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STABILITY, BOUNDEDNESS AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

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Abstract. Criteria are established for uniform asymptotic stability, boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions of certain third order nonlinear differential equations with the restoring nonlinear terms depend on t and multiplied by functions of t. By constructing a complete Lyapunov function, Lyapunov second method, the technique of Antoisewicz [8] and the limit point of Yoshizawa [22] are employed to obtain the results. Recent results on third order nonlinear differential equations and the results which have been discussed in [16] are special cases of our results.

1. Introduction

Mathematical models of most natural, scientific and industrial phenomena result in nonlinear differential equations which are not readily solvable, therefore the determination of behaviour of solutions has attracted the attention of researches over the years. In particular, the determination of stability, boundedness, periodicity and asymptotic behaviour of solutions of third order non autonomous differential equations where the restoring nonlinear terms do not depend explicitly on the independent real variable t have been investigated by many researchers. See for instance: Ademola *et al,* [6] and Ezeilo [14] worked on stability of solutions; Ademola and Arawomo [1], [2], [3], [4] and Swick [17] on stability and boundedness of solutions; Ademola *et al,* [5],

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Chukwu [9], Ezeilo [10]-[13], Mehri and Shadman [15] and Tejumola [18] on boundedness of solutions; Tunç [19] worked on asymptotic behaviour of solutions; Reissig *et al,* [16] and Yoshizawa [21]-[23] which contain the general results on the subject matter. Complete, incomplete and Yoshizawa functions were constructed and used by these authors to obtain their results.

However, the problem of uniform asymptotic stability, boundedness and asymptotic behaviour of solutions of certain third order nonlinear differential equations, where the restoring nonlinear terms depend and multiplied by functions of t, has so far remained open. The aim of this paper therefore is to tackle this problem. Motivation for this study comes from the works of Ademola and Arawomo [1], [3], [4], Mehri and Shadman [15] and Tunç [20] where uniform stability and boundedness results of third order differential equations were proved. In this paper, we shall investigate uniform asymptotic stability (when $p(t,x,y,z) = 0$), when $p(t,x,y,z) \neq 0$ boundedness and asymptotic behaviour of solutions of the nonlinear differential equation

$$\dddot{x} + f(t,x,\dot{x},\ddot{x})\ddot{x} + q(t)g(x,\dot{x}) + r(t)h(x,\dot{x},\ddot{x}) = p(t,x,\dot{x},\ddot{x}), \tag{1.1}$$

or its equivalent system

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = p(t,x,y,z) - f(t,x,y,z)z - q(t)g(x,y) - r(t)h(x,y,z), \tag{1.2}$$

where the functions f, g, h, p, q and r are continuous in their respective arguments and the derivatives $\frac{\partial}{\partial t}f(t,x,y,z) = f_t(t,x,y,z)$, $\frac{\partial}{\partial x}f(t,x,y,z) = f_x(t,x,y,z)$, $\frac{\partial}{\partial z}f(t,x,y,z) = f_z(t,x,y,z)$, $\frac{\partial}{\partial x}g(x,y) = g_x(x,y)$, $\frac{\partial}{\partial x}h(x,y,z) = h_x(x,y,z)$, $\frac{\partial}{\partial y}h(x,y,z) = h_y(x,y,z)$, $\frac{\partial}{\partial z}h(x,y,z) = h_z(x,y,z)$, $\frac{dq(t)}{dt} = \dot{q}(t)$ and $\frac{dr(t)}{dt} = \dot{r}(t)$ exist and are continuous for all t, x, y and z. Condition for existence and uniqueness will be assumed and the dots as elsewhere, stand for differentiation with respect to the independent real variable t.

2. Main Results

We have the following results for the system of first order differential equations (1.2).

Theorem 2.1. *Further to the basic assumptions on the functions f, g, h, p and r, suppose that $a, a_1, b, b_1, c, \delta_0, \delta_1$ are positive constants and for all $t \geq 0$:*

(i) $a \leq f(t,x,y,z) \leq a_1$ *for all x, y and z;*

(ii) $b \leq g(x,y)/y \leq b_1$ *for all* x *and* $y \neq 0$;

(iii) $h(0,0,0) = 0$, $\delta_0 \leq h(x,y,z)/x$ *for all* $x \neq 0, y$ *and* z;

(iv) $\delta_1 \leq r(t) \leq q(t)$, $\dot{q}(t) \leq \dot{r}(t) \leq 0$;

(v) $f_t(t,x,y,0) \leq 0$, $g_x(x,y) \leq 0$, $yf_x(t,x,y,0) \leq 0$, $h_x(x,0,0) \leq c$ *for all* x *and* $c < ab$;

(vi) $yf_z(t,x,y,z) \geq 0$, $h_y(x,y,0) \geq 0$, $h_z(x,0,z) \geq 0$ *for all* x, y *and* z;

(vii) $\int_0^\infty |p(t,x,y,z)|dt < \infty$.

