DIFFERENTIAL SANDWICH THEOREMS OF $p$-VALENT FUNCTIONS ASSOCIATED WITH A CERTAIN FRACTIONAL DERIVATIVE OPERATOR

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Abstract. In the present paper we derive some subordination and superordination results for $p$-valent functions in the open unit disk by using certain fractional derivative operator. Some special cases are also considered.

1. Introduction and Preliminaries

Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{ z : |z| < 1 \}$ and let $\mathcal{H}[a, p]$ denote the subclass of the functions $f \in \mathcal{H}(\mathbb{U})$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \ldots \quad (a \in \mathbb{C}, \ p \in \mathbb{N}).$$

Also, let $\mathcal{A}(p)$ be the class of functions $f \in \mathcal{H}(\mathbb{U})$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad p \in \mathbb{N}$$

and set $\mathcal{A} \equiv \mathcal{A}(1)$.

Let $f, g \in \mathcal{H}(\mathbb{U})$. We say that the function $f$ is subordinate to $g$, if there exist a Schwarz function $w$, analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1 \ (z \in \mathbb{U})$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{U}$.
This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$. It is well known that, if the function $g$ is univalent in $U$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $p(z), h(z) \in \mathcal{H}(U)$, and let $\Phi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If $p(z)$ and $\Phi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions, and if $p(z)$ satisfies the second-order superordination

$$(1.2) \quad h(z) \prec \Phi(p(z), zp'(z), z^2p''(z); z)$$

then $p(z)$ is called to be a solution of the differential superordination (1.2). (If $f(z)$ is subordinatnate to $g(z)$, then $g(z)$ is called to be superordinate to $f(z)$). An analytic function $q(z)$ is called a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfies (1.2). An univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (1.2) is said to be the best subordinant.

Recently, Miller and Mocanu [5] obtained conditions on $h(z), q(z)$ and $\Phi$ for which the following implication holds true:

$$h(z) \prec \Phi(p(z), zp'(z), z^2p''(z); z) \implies q(z) \prec p(z)$$

with the results of Miller and Mocanu [5], Bulboac˘ a [2] investigated certain classes of first order differential superordinations as well as superordination-preserving integral operators [3]. Ali et al. [1] used the results obtained by Bulboac˘ a [3] and gave the sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in $U$ with $q_1(0) = 1$ and $q_2(0) = 1$. Shanmugam et al. [8] obtained sufficient conditions for a normalized analytic functions to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{(f(z))^2} \prec q_2(z)$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in $U$ with $q_1(0) = 1$ and $q_2(0) = 1$.

Let $\mathbf{2F1}(a, b; c; z)$ be the Gauss hypergeometric function defined for $z \in U$ by (see Srivastava and Karlsson [9])

$$(1.3) \quad \mathbf{2F1}(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_nn!} z^n$$
where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

\begin{equation}
(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 
1, & \text{when } n = 0, \\
\lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1), & \text{when } n \in \mathbb{N}.
\end{cases}
\end{equation}

for $\lambda \neq 0, -1, -2, \ldots$.

We recall the following definitions of fractional derivative operators which were used by Owa [6], (see also [7]) as follows:

**Definition 1.1.** The fractional derivative operator of order $\lambda$ is defined by

\begin{equation}
D_\lambda^zf(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z f(\xi)(z-\xi)^{-\lambda}d\xi
\end{equation}

where $0 \leq \lambda < 1$, $f(z)$ is analytic function in a simply-connected region of the $z$-plane containing the origin, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

**Definition 1.2.** Let $0 \leq \lambda < 1$, and $\mu, \eta \in \mathbb{R}$. Then, in terms of the familiar Gauss’s hypergeometric function $2F_1$, the generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is

\begin{equation}
J_{0,z}^{\lambda,\mu,\eta}f(z) = \frac{d}{dz} \left( \frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\xi)^{-\lambda}f(\xi)2F_1(\mu-\lambda,1-\eta;1-\lambda;1-\xi/z) d\xi \right)
\end{equation}

where $f(z)$ is analytic function in a simply-connected region of the $z$-plane containing the origin, with the order $f(z) = O(|z|^\varepsilon)$, $z \to 0$, where $\varepsilon > \max\{0,\mu-\eta\}-1$ and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

