and let $d_0(z), d_1(z), b(z)$ be analytic functions of finite iterated p-order in Δ such that at least one of d_0, d_1 does not vanish identically and that $h \not\equiv 0$. If f is a finite iterated p-order solution of (3.1) such that*

$$\max \{\rho_p(A_j) \;\; (j = 0, 1), \rho_p(d_j) \;\; (j = 0, 1), \rho_p(b), \rho_p(F)\} < \rho_p(f), \tag{3.5}$$

then the differential polynomial

$$g_f = d_1 f' + d_0 f + b \tag{3.6}$$

satisfies

$$\rho_p(g_f) = \rho_p(f).$$

Remark 3.1. The condition (3.5) is necessary because if we consider the differential equation

$$f'' - z \exp_2\left\{\left(\frac{1}{1-z}\right)^2\right\} f' + \exp_2\left\{\left(\frac{1}{1-z}\right)^2\right\} f = \exp_2\left\{\left(\frac{1}{1-z}\right)^2\right\}, \tag{3.7}$$

then it is easy to see that $f(z) = z+1$ is a solution of (3.7) and by taking $d_j = \exp_2\left\{\left(\frac{1}{1-z}\right)^2\right\}$ $(j=0,1)$, $b = \exp_2\left\{\left(\frac{1}{1-z}\right)^2\right\}$, we obtain that $\rho_2(g_f) = 2 > \rho_2(f) = 0$.

Remark 3.2. In Theorem 3.1, if we don't have the condition $h \not\equiv 0$, then the conclusion of Theorem 3.1 can not holds. For example, if $d_0^{'} - d_1 A_0 \equiv 0$ and $d_1^{'} + d_0 - d_1 A_1 \equiv 0$, then $h \equiv 0$. Differentiating both sides of equation (3.6) and replacing $f^{''}$ with $f^{''} = F - A_1 f^{'} - A_0 f$, we obtain

$$g_f^{'} - b^{'} - d_1 F = \left(d_1^{'} + d_0 - d_1 A_1\right) f^{'} + \left(d_0^{'} - d_1 A_0\right) f. \tag{3.8}$$

By (3.8) we obtain $g_f^{'} - b^{'} - d_1 F \equiv 0$, it follows that $\rho_p(g_f) = \rho_p\left(g_f^{'}\right) = \rho_p\left(b^{'} + d_1 F\right) < \rho_p(f)$.

Theorem 3.2. *Assume that the assumptions of Theorem 3.1 hold, and let $\varphi(z)$ be an analytic function in Δ with $\rho_p(\varphi) < \rho_p(f)$ such that $\eta(z)$ is not a solution of* (3.1). *If f is a finite iterated p-order solution of* (3.1) *such that* (3.5) *holds, then the differential polynomial $g_f = d_1 f^{'} + d_0 f + b$ satisfies*

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f).$$

Remark 3.3. In Theorem 3.2, if we don't have the condition $\eta(z)$ is not a solution of (3.1), then the conclusion of Theorem 3.2 does not holds. For example, the functions $f_1(z) = 2 - z$ and $f_2(z) = (1-z)\exp\left(\exp\frac{1}{1-z}\right) + 1$ are linearly independent solutions of the equation

$$f^{''} + A_1(z) f^{'} + A_0(z) f = -\frac{\exp\frac{1}{1-z}}{(1-z)^3} - \frac{1}{(1-z)^3}, \tag{3.9}$$

where

$$A_0(z) = -\frac{\exp\frac{1}{1-z}}{(1-z)^3} - \frac{1}{(1-z)^3} \quad \text{and} \quad A_1(z) = -\frac{\exp\frac{1}{1-z}}{(1-z)^2} - \frac{1}{(1-z)^2}.$$