Then the solutions of (1.2) *are uniformly ultimately bounded.*

Theorem 2.2. *If in addition to the assumptions of Theorem 2.1,* $g(0,0) = 0$, *then every solution* $(x(t), y(t), z(t))$ *of* (1.2) *is uniformly bounded and satisfies*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0, \quad \lim_{t\to\infty} z(t) = 0. \tag{2.1}$$

Theorem 2.3. *Under the assumptions of Theorem 2.1, any solution* $(x(t), y(t), z(t))$ *of* (1.2) *with initial condition*

$$x(0) = x_0,\ y(0) = y_0,\ z(0) = z_0 \tag{2.2}$$

satisfies

$$|x(t)| \leq D,\ |y(t)| \leq D,\ |z(t)| \leq D \tag{2.3}$$

for all $t \geq 0$, *where the constant* $D > 0$ *depends on* $a, b, c, \delta_0, \delta_1$ *as well as on* t_0, x_0, y_0, z_0 *and on the function* p *appearing in* (1.2).

If the function $p(t,x,y,z) = 0$, (1.2) becomes

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -f(t,x,y,z)z - q(t)g(x,y) - r(t)h(x,y,z) \tag{2.4}$$

and we have the following results.

Theorem 2.4. *If* $g(0,0) = 0$ *and assumptions* (i)-(vi) *of Theorem 2.1 hold, then the trivial solution of* (2.4) *is uniformly asymptotically stable.*

Corollary 2.1. *Under the hypotheses of Theorem 2.1, the solutions of* (1.2) *are ultimately bounded.*

Remark 2.1.

(i) If the function $p(t,x,y,z) \equiv p(t) \neq 0$, $p : \mathbb{R}^+ \to \mathbb{R}$, $\mathbb{R}^+ = [0,\infty)$, $\mathbb{R} = (-\infty,\infty)$, Theorem 2.1, Theorem 2.2 and Theorem 2.3 hold true for the special case

$$\dddot{x} + f(t,x,\dot{x},\ddot{x})\ddot{x} + q(t)g(x,\dot{x}) + r(t)h(x,\dot{x},\ddot{x}) = p(t).$$

(ii) Whenever $f(t,x,y,z) \equiv \psi(t)f(x,y,z)$ the hypotheses and conclusions of Theorem 2.1 and Theorem 2.4 coincide with that discussed by Ademola and Arawomo [3]. The main tool (the Lyapunov function) used in this investigation is a direct generalization of that used in [1]-[6].

(iii) The results of Mehri and Shadman [15] and Tunç [19], [20] are special cases of our results.

(iv) Whenever $g(x,y) = g(y)$ and $h(x,y,z) = h(x)$ our hypotheses and conclusion coincide with [4] except the inequality between $\dot{q}(t)$ and $\dot{r}(t)$ which is reversed here. Thus, these results include and extend [4].

3. Preliminary Lemma

The main tool used in the proofs of the results is the function $V = V(t,x,y,z)$ defined as

$$V = e^{-P_*(t)}U \tag{3.1a}$$

where

$$P_*(t) = \int_0^t |p(\mu,x,y,z)|d\mu \tag{3.1b}$$

and

$$\begin{aligned} 2U &= 2(\alpha+a)r(t)\int_0^x h(\xi,0,0)d\xi + 4q(t)\int_0^y g(x,\tau)d\tau \\ &+ 2(\alpha+a)yz + 2z^2 + 2(\alpha+a)\int_0^y \tau f(t,x,\tau,0)d\tau + \beta y^2 + b\beta x^2 \\ &+ 4r(t)yh(x,0,0) + 2a\beta xy + 2\beta xz \end{aligned} \tag{3.1c}$$

where α and β are positive fixed constants satisfying

$$b^{-1}c < \alpha < a \tag{3.1d}$$

and

$$0 < \beta < \min\left\{(ab-c)a^{-1}, (ab-c)\delta_1\gamma_0^{-1}, \frac{1}{2}(a-\alpha)\gamma_1^{-1}\right\} \tag{3.1e}$$

where

$$\gamma_0 := 1 + a + \delta_0^{-1}\delta_1^{-1}\left(q(t)\frac{g(x,y)}{y} - b\right)^2 \text{ and } \gamma_1 := 1 + \delta_0^{-1}\delta_1^{-1}\left(f(t,x,y,z) - a\right)^2,$$

