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COMMON FIXED POINT THEOREM IN PROBABILISTIC METRIC SPACE

B. D. PANT *, SUNNY CHAUHAN † AND QAMAR ALAM †

Abstract. The notion of weakly compatible maps introduced by Singh and Jain [A fixed point theorem in Menger space through weak compatibility, J. Math. Anal. Appl. **301** (2) (2005), 439–448] in Menger spaces. In this paper, we prove a common fixed point theorem for weakly compatible maps in Menger space without appeal to continuity.

1. Introduction

In 1942, K. Menger [4] introduced the notion of a probabilistic metric space (shortly, PM-space). The idea thus appears that, instead of a single positive number, we should associate a distribution function with the point pairs. Thus the concept of a PM-space corresponds to the situations when we do not know the distance between the points, i.e., the distance between the points is inexact. Rather than a single real number, we know only probabilities of possible values of this distance. Such a probabilistic generalization of a metric space appears to be well adapted for the investigation of physical quantities and physiological threshold. The study of these spaces was expanded rapidly with the pioneering works of Schweizer and Sklar [9]. It is also of fundamental importance in probabilistic functional analysis, nonlinear analysis and applications [1].

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In 1972, Sehgal and Bharucha-Reid [10] initiated the study of contraction mappings on PM-spaces. In 1991, Mishra [5] extended the notion of compatibility (introduced by Jungck [2] in metric spaces) and Singh and Jain [12] extended the notion of weak compatibility (introduced by Jungck and Rhoades [3] in metric spaces) to PM-spaces. It is worth to mention that each pair of compatible self-maps is weakly compatible but the converse need not be true.

In this paper we establish a common fixed point theorem for weakly compatible maps in Menger space without appeal to continuity. For terminology and notions used in this paper, we refer to [6], [7], [8] and [11].

2. Preliminaries

Definition 2.1. [9] A triangular norm T (shortly t-norm) is a binary operation on the unit interval $[0,1]$ such that for all $a,b,c,d \in [0,1]$ and the following conditions are satisfied:

(a) $\mathrm{T}(a,1) = a$, for all $a \in [0,1]$;
(b) $\mathrm{T}(a,b) = \mathrm{T}(b,a)$;
(c) $\mathrm{T}(a,b) \leq \mathrm{T}(c,d)$ for $a \leq c$, $b \leq d$;
(d) $\mathrm{T}(\mathrm{T}(a,b),c) = \mathrm{T}(a,\mathrm{T}(b,c))$.

The following are the four basic t-norms:

(a) The minimum t-norm: $\mathrm{T}_M(a,b) = \min\{a,b\}$.
(b) The product t-norm: $\mathrm{T}_P(a,b) = a.b$.
(c) The Lukasiewicz t-norm: $\mathrm{T}_L(a,b) = \max\{a+b-1,0\}$.
(d) The weakest t-norm, the drastic product:

$$\mathrm{T}_D(a,b) = \begin{cases} \min\{a,b\}, & \text{if } \max\{a,b\} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

In respect of above mentioned t-norms, we have the following ordering:

$$\mathrm{T}_D(a,b) < \mathrm{T}_L(a,b) < \mathrm{T}_P(a,b) < \mathrm{T}_M(a,b).$$

Throughout this paper, T stands for an arbitrary continuous t-norm.

Definition 2.2. [9] A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in R} F(t) = 0$ and $\sup_{t\in R} F(t) = 1$.

We shall denote by $\Im$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \le 0; \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.3. [9] The ordered pair $(X, \mathfrak{F})$ is called a PM-space if X is a nonempty set of elements and $\mathfrak{F}$ is a mapping from $X \times X$ to $\Im$, the collection of all distribution functions. The value of $\mathfrak{F}$ at $(x, y) \in X \times X$ is represented by $F_{x,y}$. The functions $F_{x,y}$ are assumed to satisfy the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

(a) $F_{x,y}(t) = 1$ for all $t > 0$ if and only $x = y$;
(b) $F_{x,y}(0) = 0$;
(c) $F_{x,y}(t) = F_{y,x}(t)$;
(d) if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t + s) = 1$.

The ordered triple $(X, \mathfrak{F}, \mathrm{T})$ is called a Menger space if $(X, \mathfrak{F})$ is a PM-space, T is a t-norm and the following inequality holds:

(e) $F_{x,y}(t + s) \ge \mathrm{T}(F_{x,z}(t), F_{z,y}(s))$, for all $x, y, z \in X$ and $t, s > 0$.

Every metric space (X, d) can always be realized as a PM-space by considering $\mathfrak{F} : X \times X \to \Im$ defined by $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$.

Definition 2.4. [9] Let $(X, \mathfrak{F}, \mathrm{T})$ be a Menger space with continuous t-norm.