**Definition 1.3.** Under the hypotheses of Definition 1.2, the fractional derivative operator $J_{0,z}^{\lambda+m,\mu+m,\eta+m}$ of a function $f(z)$ is defined by

\begin{equation}
J_{0,z}^{\lambda+m,\mu+m,\eta+m}f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\eta}f(z).
\end{equation}

Notice that

\begin{equation}
J_{0,z}^{\lambda,\lambda,\eta}f(z) = D_\lambda^zf(z), \quad 0 \leq \lambda < 1.
\end{equation}

With the aid of the above definitions, we define a modification of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}$ by

\begin{equation}
M_{0,z}^{\lambda,\mu,\eta}f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\mu)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^\mu J_{0,z}^{\lambda,\mu,\eta}f(z)
\end{equation}
for \( f(z) \in \mathcal{A}(p) \) and \( \lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N} \). Then it is observed that \( M_{0,z}^{\lambda,\mu,\eta} f(z) \) maps \( \mathcal{A}(p) \) onto itself as follows:

\[
M_{0,z}^{\lambda,\mu,\eta} f(z) = z^p + \sum_{n=1}^{\infty} \delta_n(\lambda, \mu, \eta, p) a_{p+n} z^{p+n}
\]

where

\[
\delta_n(\lambda, \mu, \eta, p) = \frac{(p+1)_n(p+1-\mu+\eta)_n}{(p+1-\mu)_n(p+1-\lambda+\eta)_n}.
\]

It is easily verified from (1.10) that

\[
z \left( M_{0,z}^{\lambda,\mu,\eta} f(z) \right)' = (p-\mu)M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z) + \mu M_{0,z}^{\lambda,\mu,\eta} f(z).
\]

Notice that

\[
M_{0,z}^{0,0,\eta} f(z) = f(z)
\]

and

\[
M_{0,z}^{1,1,\eta} f(z) = \frac{zf'(z)}{p}.
\]

The object of this paper is to derive several subordination and superordination results for \( p \)-valent functions involving certain fractional derivative operator.

In order to prove our results we mention the following known results which will be used in the sequel.

**Lemma 1.1.** [7] Let \( \lambda, \mu, \eta \in \mathbb{R} \), such that \( \lambda \geq 0 \) and \( K > \max\{0, \mu - \eta\} - 1 \). Then

\[
J_{0,z}^{\lambda,\mu,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\mu+\eta+1)}{\Gamma(k-\mu+1)\Gamma(k-\lambda+\eta+1)} z^{k-\mu}.
\]

**Definition 1.4.** [5] Denote by \( Q \) the set of all functions \( f \) that are analytic and injective in \( \overline{U} - E(f) \), where

\[
E(f) = \{ \xi \in \partial U : \lim_{z \to \infty} f(z) = \infty \}
\]

and are such that \( f'(\xi) \neq 0 \) for \( \xi \in \partial U - E(f) \).

**Lemma 1.2.** [4] Let the function \( q \) be univalent in the open unit disk \( U \), and \( \theta \) and \( \varphi \) be analytic in a domain \( D \) containing \( q(U) \) with \( \varphi(w) \neq 0 \) when \( w \in q(U) \). Set \( Q(z) = zq'(z)\varphi(q(z)) \) and \( h(z) = \theta(q(z)) + Q(z) \). Suppose that

(a) \( Q \) is starlike univalent in \( U \), and

(b) \( \text{Re} \left( \frac{zh'(z)}{Q(z)} \right) > 0 \) for \( z \in U \).
If
\[ \theta(p(z)) + zp'(z)\varphi(p(z)) \preceq \theta(q(z)) + zq'(z)\varphi(q(z)) \]
then \( p(z) \prec q(z) \) and \( q \) is the best dominant.

Taking \( \theta(w) = \alpha w \) and \( \varphi(w) = \gamma \) in Lemma 1.6, Shanmugam et al. [8] obtained the following lemma.

**Lemma 1.3.** [8] Let \( q \) be univalent in the open unit disk \( U \) with \( q(0) = 1 \) and \( \alpha, \gamma \in \mathbb{C} \). Further assume that
\[ \Re \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\Re \left( \frac{\alpha}{\gamma} \right) \right\}. \]

If \( p(z) \) is analytic in \( U \), and
\[ \alpha p(z) + \gamma z p'(z) \preceq \alpha q(z) + \gamma z q'(z) \]
then \( p(z) \prec q(z) \) and \( q(z) \) is the best dominant.