$y \neq 0$, $(1+a)\delta_0\delta_1 \neq \left(q(t)\frac{g(x,y)}{y} - b\right)^2$ and $\delta_0\delta_1 \neq \left(f(t,x,y,z) - a\right)^2$. The following lemma proves beyond doubt that the function V is a Lyapunov function for the system (1.2).

Lemma 3.1. *Under the assumptions of Theorem 2.1 there exist finite constants $D_1 > 0$ and $D_2 > 0$ such that for the function V defined by (3.1), we have*

$$D_1(x^2(t) + y^2(t) + z^2(t)) \leq V(t,x,y,z) \leq D_2(x^2(t) + y^2(t) + z^2(t)) \tag{3.2a}$$

and

$$V(t,x(t),y(t),z(t)) \to +\infty \text{ as } x^2(t) + y^2(t) + z^2(t) \to \infty. \tag{3.2b}$$

In addition, there exists a positive constant D_3 such that along a solution $(x(t),y(t),z(t))$ of (1.2) we have

$$\dot{V} \leq -D_3(x^2(t) + y^2(t) + z^2(t)), \tag{3.2c}$$

for all $(t,x,y,z) \in \mathbb{R}^+ \times \mathbb{R}^3$.

Proof. If $x(t) = y(t) = z(t) = 0$ for all $t \geq 0$ in (3.1), it follows that $V(t,0,0,0) = 0$. Since $h(0,0,0) = 0$, (3.1c) can be arranged in the form

$$\begin{aligned}
2U = {} & 2b^{-1}r(t)\int_0^x \Big[(\alpha + a)b - 2h_\xi(\xi,0,0)\Big]h(\xi,0,0)d\xi + \beta y^2 \\
& + 2br(t)\left(y + b^{-1}h(x,0,0)\right)^2 + 4r(t)\int_0^y \left(\frac{q(t)}{r(t)}\frac{g(x,\tau)}{\tau} - b\right)\tau d\tau \\
& + 2\int_0^y \Big[(\alpha + a)f(t,x,\tau,0) - (\alpha^2 + a^2)\Big]\tau d\tau \\
& + \beta(b - \beta)x^2 + (\alpha y + z)^2 + (\beta x + ay + z)^2.
\end{aligned}$$

Applying the hypotheses of Theorem 2.1, we have

$$\begin{aligned}
U \geq {} & \frac{1}{2}\Big[[(\alpha + a)b - 2c]\,b^{-1}\delta_0\delta_1 + \beta(b - \beta)\Big]x^2 + \frac{1}{2}(\alpha y + z)^2 \\
& + \frac{1}{2}\left[\alpha(a - \alpha) + \beta\right]y^2 + \frac{\delta_1}{b}(\delta_0 x + by)^2 + \frac{1}{2}(\beta x + ay + z)^2.
\end{aligned} \tag{3.3}$$

In view of the inequalities in (3.1d) and (3.1e), $\alpha b - c > 0$, $ab - c > 0$, $a - \alpha > 0$ and $b - \beta > 0$. The right hand side of the estimate in (3.3) is positive definite, hence there exists a constant $\gamma_2 > 0$ such that

$$U \geq \gamma_2(x^2 + y^2 + z^2) \tag{3.4a}$$

for all $t \geq 0, x, y$ and z. By (3.1b) and condition (vii) of Theorem 2.1, there exists a constant $P_0 > 0$ such that

$$0 \leq P_*(t) \leq P_0 \tag{3.4b}$$

for all $t \geq 0$. Using estimates (3.4) in (3.1a), we obtain

$$V \geq \gamma_3(x^2 + y^2 + z^2), \tag{3.5a}$$

for all $t \geq 0, x, y, z$ where $\gamma_3 = \gamma_2 e^{-P_0} > 0$. From (3.5a), $V(t, x, y, z) = 0$ if and only if $x^2 + y^2 + z^2 = 0$, $V(t, x, y, z) > 0$ if and only if $x^2 + y^2 + z^2 \neq 0$, it follows that