Clearly $f = \frac{f_1+f_2}{2}$ is a solution of equation (3.9) with $\rho_1(f) = \infty$ and $\rho_2(f) = 1$. Set $d_1 = b \equiv 0$ and $d_0 = \frac{\exp\frac{1}{1-z}}{1-z}$. Then $g_f = d_0 f$, $h = -d_0^2$, $\eta(z) = \frac{\varphi}{d_0}$ and f satisfies (3.5) for $p = 2$. If we take $\varphi = d_0\frac{(f_1+1)}{2}$, then $\rho_2(\varphi) = 0 < \rho_2(f) = 1$, $\eta(z) = \frac{f_1+1}{2}$ is a solution of (3.9) and we have

$$\begin{aligned}\lambda_2(g_f - \varphi) &= \lambda_2\left(d_0 f - d_0\frac{(f_1+1)}{2}\right) = \lambda_2\left(d_0\frac{(f_2-1)}{2}\right)\\ &= \lambda_2\left(\frac{1}{2}\exp\frac{1}{1-z}\exp\left(\exp\frac{1}{1-z}\right)\right) = 0.\end{aligned}$$

On the other hand,

$$\rho_2(g_f) = \rho_2(d_0 f) = \rho_2\left(d_0 \frac{(f_1+f_2)}{2}\right)$$
$$= \rho_2\left(\frac{(3-z)\exp\frac{1}{1-z}}{2(1-z)} + \frac{1}{2}\exp\frac{1}{1-z}\exp\left(\exp\frac{1}{1-z}\right)\right) = 1.$$

4. Preliminary lemmas

We need the following lemmas in the proofs of our theorems.

Lemma 4.1. [7] *If f and g are meromorphic functions in Δ, $p \geqslant 1$ is an integer, then we have*

(i) $\rho_p(f) = \rho_p(1/f)$, $\rho_p(a.f) = \rho_p(f)$ $(a \in \mathbb{C} - \{0\})$;
(ii) $\rho_p(f) = \rho_p\left(f'\right)$;
(iii) $\max\{\rho_p(f+g), \rho_p(fg)\} \leqslant \max\{\rho_p(f), \rho_p(g)\}$;
(iv) *if* $\rho_p(f) < \rho_p(g)$, *then* $\rho_p(f+g) = \rho_p(g)$, $\rho_p(fg) = \rho_p(g)$.

Lemma 4.2. [10] *Let $A_0, A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ be meromorphic functions in the unit disc Δ and let $f(z)$ be a meromorphic solution of the differential equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F \tag{4.1}$$

such that $\max\{\rho_p(F), \rho_p(A_j)\ (j = 0, \ldots, k-1)\} < \rho_p(f)$. *Then* $\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f)$.

Using the same arguments as in the proof of Lemma 4.2 (see [10]), we easily obtain the following lemma.

Lemma 4.3. *Let $A_0, A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ be finite iterated p- order meromorphic functions in the unit disc Δ. If $f$is a meromorphic solution with $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho < +\infty$ of equation* (4.1), *then $\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = +\infty$ and $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho$.*

Lemma 4.4. [5] *Let f, g be meromorphic functions in the unit disc Δ such that $0 < \rho_p(f), \rho_p(g) < +\infty$ and $0 < \sigma_p(f), \sigma_p(g) < +\infty$. Then, the following two statements hold:*

(i) *If* $\rho_p(g) < \rho_p(f)$, *then*

$$\sigma_p(f+g) = \sigma_p(fg) = \sigma_p(f). \tag{4.2}$$

(ii) *If* $\rho_p(g) = \rho_p(f)$ *and* $\sigma_p(g) < \sigma_p(f)$, *then*

$$\rho_p(f+g) = \rho_p(fg) = \rho_p(f) = \rho_p(g). \tag{4.3}$$

Lemma 4.5. [16] *Let f be a meromorphic function in the unit disc Δ, and let $k \geqslant 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f), \tag{4.4}$$

where $S(r,f) = O\left(\log^+ T(r,f) + \log(\frac{1}{1-r})\right)$, *possibly outside a set* $E_1 \subset [0,1)$ *with* $\int_{E_1} \frac{dr}{1-r} < +\infty$. *If f is of finite order (namely, finite iterated 1-order) of growth, then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log(\frac{1}{1-r})\right). \tag{4.5}$$