**Lemma 1.4.** [2] Let the function \( q \) be univalent in the open unit disk \( U \), and \( \theta \) and \( \varphi \) be analytic in a domain \( D \) containing \( q(U) \) with \( \varphi(w) \neq 0 \) when \( w \in q(U) \). Suppose that
(a) \( \Re \left( \frac{\varphi'(q(z))}{\varphi(q(z))} \right) > 0 \) for \( z \in U \),
(b) \( zq'(z)\varphi(q(z)) \) is starlike univalent in \( U \).

If \( p(z) \in \mathcal{H}[q(0), 1] \cap Q \) with \( p(U) \subseteq D \), and \( \theta(p(z)) + zp'(z)\varphi(p(z)) \) is univalent in \( U \), and
\[ \theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)) \]
then \( q(z) \prec p(z) \) and \( q \) is the best subordinant.

Taking \( \theta(w) = \alpha w \) and \( \varphi(w) = \gamma \) in Lemma 1.8, Shanmugam et al. [8] obtained the following lemma.

**Lemma 1.5.** [8] Let \( q \) be univalent in the open unit disk \( U \) with \( q(0) = 1 \). Let \( \alpha, \gamma \in \mathbb{C} \) and \( \Re \left( \frac{\alpha}{\gamma} \right) > 0 \). If \( p(z) \in \mathcal{H}[q(0), 1] \cap Q \), \( \alpha p(z) + \gamma z p'(z) \) is univalent in \( U \), and
\[ \alpha q(z) + \gamma z q'(z) \prec \alpha p(z) + \gamma z p'(z) \]
then \( q(z) \prec p(z) \) and \( q(z) \) is the best subordinant.
2. Subordination and superordination for $p$-valent functions

We begin with the following result involving differential subordination between analytic functions.

**Theorem 2.1.** Let $q$ be univalent in $U$ with $q(0) = 1$, and suppose that

\[ \text{Re} \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\text{Re} \left( \frac{1}{\gamma} \right) \right\}. \tag{2.1} \]

If $f(z) \in A(p)$, and

\[ \Phi_{\lambda, \mu, \eta}(\gamma, f)(z) = \gamma \left[ (p - \mu) - (p - \mu - 1) \frac{M_{0, z}^{\lambda, \mu, \eta} f(z) M_{0, z}^{\lambda + 1, \mu + 2, \eta + 2} f(z)}{(M_{0, z}^{\lambda + 1, \mu + 1, \eta + 1} f(z))^2} \right] \]

\[ + (1 - \gamma) \frac{M_{0, z}^{\lambda, \mu, \eta} f(z)}{M_{0, z}^{\lambda + 1, \mu + 1, \eta + 1} f(z)} \tag{2.2} \]

and if $q$ satisfies the following subordination:

\[ \Phi_{\lambda, \mu, \eta}(\gamma, f)(z) \prec q(z) + \gamma zq'(z) \tag{2.3} \]

($\lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N}; \gamma \in \mathbb{C}$) then

\[ \frac{M_{0, z}^{\lambda, \mu, \eta} f(z)}{M_{0, z}^{\lambda + 1, \mu + 1, \eta + 1} f(z)} \prec q(z) \tag{2.4} \]

and $q$ is the best dominant.

**Proof.** Let the function $p(z)$ be defined by

\[ p(z) = \frac{M_{0, z}^{\lambda, \mu, \eta} f(z)}{M_{0, z}^{\lambda + 1, \mu + 1, \eta + 1} f(z)}. \]

So, by a straightforward computation, we have

\[ \frac{zp'(z)}{p(z)} = \frac{z(M_{0, z}^{\lambda, \mu, \eta} f(z))'}{M_{0, z}^{\lambda, \mu, \eta} f(z)} - \frac{z(M_{0, z}^{\lambda + 1, \mu + 1, \eta + 1} f(z))'}{M_{0, z}^{\lambda + 1, \mu + 1, \eta + 1} f(z)}. \tag{2.5} \]