$$V(t, x, y, z) \to +\infty \text{ as } x^2 + y^2 + z^2 \to \infty. \tag{3.5b}$$

Moreover, $h(0, 0, 0) = 0$ implies that $h(x, 0, 0) \leq cx$ for all $x \neq 0$, also $\dot{q}(t) \leq \dot{r}(t) \leq 0$ imply that $q(t) \leq q_0$ and $r(t) \leq r_0$ where $q_0 = q(0) > 0$ and $r_0 = r(0) > 0$ are positive constants. From these estimates, the upper bounds for the functions f and g, and the obvious inequality $2xy \leq x^2 + y^2$, Eq. (3.1c) becomes

$$U \leq \gamma_4(x^2 + y^2 + z^2) \tag{3.6a}$$

where $\gamma_4 := \frac{1}{2}\max\{\gamma_{41}, \gamma_{42}, \gamma_{43}\}$, $\gamma_{41} := (\alpha + a + 2)cr_0 + (a + b + 1)\beta$, $\gamma_{42} := (\alpha + a)(a_1 + 1) + \beta(a + 1) + 2(b_1 q_0 + cr_0)$ and $\gamma_{43} := \alpha + \beta + a + 2$. From the estimates (3.4b) and (3.6a), Eq. (3.1a) yields

$$V \leq \gamma_4(x^2 + y^2 + z^2), \tag{3.6b}$$

for all $t \geq 0, x, y$ and z. Now the derivative of the function V defined in (3.1) along a solution of (1.2) is defined as

$$\dot{V}_{(1.2)} = -e^{-P_*(t)}\Big[U|p(t, x, y, z)| - \dot{U}_{(1.2)}\Big] \tag{3.7}$$

where $P_*(t)$ and $U = U(t, x(t), y(t), z(t))$ are the functions defined by (3.1b) and (3.1c) respectively and $\dot{U}_{(1.2)}$, after simplification, is defined as

$$\begin{aligned} \dot{U}_{(1.2)} \leq{}& W_1 + W_2 + a\beta y^2 + 2\beta yz + (\beta x + (\alpha + a)y + 2z)\, p(t, x, y, z) \\ & - W_3 - W_4 - \beta\left[q(t)\frac{g(x, y)}{y} - b\right]xy - \beta[f(t, x, y) - a]xz, \end{aligned} \tag{3.8}$$

where

$$
\begin{aligned}
W_1 &:= (\alpha+a)\dot r(t)\int_0^x h(\xi,0,0)d\xi + 2\dot q(t)\int_0^y g(x,\tau)d\tau + 2\dot r(t)yh(x,0,0),\\
W_2 &:= (\alpha+a)\int_0^y \tau f_t(t,x,\tau,0)d\tau + 2q(t)y\int_0^y g_x(x,\tau)d\tau\\
&\quad + (\alpha+a)y\int_0^y \tau f_x(t,x,\tau,0)d\tau,\\
W_3 &:= r(t)\left[(\alpha+a)y+2z\right]\left[h(x,y,z)-h(x,0,0)\right]\\
&\quad + (\alpha+a)yz\left[f(t,x,y,z)-f(t,x,y,0)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
W_4 &:= \beta r(t)xh(x,y,z) + r(t)\left[(\alpha+a)\frac{q(t)}{r(t)}\frac{g(x,y)}{y} - 2h_x(x,0,0)\right]y^2\\
&\quad + \left[2f(t,x,y,z)-(\alpha+a)\right]z^2.
\end{aligned}
$$

Now rearranging the terms in W_1, we have

$$W_1 = \dot q(t)W_{11}$$

where

$$W_{11} := (\alpha+a)\frac{\dot r(t)}{\dot q(t)}\int_0^x h(\xi,0,0)d\xi + 2\int_0^y g(x,\tau)d\tau + 2\frac{\dot r(t)}{\dot q(t)}yh(x,0,0).$$