Lemma 4.6. [4] *Let f be a meromorphic function in the unit disc Δ for which* $i(f) = p \geqslant 1$ *and* $\rho_p(f) = \beta < +\infty$, *and let $k \geqslant 1$ be an integer. Then for any $\varepsilon > 0$,*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right) \tag{4.6}$$

holds for all r outside a set $E_2 \subset [0,1)$ *with* $\int_{E_2} \frac{dr}{1-r} < +\infty$.

Lemma 4.7. [1] *Let* $g : (0,1) \to \mathbb{R}$ *and* $h : (0,1) \to \mathbb{R}$ *be monotone increasing functions such that* $g(r) \leqslant h(r)$ *holds outside of an exceptional set* $E_3 \subset [0,1)$ *of finite logarithmic measure. Then there exists a* $d \in (0,1)$ *such that if* $s(r) = 1 - d(1-r)$, *then* $g(r) \leqslant h(s(r))$ *for all* $r \in [0,1)$.

Lemma 4.8. *Let f be a meromorphic function in the disc Δ with iterated p-order* $0 < \rho_p(f) < \infty$ *and iterated p-type* $0 < \sigma_p(f) < \infty$. *Then for any given* $\beta < \sigma_p(f)$, *there exists a subset E_4 of $[0,1)$ that has an infinite logarithmic measure such that* $\log_{p-1} T(r,f) > \beta\left(\frac{1}{1-r}\right)^{\rho_p(f)}$ *holds for all* $r \in E_4$.

Proof. By the definition of iterated p-order and iterated p-type, there exists an increasing sequence $\{r_m\}_{m=1}^{\infty} \subset [0,1)$ $(r_m \to 1^-)$ satisfying $\frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m < r_{m+1}$ and

$$\lim_{r_m \to 1^-} \frac{\log_{p-1} T(r_m, f)}{\left(\frac{1}{1-r_m}\right)^{\rho_p(f)}} = \sigma_p(f). \tag{4.7}$$

Then there exists a positive integer m_0 such that for all $m > m_0$ and for any given $0 < \varepsilon < \sigma_p(f) - \beta$, we have

$$\log_{p-1} T(r_m, f) > (\sigma_p(f) - \varepsilon)\left(\frac{1}{1 - r_m}\right)^{\rho_p(f)}. \tag{4.8}$$

For any given $\beta < \sigma_p(f) - \varepsilon$, there exists a positive integer m_1 such that for all $m > m_1$ we have

$$\left(1 - \frac{1}{m}\right)^{\rho_p(f)} > \frac{\beta}{\sigma_p(f) - \varepsilon}. \tag{4.9}$$

Take $m \geqslant m_2 = \max\{m_0, m_1\}$. By (4.8) and (4.9), for any $r \in [r_m, \frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m]$, we have

$$\begin{aligned} \log_{p-1} T(r, f) &\geqslant \log_{p-1} T(r_m, f) > (\sigma_p(f) - \varepsilon)\left(\frac{1}{1 - r_m}\right)^{\rho_p(f)} \\ &\geqslant (\sigma_p(f) - \varepsilon)\left(1 - \frac{1}{m}\right)^{\rho_p(f)}\left(\frac{1}{1 - r}\right)^{\rho_p(f)} > \beta\left(\frac{1}{1 - r}\right)^{\rho_p(f)}. \end{aligned} \tag{4.10}$$