Using the identity (1.12), a simple computation shows that

\[ \gamma \left[ (p - \mu) - (p - \mu - 1) \frac{M_{0, z}^{\lambda, \mu, \eta} f(z) M_{0, z}^{\lambda + 2, \mu + 2, \eta + 2} f(z)}{(M_{0, z}^{\lambda + 1, \mu + 1, \eta + 1} f(z))^2} \right] \]

\[ + (1 - \gamma) \frac{M_{0, z}^{\lambda, \mu, \eta} f(z)}{M_{0, z}^{\lambda + 1, \mu + 1, \eta + 1} f(z)} = p(z) + \gamma z p'(z). \]

The assertion (2.4) of Theorem 2.1 now follows by an application of Lemma 1.3, with $\alpha = 1$. \qed
Remark 2.1. For the choice \( q(z) = \frac{1 + Az}{1 + Bz}, -1 \leq B < A \leq 1 \), in Theorem 2.1, we get the following Corollary.

**Corollary 2.1.** Let \(-1 \leq B < A \leq 1\), and suppose that

\[
\text{Re} \left( \frac{1 - Bz}{1 + Bz} \right) > \max \left\{ 0, -\text{Re} \left( \frac{1}{\gamma} \right) \right\}.
\]

If \( f(z) \in A(p) \) and

\[
\Phi_{\lambda,\mu,\eta}(\gamma, f)(z) < \frac{1 + Az}{1 + Bz} + \frac{\gamma(A - B)z}{(1 + Bz)^2}
\]

(\( \lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N}; \gamma \in \mathbb{C} \)) where \( \Phi_{\lambda,\mu,\eta}(\gamma, f)(z) \) is as defined in (2.2), then

\[
\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} < \frac{1 + Az}{1 + Bz}
\]

and \( \frac{1 + Az}{1 + Bz} \) is the best dominant.

Next, by appealing to Lemma 1.5 of the preceding section, we prove the following.

**Theorem 2.2.** Let \( q \) be convex in \( U \) and \( \gamma \in \mathbb{C} \) with \( \text{Re} \gamma > 0 \). If \( f(z) \in A(p) \),

\[
0 \neq \frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \in \mathcal{H}[1, 1] \cap Q
\]

and \( \Phi_{\lambda,\mu,\eta}(\gamma, f)(z) \) is univalent in \( U \), then

\[
q(z) + \gamma zq'(z) < \Phi_{\lambda,\mu,\eta}(\gamma, f)(z)
\]

(\( \lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N} \)) implies

\[
q(z) < \frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}
\]

and \( q \) is the best subordinant where \( \Phi_{\lambda,\mu,\eta}(\gamma, f)(z) \) is as defined in (2.2).

**Proof.** Let the function \( p(z) \) be defined by

\[
p(z) = \frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}.
\]

Then from the assumption of Theorem 2.2, the function \( p(z) \) is analytic in \( U \) and (2.5) holds. Hence, the subordination (2.7) is equivalent to

\[
q(z) + \gamma zq'(z) < p(z) + \gamma zp'(z).
\]

The assertion (2.8) of Theorem 2.2 now follows by an application of Lemma 1.5. \( \square \)
Combining Theorem 2.1 and Theorem 2.2, we get the following sandwich theorem.

**Theorem 2.3.** Let \( q_1 \) and \( q_2 \) be convex functions in \( U \) with \( q_1(0) = q_2(0) = 1 \). Let \( \gamma \in \mathbb{C} \) with \( \text{Re}\gamma > 0 \). If \( f(z) \in A(p) \) such that

\[
\frac{M_{\lambda+1,\mu+1,\eta+1}^{\lambda,\mu,\eta}f(z)}{M_{\lambda,\mu,\eta}^{\lambda+1,\mu+1,\eta+1}f(z)} \in \mathcal{H}[1,1] \cap Q
\]

and \( \Phi_{\lambda,\mu,\eta}(\gamma, f)(z) \) is univalent in \( U \), then

\[
q_1(z) + \gamma z q_1'(z) \prec \Phi_{\lambda,\mu,\eta}(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z)
\]

\((\lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N})\) implies

\[
q_1(z) \prec \frac{M_{\lambda+1,\mu+1,\eta+1}^{\lambda,\mu,\eta}f(z)}{M_{\lambda,\mu,\eta}^{\lambda+1,\mu+1,\eta+1}f(z)} \prec q_2(z)
\]

and \( q_1 \) and \( q_2 \) are respectively the best subordinant and the best dominant where \( \Phi_{\lambda,\mu,\eta}(\gamma, f)(z) \) is as defined in (2.2).