Since $\dot q(t)\le \dot r(t)$, for all $t\ge 0$, $h(0,0,0)=0$, $h(x,0,0)\ge \delta_0 x$ $(x\ne 0)$, $g(x,y)\ge by$ for all x and $y\ne 0$ and $h_x(x,0,0)\le c$ for all x it follows that

$$W_{11} \ge \frac{1}{2b}(\alpha b - c + ab - c)\delta_0 x^2 + b^{-1}(\delta_0 x + by)^2.$$

By (3.1d), $ab>c$ and $\alpha b>c$ so that W_{11} is positive semi definite for all x and y. Thus, $\dot q(t)\le 0$ for all $t\ge 0$ implies that

$$W_1 = \dot q(t)W_{11} \le 0,$$

for all $t\ge 0, x$ and y. Also, using the assumptions of Theorem 2.1, we have

$$W_2 \le 0$$

for all $t\ge 0, x$ and y.

Moreover, by the mean value theorem, we have

$$W_3 = r(t)\Big[(\alpha+a)\,y^2h_y(x,\theta_1 y,0) + 2z^2h_z(x,0,\theta_2 z)\Big] + (\alpha+a)z^2yf_z(t,x,y,\theta_3 z) \ge 0,$$

$0\le\theta_i\le 1$ $(i=1,2,3)$ for all $t\ge 0, x, y$ and z, but $W_3=0$ if $y=0=z$.

Also, by hypotheses of Theorem 2.1, we have

$$W_4 \geq \beta\delta_0\delta_1 x^2 + \delta_1\left[(\alpha + a)\,b - 2c\right]y^2 + (a - \alpha)z^2,$$

for all $t \geq 0, x, y$ and z. Using estimates W_i $(i = 1, 2, 3, 4)$ and the fact that $2yz \leq y^2 + z^2$ in (3.8), we obtain

$$\begin{aligned} \dot{U}_{(1.2)} \leq{}& \gamma_5\left(|x| + |y| + |z|\right)|p(t, x, y, z)| - W_5 - W_6 - \frac{1}{2}\beta\delta_0\delta_1 x^2 \\ &- \delta_1(\alpha b - c + ab - c - \beta\delta_1^{-1}(1 + a))y^2 - (a - \alpha - \beta)z^2, \end{aligned} \tag{3.9}$$

where $\gamma_5 = \max\{\beta, \alpha + a, 2\}$, $W_5 := \frac{1}{4}\beta\delta_0\delta_1 x^2 + \beta\left[q(t)\frac{g(x,y)}{y} - b\right]xy$ and $W_6 := \frac{1}{4}\beta\delta_0\delta_1 x^2 + \beta[f(t, x, y, z) - a]xz$. Completing the squares in the right hand sides of W_5 and W_6 estimate (3.9) becomes

$$\begin{aligned} \dot{U}_{(1.2)} \leq{}& \gamma_5\left(|x| + |y| + |z|\right)|p(t, x, y, z)| - \left[x + 2\delta_0^{-1}\delta_1^{-1}\left(q(t)\frac{g(x, y)}{y} - b\right)y\right]^2 \\ &- \left[x + 2\delta_0^{-1}\delta_1^{-1}(f(t, x, y, z) - a)z\right]^2 - \frac{1}{2}\beta\delta_0\delta_1 x^2 - \delta_1(\alpha b - c)y^2 \\ &- \frac{1}{2}(a - \alpha)z^2 - \left\{\delta_1(ab - c) - \beta\left[1 + a + \delta_0^{-1}\delta_1^{-1}\left(q(t)\frac{g(x, y)}{y} - b\right)^2\right]\right\}y^2 \\ &- \left\{\frac{1}{2}(a - \alpha) - \beta\left[1 + \delta_0^{-1}\delta_1^{-1}\left(f(t, x, y, z) - a\right)^2\right]\right\}z^2. \end{aligned}$$

From estimates (3.1d) and (3.1e), this inequality becomes

$$\dot{U}_{(1.2)} \leq \gamma_5((|x| + |y| + |z|)|p(t, x, y, z)| - \gamma_6(x^2 + y^2 + z^2), \tag{3.10}$$

where $\gamma_6 = \min\{\frac{1}{2}\beta\delta_0\delta_1, \delta_1(\alpha b - c), \frac{1}{2}(a - \alpha)\}$. Using estimates (3.4a) and (3.10) in (3.7), we have

$$\begin{aligned} \dot{V}_{(1.2)} \leq -e^{-P_*(t)}\bigg(\gamma_2(x^2 + y^2 + z^2) - \gamma_5(|x| + |y| + |z|)\bigg)|p(t, x, y, z)| \\ - \gamma_6 e^{-P_*(t)}(x^2 + y^2 + z^2). \end{aligned} \tag{3.11}$$