Set $E_4 = \bigcup\limits_{m=m_2}^{\infty}\left[r_m, \frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m\right]$, then there holds

$$m_l E_4 = \sum_{m=m_2}^{\infty} \int\limits_{r_m}^{\frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m} \frac{dt}{1 - t} = \sum_{m=m_2}^{\infty} \log \frac{m}{m - 1} = \infty.$$

$\square$

Lemma 4.9. [7], [17] *Let $p \geqslant 1$ be an integer. If $A_0(z), \ldots, A_{k-1}(z)$ are analytic functions in the unit disc Δ, then every solution $f \not\equiv 0$ of* (1.1) *satisfies*

$$\rho_{M,p+1}(f) \leqslant \max\{\rho_{M,p}(A_j) : j = 0, 1, \ldots, k - 1\}. \tag{4.11}$$

Lemma 4.10. *Let $A_0(z), A_1(z)$ be analytic functions in Δ and $i(A_0) = p$ ($p \geqslant 1$ is an integer) such that $\rho_p(A_0) = \rho$ $(0 < \rho < \infty)$, $\sigma_p(A_0) = \sigma$ $(0 < \sigma < \infty)$, and let $\rho_p(A_1) < \rho_p(A_0)$ and $\sigma_p(A_1) < \sigma_p(A_0)$ if $\rho_p(A_0) = \rho_p(A_1)$. If $f \not\equiv 0$ is a solution of* (2.1)*, then*

$$\rho_p(f) = \infty, \ \rho_p(A_0) \leqslant \rho_{p+1}(f) = \rho_{M,p+1}(f) \leqslant \alpha_M, \tag{4.12}$$

where $\alpha_M = \max\{\rho_{M,p}(A_j) : j = 0, 1\}$.

Proof. If $\rho_p(A_1) < \rho_p(A_0)$, then the result can be deduced by Theorem 1.1. We prove only the case when $\rho_p(A_0) = \rho_p(A_1) = \rho$ and $\sigma_p(A_1) < \sigma_p(A_0)$. Since $f \not\equiv 0$, then by (2.1)

$$A_0 = -\left(\frac{f''}{f} + A_1 \frac{f'}{f}\right). \tag{4.13}$$

Suppose that f is of finite iterated p-order $\rho_p(f) = \beta < +\infty$, then by Lemma 4.6

$$T(r, A_0) \leqslant T(r, A_1) + O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right) \quad (\varepsilon > 0) \tag{4.14}$$

holds for all r outside a set $E_2 \subset [0, 1)$ with $\int_{E_2} \frac{dr}{1-r} < +\infty$. Then by (4.14) and Lemma 4.7 we obtain

$$\sigma_p(A_0) \leqslant \sigma_p(A_1). \tag{4.15}$$

This is a contradiction. Hence $\rho_p(f) = \infty$. By Lemma 4.9, we have

$$\rho_{M,p+1}(f) \leqslant \alpha_M \tag{4.16}$$

and since $\rho_{M,p+1}(f) = \rho_{p+1}(f)$, we obtain

$$\rho_{p+1}(f) \leqslant \alpha_M. \tag{4.17}$$

On the other hand, since $\rho_p(f) = \infty$, then by Lemma 4.5

$$T(r, A_0) \leqslant T(r, A_1) + O\left(\log^+ T(r, f)\right) + O\left(\log\left(\frac{1}{1-r}\right)\right) \tag{4.18}$$

holds for all r outside a set $E_1 \subset [0, 1)$ with $\int_{E_1} \frac{dr}{1-r} < +\infty$. By $\sigma_p(A_1) < \sigma_p(A_0)$, we choose α_0, α_1 satisfying $\sigma_p(A_1) < \alpha_1 < \alpha_0 < \sigma_p(A_0)$ such that for $r \to 1^-$, we have

$$T(r, A_1) \leqslant \exp_{p-1}\left\{\alpha_1 \left(\frac{1}{1-r}\right)^{\rho}\right\}. \tag{4.19}$$