**Remark 2.2.** For \( \lambda = \mu = 0 \) in Theorem 2.3, we get the following result.

**Corollary 2.2.** Let \( q_1 \) and \( q_2 \) be convex functions in \( U \) with \( q_1(0) = q_2(0) = 1 \). Let \( \gamma \in \mathbb{C} \) with \( \text{Re}\gamma > 0 \). If \( f(z) \in A(p) \) such that

\[
\frac{pf(z)}{zf'(z)} \in \mathcal{H}[1,1] \cap Q
\]

and let

\[
\Phi_1(\gamma, f)(z) = \gamma p \left[ 1 - \frac{f''(z)f(z)}{(f'(z))^2} \right] + p(1 - \gamma) \frac{f(z)}{zf'(z)} , \ p \in \mathbb{N}
\]

is univalent in \( U \), then

\[
q_1(z) + \gamma z q_1'(z) \prec \Phi_1(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z)
\]

implies

\[
q_1(z) \prec \frac{pf(z)}{zf'(z)} \prec q_2(z)
\]

and \( q_1 \) and \( q_2 \) are respectively the best subordinant and the best dominant.
Theorem 2.4. Let \( q \) be univalent in \( \mathbb{U} \) with \( q(0) = 1 \), and assume that (2.1) holds. Let \( f(z) \in A(p) \), and
\[
\Psi_{\lambda,\mu,\eta}(\gamma, f)(z) = [1 + \gamma(\mu - p - 1)] \frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2}{z p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} + 2 \gamma(p - \mu) \frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z p}.
\]

(2.11)
\[
- \gamma(p - \mu - 1) \frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2 M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{z p (M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z))^2}.
\]

If \( q \) satisfies the following subordination:
\[
\Psi_{\lambda,\mu,\eta}(\gamma, f)(z) < q(z) + \gamma z q'(z)
\]
(\( \lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N}; \gamma \in \mathbb{C} \)) then
\[
\frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2}{z p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} < q(z)
\]
(2.12)
and \( q \) is the best dominant.

Proof. Let the function \( p(z) \) be defined by
\[
p(z) = \left( \frac{M_{0,z}^{\lambda,\mu,\eta} f(z))^2}{z p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \right).
\]
So, by a straightforward computation, we have
\[
\frac{zp'(z)}{p(z)} = 2z(M_{0,z}^{\lambda,\mu,\eta} f(z))' - z - \frac{z(M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z))'}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}.
\]

(2.13)
Using the identity (1.12), a simple computation shows that
\[
[1 + \gamma(\mu - p - 1)] \frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2}{z p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} + 2 \gamma(p - \mu) \frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z p}.
\]

(2.14)
\[
- \gamma(p - \mu - 1) \frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2 M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{z p (M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z))^2} = p(z) + \gamma z p'(z).
\]
The assertion (2.12) of Theorem 2.4 now follows by an application of Lemma 1.3, with \( \alpha = 1 \).

Remark 2.3. For the choice \( q(z) = \frac{1 + A z}{1 + B z}, -1 \leq B < A \leq 1 \), in Theorem 2.4, we get the following result.

Corollary 2.3. Let \(-1 \leq B < A \leq 1 \), and assume that (2.6) holds. If \( f(z) \in A(p) \) and
\[
\Psi_{\lambda,\mu,\eta}(\gamma, f)(z) < \frac{1 + A z}{1 + B z} + \frac{\gamma(A - B)z}{(1 + B z)^2}
\]
\((\lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N}; \gamma \in \mathbb{C})\) where \(\Psi_{\lambda,\mu,\eta}(\gamma, f)(z)\) is as defined in (2.11), then

\[
\frac{(M_{0, z}^{\lambda, \mu, \eta} f(z))^2}{z^p M_{0, z}^{\lambda + 1, \mu + 1, \eta + 1} f(z)} < \frac{1 + Az}{1 + Bz}
\]

and \(\frac{1+A_z}{1+Bz}\) is the best dominant.

Next, by appealing to Lemma 1.5 of the preceding section, we prove the following.