Now, since $(|x| + |y| + |z|)^2 \leq 3(x^2 + y^2 + z^2)$, choosing $(x^2 + y^2 + z^2)^{1/2} \geq 3^{1/2}\gamma_2^{-1}\gamma_5$ and noting hypothesis (vii) of Theorem 2.1, we have

$$\dot{V}_{(1.2)} \leq -\gamma_7(x^2 + y^2 + z^2) \tag{3.12}$$

for all $t \geq 0, x, y$ and z, where $\gamma_7 = \gamma_5 e^{-P_*(\infty)} > 0$. This completes the proof of the lemma. $\square$

4. Proof of the Main Results

Proof of Theorem 2.1. Let $(x(t), y(t), z(t))$ be any solution of (1.2). From estimates (3.5a), (3.5b), (3.6b) and (3.12), the assumptions of Theorem 10.4 pp. 42 in [23] hold, thus by Theorem 10.4 the solutions of (1.2) are uniformly ultimately bounded. This proves the theorem. □

Proof of Theorem 2.2. The proof of this theorem depends on the continuously differentiable function V defined by (3.1). Let $(x(t), y(t), z(t))$ be any solution of (1.2). From estimate (3.12), we have $\dot{V}_{(1.2)} \leq 0$ for all $(t, x, y, z) \in \mathbb{R}^+ \times \mathbb{R}^3$, thus from this inequality, estimates (3.5) and (3.6b), the solutions of (1.2) are uniformly bounded (see [23] Theorem 10.2 pp.38-39). Next, let $W(X) = \gamma_7(x^2 + y^2 + z^2)$ clearly $W(X) \geq 0$ for all $X \in \mathbb{R}^3$. Consider the set

$$\Omega := \{X = (x, y, z) \in \mathbb{R}^3 |\ W(X) = 0\}.$$

From the continuity of the function $W(X)$, the set Ω is closed and $W(X)$ is positive definite with respect to Ω and

$$\dot{V}_{(1.2)}(t, X) \leq -W(X)$$

for all $(t, X) \in \mathbb{R}^+ \times \mathbb{R}^3$. Furthermore, the system (1.2) can be recast in the form

$$\dot{X} = F(t, X) + G(t, X) \tag{4.1}$$

where $X = (x, y, z)^T$, $F(t, X) = (y, z, -f(t, x, y, z)z - q(t)g(x, y) - r(t)h(x, y, z))^T$ and $G(t, X) = (0, 0, p(t, x, y, z))^T$. From the continuity and boundedness of the functions f, g, h, q and r it follows that the function $F(t, X)$ is bounded for all t when X belong to an arbitrary compact set in $\mathbb{R}^3$. Also, in view of the hypotheses of Theorem 2.2 the function $F(t, X)$ tends to $F(X)$ as $t \to \infty$. It is easy to show that the conditions (a) and (b) of Theorem 14.2 pp. 61 in [23] hold true, thus every solution of (1.2) approaches the largest invariant set of

$$\dot{X} = F(X) \tag{4.2}$$

contained in Ω as $t \to \infty$. Since $W(X) = 0$ on Ω and by hypotheses of Theorem 2.2 $h(0, 0, 0) = 0 = g(0, 0)$, Eq. (4.2) becomes

$$(\dot{x}, \dot{y}, \dot{z})^T = (0, 0, 0)^T$$

and the solution is given by

$$(x, y, z)^T = (\gamma_8, \gamma_9, \gamma_{10})^T$$

where γ_i $(i = 8, 9, 10)$ is a constant. To remain in Ω, $\gamma_8 = \gamma_9 = \gamma_{10} = 0$. Hence, the largest semi invariant set contained in Ω as $t \to \infty$ is the set $\{(0,0,0)\}$. This completes the proof of theorem. $\square$