By Lemma 4.8, there exists a subset $E_4 \subset [0, 1)$ of infinite logarithmic measure such that

$$T(r, A_0) > \exp_{p-1}\left\{\alpha_0 \left(\frac{1}{1-r}\right)^{\rho}\right\}. \tag{4.20}$$

By (4.18)-(4.20) we obtain for all $r \in E_4 - E_1$

$$\exp_{p-1}\left\{\alpha_0 \left(\frac{1}{1-r}\right)^{\rho}\right\} \leqslant \exp_{p-1}\left\{\alpha_1 \left(\frac{1}{1-r}\right)^{\rho}\right\}$$

$$+O\left(\log^+ T(r, f)\right) + O\left(\log\left(\frac{1}{1-r}\right)\right). \tag{4.21}$$

Since $0 < \alpha_1 < \alpha_0$, from (4.21), we have for all $r \in E_4 - E_1$

$$(1 - o(1)) \exp_{p-1}\left\{\alpha_0 \left(\frac{1}{1-r}\right)^{\rho}\right\} \leqslant O\left(\log^+ T(r, f)\right) + O\left(\log\left(\frac{1}{1-r}\right)\right). \tag{4.22}$$

Using (4.22) and Lemma 4.7, we obtain

$$\rho_p(A_0) \leqslant \rho_{p+1}(f).$$

Hence $\rho_p(A_0) \leqslant \rho_{p+1}(f) = \rho_{M,p+1}(f) \leqslant \alpha_M$. □

5. Proof of Theorem 2.1

By Lemma 4.10, we have $\rho_p(f) = \infty$ and

$$\rho_p(A_0) \leqslant \rho_{p+1}(f) = \rho_{M,p+1}(f) \leqslant \alpha_M. \tag{5.1}$$

Differentiating both sides of equation (2.2) and replacing f'' with $f'' = -A_1 f' - A_0 f$, we obtain

$$g_f' = \left(d_1' + d_0 - d_1 A_1\right) f' + \left(d_0' - d_1 A_0\right) f. \tag{5.2}$$

Set

$$\alpha_0 = d_0' - d_1 A_0,\ \alpha_1 = d_1' + d_0 - d_1 A_1. \tag{5.3}$$

Then by (2.2), (5.2) and (5.3), we have

$$d_1 f' + d_0 f = g_f,\ \alpha_1 f' + \alpha_0 f = g_f'. \tag{5.4}$$

Set

$$h = d_1\alpha_0 - d_0\alpha_1 = d_1\left(d_0' - d_1 A_0\right) - d_0\left(d_1' + d_0 - d_1 A_1\right). \tag{5.5}$$

First we suppose that $d_1 \not\equiv 0$. Now check all the terms of h, we can write

$$h = d_1 d_0' - d_1^2 A_0 - d_0 d_1' - d_0^2 + d_0 d_1 A_1.$$

By $d_1 \not\equiv 0$, $A_0 \not\equiv 0$ and Lemma 4.4 we have $\rho_p(h) = \rho_p(A_0) > 0$, hence $h \not\equiv 0$.

Now suppose $d_1 \equiv 0$, $d_0 \not\equiv 0$ we get $h = -d_0^2 \not\equiv 0$. By $h \not\equiv 0$, (5.4) and (5.5), we obtain

$$f = \frac{d_1 g_f' - \alpha_1 g_f}{h}. \tag{5.6}$$

If $\rho_p(g_f) < \infty$, then by (5.6) and Lemma 4.1, we get $\rho_p(f) < \infty$ and this is a contradiction. Hence $\rho_p(g_f) = \infty$.

Now, we prove that $\rho_{p+1}(g_f) = \rho_{p+1}(f)$. By (2.2) and Lemma 4.1, we get $\rho_{p+1}(g_f) \leqslant \rho_{p+1}(f)$ and by (5.6) we have $\rho_{p+1}(f) \leqslant \rho_{p+1}(g_f)$. This yield $\rho_{p+1}(g_f) = \rho_{p+1}(f)$.

