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AN EVEN EASIER PROOF OF MONOTONICITY OF STOLARSKY MEANS

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Abstract. A new, very simple proof of monotonicity of Stolarsky means is presented in the paper.

1. Introduction

The Extended Mean Values (many authors call them Stolarsky means) appeared in the literature for the first time in Stolarsky's paper [14]. They are defined for positive x, y and real r, s by

$$E(r,s) = E(r,s;x,y) = \begin{cases} \left(\dfrac{s}{r}\cdot\dfrac{x^r-y^r}{x^s-y^s}\right)^{1/(r-s)}, & sr(s-r)(x-y)\neq 0, \\ \left(\dfrac{1}{r}\cdot\dfrac{x^r-y^r}{\log x-\log y}\right)^{1/r}, & r(x-y)\neq 0,\ s=0, \\ e^{-1/r}\left(x^{x^r}/y^{y^r}\right)^{1/(x^r-y^r)}, & r=s,\ r(x-y)\neq 0, \\ \sqrt{xy}, & r=s=0,\ x-y\neq 0, \\ x, & x=y \end{cases} \tag{1.1}$$

and attract attention of many mathematicians, because they contain well-known means such as power means $M_r(x,y) = E(2r,r;x,y)$, the logarithmic mean $L(x,y) = E(1,0;x,y) = \frac{x-y}{\log x-\log y}$ and Heronian means $H_n(x,y) = E(1+1/n, 1/n; x,y)$. Monotonicity of E is one of the first subjects to deal with.

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Theorem 1.1. *For arbitrary $r, s \in \mathbb{R}$ $E(r, s; x, y)$ strictly increases in x and y. For fixed $x \neq y$ $E(r, s; x, y)$ strictly increases in r and s.*

The first part was discovered by Stolarsky ([14]), Leach and Sholander ([5]) were the first to show the second part. Over last 30 years mathematicians have made serious progress investigating properties of Stolarsky means, arriving at many new proofs.

Observe that in case $r \neq s$ we can represent E as

$$E(r, s; x, y) = \left(\frac{L(x^r, y^r)}{L(x^s, y^s)}\right)^{1/(r-s)} = \left(\frac{\int_y^x t^{r-1} dt}{\int_y^x t^{s-1} dt}\right)^{1/(r-s)}, \tag{1.2}$$

which leads to

$$\log E(r, r; x, y) = \lim_{s \to r} \frac{\log L(x^r, y^r) - \log L(x^s, y^s)}{r - s} = \frac{d}{dr} \log L(x^r, y^r), \tag{1.3}$$

and this, in turn, yields

$$\log E(r, s; x, y) = \frac{\log L(x^r, y^r) - \log L(x^s, y^s)}{r - s} = \frac{\int_s^r \log E(t, t; x, y) dt}{r - s}. \tag{1.4}$$

Thus, we see that means $E(t, t; x, y)$ (known as *identric means*) are of particular importance.

The easiest way to prove monotonicity in r, s is by the following, elementary characterisation of convex functions (see e.g. [1, p. 26]).

Lemma 1.1. *A function f is convex if and only if the divided difference*

$$g(x, y) = \frac{f(x) - f(y)}{x - y} \quad \textit{for} \quad x \neq y$$

increases in both variables.

As a matter of fact, all known direct proofs use some version of the above lemma. Stolarsky ([14]) proved monotonicity in r, s using (1.4) and showing by straightforward differentiation that $\log E(t, t; x, y)$ is strictly increasing in t.
In 1983 Leach and Sholander ([6]) and later Páles and Czinder ([7], [4]) found sufficient and necessary conditions for $E(a, b; x, y) \leq E(c, d; x, y)$ to hold for every x, y. Monotonicity in r, s is a simple corollary of their results.
Proofs in [11], [17] and [9] base on convexity (demonstrated by more or less complicated calculations) of the function $\varphi(t) = \log \frac{x^t - y^t}{t}$ and Lemma 1.1.

The first proof of monotonicity in x, y (by Leach and Sholander [5]) uses the fact that if $a+b+c=0$ and $a\alpha+b\beta+c\gamma=0$, then for $0<\theta\neq 1$

$$\operatorname{sgn}(a\theta^\alpha+b\theta^\beta+c\theta^\gamma)=-\operatorname{sgn}(abc).$$

Recently, several proofs of monotonicity in x, y have been published using differentiation of specially selected functions ([9], [3]). In [15] the author represented $E(r,s;x,1)$ as the composition of four strictly monotone functions: x^s, $\frac{x^{r/s}-1}{x-1}$, $\frac{s}{r}x$, $x^{1/(r-s)}$ and applied the Chain Rule.

Qi used the integral representation (1.2) introducing generalized weighted mean values

$$M_{p,f}(r,s;x,y)=\left(\frac{\int_x^y p(t)f^r(t)dt}{\int_x^y p(t)f^s(t)dt}\right)^{1/(r-s)},$$

where p is nonnegative and f is a positive continuous function. They are always increasing in r, s and monotone in x, y if and only if f is monotone. Several proofs of this fact can be found in [8], [12], [10], [2] and [15].

Another approach to the subject can be found in [16], where Yang considered means of the form $\mathcal{H}_f(r,s;x,y)=\left(\frac{f(x^r,y^r)}{f(x^s,y^s)}\right)^{1/(r-s)}$ and found sufficient conditions for f to guarantee monotonicity of $\mathcal{H}_f$. Needless to say, the logarithmic mean satisfies these conditions.

In [13] the author gave another version of proof in r, s. Let us complete his paper by another simple proof of monotonicity in x, y. In our proof we shall need the integral representation (1.4) and Lemma 1.1.

2. Proof of monotonicity in x, y

Proof. Note that since for fixed $r\neq 0$ the function e^{-tr} is strictly convex, $\frac{e^{-rt}-1}{t}$ strictly increases by Lemma 1.1, hence so does $\frac{t}{1-e^{-rt}}$. On the other hand

$$\log E(r,r;e^t,1)=-\frac{1}{r}+\frac{te^{rt}}{e^{rt}-1}=-\frac{1}{r}+\frac{t}{1-e^{-rt}},$$

and homogeneity of Stolarsky means in x, y implies that identric means are strictly increasing (case $r=0$ is trivial). Employing (1.4), we see that this property extends to all parameters r, s. $\square$

3. Remark on Gini means

Simić's paper deals also with Gini means

$$G(r,s;x,y)=\left(\frac{x^r+y^r}{x^s+y^s}\right)^{1/(r-s)}.$$

For completness let us note that the Gini means in general are not monotone in x,y. Indeed, if r,s are positive, then $G(r,s;0^+,1)=1=G(r,s;1,1)$.

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