Proof of Theorem 2.3. Let $(x(t), y(t), z(t))$ be any solution of (1.2). We apply the strategy introduced in [8] to establish (2.3) for all $t \geq 0$. Noting that $x^2+y^2+z^2 \geq 0$, $|x| \leq 1 + x^2$, $|y| \leq 1 + y^2$ and $|z| \leq 1 + z^2$ estimate (3.11) becomes

$$\dot{V}_{(1.2)} \leq \gamma_5 e^{-P_*(t)}\Big[3 + (x^2 + y^2 + z^2)\Big]|p(t, x, y, z)|,$$

for all $t \geq 0, x, y$ and z. In view of estimates (3.4b) and (3.5a) this inequality becomes

$$\dot{V}_{1.2} - \gamma_3^{-1}\gamma_5|p(t, x, y, z)|V \leq 3\gamma_5|p(t, x, y, z)|.$$

Solving this first order differential inequality using integrating factor $\exp[-\gamma_3^{-1}\gamma_5 P_*(t)]$, we obtain

$$V(t, x, y, z) \leq \gamma_6 \tag{4.3}$$

for all $t \geq 0, x, y$ and z, where $\gamma_6 := [V(t_0, x_0, y_0, z_0) + 3\gamma_5 P_0]\exp[\gamma_3^{-1}\gamma_5 P_0]$. Now from the inequalities in (3.5a) and (4.3), estimate (2.3) follows immediately with $\gamma_3^{-1}\gamma_6 \equiv D$. This completes the proof of the theorem. $\square$

Proof of Theorem 2.4. The usual limit point argument as contained in [22] is used to show that if Lemma 3.1 holds, then $U(t) = U(t, x, y, z) \to 0$ as $t \to \infty$. Setting $p(t, x, y, z) = 0$ in (3.1b), the function V in (3.1a) coincide with U defined in (3.1c). From estimate (3.4a), we find that $U(t, x, y, z) = 0$ if and only if $x^2 + y^2 + z^2 = 0$, $U(t, x, y, z) > 0$ if and only if $x^2 + y^2 + z^2 \neq 0$ and $U(t, x, y, z) \to +\infty$ if and only if $x^2 + y^2 + z^2 \to \infty$. The remaining of this proof follows the strategy indicated in [6]. $\square$

Example 4.1. A particular case of equation (1.1), is given by the following third order nonlinear ordinary differential equation

$$\begin{aligned}
&\dddot{x} + 4\ddot{x} + \frac{\ddot{x}}{1 + t^2 + |x\dot{x}| + \exp(1/(1 + |\dot{x}\ddot{x}|))}\\
&+ \left(\frac{1}{4} + \frac{1}{1+t^2}\right)\left(3\dot{x} + \frac{\dot{x}}{1 + |x\dot{x}|}\right)\\
&+ \left(\frac{1}{4} + \frac{1}{2+t^2}\right)\left(5x + \frac{x}{1 + \exp(1/(1 + |x|\dot{x}^2|\ddot{x}|))}\right)\\
&= \frac{1}{1 + t^2 + x^2 + \dot{x}^2 + \ddot{x}^2}.
\end{aligned} \tag{4.4}$$

Equation (4.4) is equivalent to

$$\begin{aligned}
&\dot{x} = y,\ \dot{y} = z,\\
&\dot{z} = \frac{1}{1+t^2+x^2+y^2+z^2} - 4z\\
&\quad - \frac{z}{1+t^2+|xy|+\exp(1/(1+|yz|))} - \left(\frac{1}{4}+\frac{1}{1+t^2}\right)\left(3y+\frac{y}{1+|xy|}\right)\\
&\quad - \left(\frac{1}{4}+\frac{1}{2+t^2}\right)\left(5x+\frac{x}{1+\exp(1/(1+|x|y^2|z|))}\right).
\end{aligned} \tag{4.5}$$

Comparing (1.2) and (4.5), we have the following:

(a) The function $f(t,x,y,z)$ is defined as

$$4 - \frac{1}{1+t^2+|xy|+\exp(1/(1+|yz|))}. \tag{4.6}$$

(i) Now, since $0 \le \dfrac{1}{1+t^2+|xy|+\exp(1/(1+|yz|))} \le 1$ for all $t \ge 0, x, y$ and z, it follows that

$$4 \le f(t,x,y,z) \le 5$$

for all $t \ge 0, x, y$ and z, where $a = 4 > 0$ and $a_1 = 5 > 0$.

(ii) From (4.6), we have

$$f_t(t,x,y,z) = \frac{-2t}{[1+t^2+|xy|+\exp(1/(1+|yz|))]^2} \le 0$$

for all $t \ge 0, x, y$ and z.