6. Proof of Theorem 2.2

If $\rho_p(A_1) < \rho_p(A_0)$, then the result can be deduced by Theorem 1.2. We prove only the case when $\rho_p(A_0) = \rho_p(A_1) = \rho$ and $\sigma_p(A_1) < \sigma_p(A_0)$. By Lemma 4.10, we have $\rho_p(f) = \infty$ and

$$\rho_p(A_0) \leqslant \rho_{p+1}(f) = \rho_{M,p+1}(f) \leqslant \max\{\rho_{M,p}(A_0), \rho_{M,p}(A_1)\}$$

and by Theorem 2.1 we get $\rho_p(g_f) = \rho_p(f) = \infty$ and $\rho_{p+1}(g_f) = \rho_{p+1}(f)$. Set $w(z) = d_1 f' + d_0 f - \varphi$. Then, by $\rho_p(\varphi) < \infty$, we have $\rho_p(w) = \rho_p(g_f) = \rho_p(f) = \infty$ and $\rho_{p+1}(w) = \rho_{p+1}(g_f) = \rho_{p+1}(f)$. In order to prove $\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f) = \infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f)$, we need to prove only $\overline{\lambda}_p(w) = \lambda_p(w) = \rho_p(f) = \infty$ and $\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho_{p+1}(f)$. Using $g_f = w + \varphi$, we get from (5.6)

$$f = \frac{d_1 w' - \alpha_1 w}{h} + \psi, \tag{6.1}$$

where

$$\psi = \frac{d_1 \varphi' - \left(d_1' + d_0 - d_1 A_1\right)\varphi}{h}. \tag{6.2}$$

Substituting (6.1) into equation (2.1), we obtain

$$\frac{d_1}{h} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w = -\left(\psi'' + A_1(z)\psi' + A_0(z)\psi\right) = A, \tag{6.3}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions with $\rho_p(\phi_j) < \infty$ $(j = 0, 1, 2)$.

Now, we prove that $\psi(z) \not\equiv 0$. Assume that $\psi(z) \equiv 0$. Then from (6.2), we obtain that

$$d_1' + d_0 - d_1 A_1 = d_1 \frac{\varphi'}{\varphi}. \tag{6.4}$$

First, if $d_1 \equiv 0$, then by (6.4), we get $d_0 \equiv 0$ and this is a contradiction. Now if $d_1 \not\equiv 0$, since $\rho_p(\varphi) = \mu < +\infty$, then by Lemma 4.6 and (6.4), we get

$$\begin{aligned} T\left(r, d_1' + d_0 - d_1 A_1\right) &= m\left(r, d_1' + d_0 - d_1 A_1\right) \\ &\leqslant m(r, d_1) + O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\mu+\varepsilon}\right) \\ &= T(r, d_1) + O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\mu+\varepsilon}\right) \quad (\varepsilon > 0) \end{aligned} \tag{6.5}$$

holds for all r outside a set $E_2 \subset [0,1)$ with $\int_{E_2} \frac{dr}{1-r} < +\infty$. Then by (6.5) and Lemma 4.7, we obtain

$$\rho_p(A_1) = \rho_p\left(d_1^{'} + d_0 - d_1 A_1\right) \leqslant \rho_p(d_1) < \rho_p(A_0)$$

and this is a contradiction. Hence $\psi(z) \not\equiv 0$.

By $\psi(z) \not\equiv 0$ and $\rho_p(\psi) < \infty$, it follows by Theorem 2.1 that $A \not\equiv 0$. Then by $h \not\equiv 0$ and Lemma 4.3, we obtain $\overline{\lambda}_p(w) = \lambda_p(w) = \rho_p(w) = \infty$ and $\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho_{p+1}(w)$, that is, $\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(g_f) = \rho_p(f) = \infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_{M,p+1}(f)$.