**Theorem 2.5.** Let \(q\) be convex in \(U\), and \(\gamma \in \mathbb{C}\) with \(\text{Re} \gamma > 0\). If \(f(z) \in A(p)\),

\[
0 \neq \frac{(M_{0, z}^{\lambda, \mu, \eta} f(z))^2}{z^p M_{0, z}^{\lambda + 1, \mu + 1, \eta + 1} f(z)} \in \mathcal{K}[1, 1] \cap Q
\]

and \(\Psi_{\lambda,\mu,\eta}(\gamma, f)(z)\) is univalent in \(U\), then

\[
q(z) + \gamma z q'(z) < \Psi_{\lambda,\mu,\eta}(\gamma, f)(z)
\]

\((\lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N})\) implies

\[
q(z) < \frac{(M_{0, z}^{\lambda, \mu, \eta} f(z))^2}{z^p M_{0, z}^{\lambda + 1, \mu + 1, \eta + 1} f(z)}
\]

and \(q\) is the best subordinant where \(\Psi_{\lambda,\mu,\eta}(\gamma, f)(z)\) is as defined in (2.11).

**Proof.** Let the function \(p(z)\) be defined by

\[
p(z) = \frac{(M_{0, z}^{\lambda, \mu, \eta} f(z))^2}{z^p M_{0, z}^{\lambda + 1, \mu + 1, \eta + 1} f(z)}.
\]

Then from the assumption of Theorem 2.5, the function \(p(z)\) is analytic in \(U\) and (2.13) holds. Hence, the subordination (2.15) is equivalent to

\[
q(z) + \gamma z q'(z) < p(z) + \gamma z p'(z).
\]

The assertion (2.16) of Theorem 2.5 now follows by an application of Lemma 1.5. □

Combining Theorem 2.4 and Theorem 2.5, we get the following sandwich theorem.

**Theorem 2.6.** Let \(q_1\) and \(q_2\) be convex functions in \(U\) with \(q_1(0) = q_2(0) = 1\). Let \(\gamma \in \mathbb{C}\) with \(\text{Re} \gamma > 0\). If \(f(z) \in A(p)\) such that

\[
\frac{(M_{0, z}^{\lambda, \mu, \eta} f(z))^2}{z^p M_{0, z}^{\lambda + 1, \mu + 1, \eta + 1} f(z)} \in \mathcal{K}[1, 1] \cap Q
\]

and \(\Psi_{\lambda,\mu,\eta}(\gamma, f)(z)\) is univalent in \(U\), then

\[
q_1(z) + \gamma z q_1'(z) < \Psi_{\lambda,\mu,\eta}(\gamma, f)(z) < q_2(z) + \gamma z q_2'(z)
\]
(λ ≥ 0; μ < p + 1; η > max(λ, μ) − p − 1; p ∈ N) implies
\[ q_1(z) \prec \frac{(M_{0,z}^{λ,μ}f(z))^2}{z^p M_{0,z}^{λ+p,μ+p+1}f(z)} \prec q_2(z) \]
and \( q_1 \) and \( q_2 \) are respectively the best subordinant and the best dominant where
\[ \Psi_{λ,μ,η}(γ, f)(z) \]
is as defined in (2.11).

**Remark 2.4.** For \( λ = μ = 0 \) in Theorem 2.6, we get the following result.

**Theorem 2.7.** Let \( q_1 \) and \( q_2 \) be convex functions in \( U \) with \( q_1(0) = q_2(0) = 1 \). Let \( γ \in \mathbb{C} \) with \( \text{Re} γ > 0 \). If \( f(z) \in A(p) \) such that
\[ \frac{p(f(z))^2}{z^{p+1}f'(z)} \in H[1, 1] \cap Q \]
and let
\[ \Psi_1(γ, f)(z) = [1 − γ(p + 1)] \frac{p(f(z))^2}{z^{p+1}f'(z)} + 2γp \frac{f(z)}{z^p} − γp f''(z)(f(z))^2 \]
is univalent in \( U \), then
\[ q_1(z) + γzq_1'(z) \prec \Psi_1(γ, f)(z) \prec q_2(z) + γzq_2'(z) \]
implies
\[ q_1(z) \prec \frac{p(f(z))^2}{z^{p+1}f'(z)} \prec q_2(z) \]
and \( q_1 \) and \( q_2 \) are respectively the best subordinant and the best dominant.

**References**


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