(iii) The derivative of the function in (4.6) with respect to $x > 0$ is

$$f_x(t,x,y,z) = \frac{-|y|}{[1+t^2+|xy|+\exp(1/(1+|yz|))]^2}$$

and

$$yf_x(t,x,y,z) = \frac{-y^2}{[1+t^2+|xy|+\exp(1/(1+|yz|))]^2} \le 0$$

for all $t \ge 0, x, y$ and z.

(iv) Also, if $z > 0$

$$yf_z(t,x,y,z) = \frac{y^2}{[1+t^2+|xy|+\exp(1/(1+|yz|))]^2} \ge 0$$

for all $t \ge 0, x, y$ and z.

(b) The function

$$g(x,y) = 3y + \frac{y}{1+|xy|}. \tag{4.7}$$

(i) Clearly $g(0,0) = 0$.

(ii) Since $0 \leq \dfrac{1}{1+|xy|} \leq 1$ for all x and y, it follows that

$$3 \leq \frac{g(x,y)}{y} \leq 4$$

for all x and $y \neq 0$, where $b = 3 > 0$ and $b_1 = 4 > 0$.

(iii) For $x > 0$, we have

$$g_x(x,y) = \frac{-y^2}{[1+|xy|]^2} \leq 0$$

for all x and y.

(c) The function $h(x,y,z)$ is defined as

$$5x + \frac{x}{1+\exp(1/(1+|x|y^2|z|))}$$

from which we have the following estimates:

(i) Clearly, $h(0,0,0) = 0$.

(ii) Since $0 \leq \dfrac{1}{1+\exp(1/(1+|x|y^2|z|))}$ for all x, y and z, it follows that

$$\frac{h(x,y,z)}{x} \geq 5$$

for all $x \neq 0, y$ and z, where $\delta_0 = 5 > 0$.

(iii) Furthermore,

$$h_x(x,y,z) - 5 = \frac{[1+|x|y^2|z|]^2[1+e^u] + |x|y^2|z|e^u}{[1+|x|y^2|z|]^2[1+e^u]^2}$$

where $u = \dfrac{1}{1+|x|y^2|z|}$. Since

$$\frac{[1+|x|y^2|z|]^2[1+e^u] + |x|y^2|z|e^u}{[1+|x|y^2|z|]^2[1+e^u]^2} \leq 1$$

for all x when $y = 0$, it follows that

$$h_x(x,0,0) \leq 6$$

for all x where $c = 6 > 0$ and $ab > c$ implies that $2 > 1$.

(iv) Also,

$$h_y(x,y,z) = \frac{2x^2|yz|e^u}{[1+e^u]^2} \geq 0$$

for all x, y and z.

(v) Similarly for $z > 0$, we have

$$h_z(x, y, z) = \frac{x^2 y^2 e^u}{[1 + e^u]^2 [1 + |x| y^2 |z|]^2} \geq 0$$

for all x, y and z.

(d) The functions $q(t)$ and $r(t)$ are

$$\frac{1}{4} + \frac{1}{1 + t^2} \text{ and } \frac{1}{4} + \frac{1}{2 + t^2}.$$

respectively.

(i) Since $\frac{1}{1 + t^2} \geq \frac{1}{2 + t^2} \geq 0$ for all $t \geq 0$, we have

$$\frac{1}{4} \leq r(t) \leq q(t)$$

for all $t \geq 0$, where $\delta_1 = \frac{1}{4} > 0$.

(ii) Differentiating the functions $q(t)$ and $r(t)$ with respect to t, we obtain

$$\dot{q}(t) = \frac{-2t}{(1 + t^2)^2} \text{ and } \dot{r}(t) = \frac{-2t}{(2 + t^2)^2}.$$

Now, since $\frac{-2t}{(1 + t^2)^2} \leq \frac{-2t}{(2 + t^2)^2} \leq 0$ for all $t \geq 0$, we obtain

$$\dot{q}(t) \leq \dot{r}(t) \leq 0$$

for all $t \geq 0$.

(e) It is not difficult to show that the function $p(t, x, y, z)$ satisfies the integral inequality

$$\int_0^\infty \left| \frac{1}{1 + t^2 + x^2 + y^2 + z^2} \right| dt < \infty$$

for all $t \geq 0, x, y$ and z. Hence, all the assumptions of the theorems are satisfied and the conclusions follow.

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