7. Proof of Theorem 3.1

Suppose that f is a solution of equation (3.1) with

$$\max\{\rho_p(A_j) \quad (j=0,1), \rho_p(d_j) \quad (j=0,1), \rho_p(b), \rho_p(F)\} < \rho_p(f) < \infty. \tag{7.1}$$

Differentiating both sides of equation (3.6) and replacing $f^{''}$ with $f^{''} = F - A_1 f^{'} - A_0 f$, we obtain

$$g_f^{'} - b^{'} - d_1 F = \left(d_1^{'} + d_0 - d_1 A_1\right) f^{'} + \left(d_0^{'} - d_1 A_0\right) f. \tag{7.2}$$

Then by (3.2), (3.6) and (7.2), we have

$$d_1 f^{'} + d_0 f = g_f - b, \ \beta_1 f^{'} + \beta_0 f = g_f^{'} - b^{'} - d_1 F. \tag{7.3}$$

Set

$$h = d_1\beta_0 - d_0\beta_1 = d_1\left(d_0^{'} - d_1 A_0\right) - d_0\left(d_1^{'} + d_0 - d_1 A_1\right). \tag{7.4}$$

By $h \not\equiv 0$, (7.3) and (7.4), we obtain

$$f = \frac{d_1\left(g_f^{'} - b^{'} - d_1 F\right) - \beta_1(g_f - b)}{h}. \tag{7.5}$$

By (3.6) and Lemma 4.1, we have $\rho_p(g_f) \leqslant \rho_p(f)$. If $\rho_p(g_f) < \rho_p(f)$, then by (7.1) and (7.5) we get

$$\rho_p(f) \leqslant \max\{\rho_p(A_j) \quad (j=0,1), \rho_p(d_j) \quad (j=0,1), \rho_p(b), \rho_p(F), \rho_p(g_f)\} < \rho_p(f),$$

which is a contradiction. Hence $\rho_p(g_f) = \rho(f)$.

8. Proof of Theorem 3.2

By Theorem 3.1, we have $\rho_p(g_f) = \rho_p(f)$. Set $w(z) = d_1 f' + d_0 f + b - \varphi$. Then, by $\rho_p(\varphi) < \rho_p(f)$ and Lemma 4.1, we have $\rho_p(w) = \rho_p(g_f) = \rho_p(f)$. In order to prove $\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f)$, we need to prove only $\overline{\lambda}_p(w) = \lambda_p(w) = \rho_p(f)$. Using $g_f = w + \varphi$, we get from (7.5)

$$f = \frac{d_1 w' - \beta_1 w}{h} + \eta, \tag{8.1}$$

where

$$\eta(z) = \frac{d_1\left(\varphi' - b' - d_1 F\right) - \beta_1(\varphi - b)}{h}. \tag{8.2}$$

Substituting (8.1) into equation (3.1), we obtain

$$\frac{d_1}{h} w''' + \tilde{\phi}_2 w'' + \tilde{\phi}_1 w' + \tilde{\phi}_0 w = F - \left(\eta'' + A_1(z)\eta' + A_0(z)\eta\right) = B, \tag{8.3}$$

where $\tilde{\phi}_j$ $(j = 0,1,2)$ are meromorphic functions with $\rho_p\left(\tilde{\phi}_j\right) < \rho_p(f)$ $(j = 0,1,2)$. Since $\eta(z)$ is not a solution of (3.1), it follows that $B \not\equiv 0$. By $\rho_p\left(\tilde{\phi}_j\right) < \rho_p(f)$ $(j = 0,1,2)$, $\rho_p\left(\frac{d_1}{h}\right) < \rho_p(f)$, $\rho_p(B) < \rho_p(f)$ and Lemma 4.2, we obtain $\overline{\lambda}_p(w) = \lambda_p(w) = \rho_p(f)$, i.e., $\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f)$.

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