(a) A sequence $\{x_n\}$ in X is said to be converge to a point x in X if and only if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer N such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \ge N$.
(b) A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer N such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n, m \ge N$.
(c) A Menger space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.5. [5] Self maps A and B of a Menger space $(X, \mathfrak{F}, \mathrm{T})$ are said to be compatible if and only if $F_{ABx_n,BAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to x$ for some x in X as $n \to \infty$.

Definition 2.6. [12] Self maps A and B of a Menger space $(X, \mathfrak{F}, \mathrm{T})$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $Ax = Bx$ for some $x \in X$, then $ABx = BAx$.

Remark 2.1. [12] Two compatible self-maps are weakly compatible, but the converse is not true. Therefore the concept of weak compatibility is more general than that of compatibility.

The following is an example of pair of self maps in a Menger space which are weakly compatible but not compatible.

Example 2.1. Let (X, d) be a metric space defined by $d(x, y) = |x-y|$, where $X = [0, 6]$ and $(X, \mathcal{F}, \mathrm{T})$ be the induced Menger space with $F_{x,y}(t) = \frac{t}{t+d(x,y)}$, for all $t > 0$. We define self maps A and B as follows:

$$A(x) = \begin{cases} 6-x, & \text{if } 0 \leq x < 3; \\ 6, & \text{if } 3 \leq x \leq 6. \end{cases}$$

$$B(x) = \begin{cases} x, & \text{if } 0 \leq x < 3; \\ 6, & \text{if } 3 \leq x \leq 6. \end{cases}$$

Taking $x_n = 3 - \frac{1}{n}$. We get $Ax_n = 3 + \frac{1}{n}$, $Bx_n = 3 - \frac{1}{n}$. Thus, $Ax_n \to 3$, $Bx_n \to 3$. Hence $x = 3$. Further $ABx_n = 3 + \frac{1}{n}$, $BAx_n = 6$. Now; $\lim_{n\to\infty} F_{ABx_n, BAx_n}(t) = \lim_{n\to\infty} F_{3+\frac{1}{n},6}(t) = \frac{t}{t+3} < 1$, for all $t > 0$. Hence (A, B) is not compatible.

Coincidence points of A and B are in $[3, 6]$. Now for any $x \in [3, 6], Ax = Bx = 6$ and $AB(x) = A(6) = 6 = B(6) = BA(x)$. Thus (A, B) is weakly compatible.

Lemma 2.1. [6], [11] *Let $(X, \mathcal{F}, \mathrm{T})$ be a Menger PM-space and define $E_{\lambda,F} : X^2 \to \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,F}(x, y) = \inf\{t > 0 : F_{x,y}(t) > 1 - \lambda\},$$

for each $\lambda \in (0, 1)$ and $x, y \in X$. Then we have

(a) *For any $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$ such that*

$$E_{\mu,F}(x_1, x_n) \leq E_{\lambda,F}(x_1, x_2) + \ldots + E_{\lambda,F}(x_{n-1}, x_n),$$

for any $x_1, \ldots, x_n \in X$.

(b) *The sequence $\{x_n\}_{n\in N}$ is convergent with respect to Menger probabilistic metric $\mathcal{F}$ if and only if $E_{\lambda,F}(x_n, x) \to 0$. Also the sequence $\{x_n\}$ is a Cauchy sequence with respect to Menger probabilistic metric $\mathcal{F}$ if and only if it is a Cauchy sequence with $E_{\lambda,F}$.*

Lemma 2.2. [5] *Let $(X, \mathcal{F}, \mathrm{T})$ be a Menger space. If there exists a constant $k \in (0, 1)$ such that*

$$F_{x,y}(kt) \geq F_{x,y}(t),$$

for all $x, y \in X$ and $t > 0$ then $x = y$.

3. Result

Theorem 3.1. *Let A, B and L be self maps on a complete Menger space $(X, \mathcal{F}, \mathrm{T})$ and satisfy the following conditions:*

(a) $L(X) \subseteq AB(X)$;
(b) $AB(X)$ *or* $L(X)$ *is a closed subset of* X;
(c) $AB = BA$ *and either* $LB = BL$ *or* $LA = AL$;
(d) *There exists* $k \in (0,1)$ *such that*

$$F_{Lx,Ly}(kt) \geq F_{ABx,ABy}(t),$$

for all $x, y \in X$ *and* $t > 0$;
(e) *The pair* (L, AB) *is weakly compatible.*

In addition assume that

$$E_{\lambda,F}(x,y) = \inf\{t > 0 : F_{x,y}(t) > 1 - \lambda\},$$

for each $\lambda \in (0,1)$ *and* $x, y \in X$.

Then A, B *and* L *have a unique common fixed point in* X.

Proof. By (a) since $L(X) \subseteq AB(X)$ for any point $x_0 \in X$ there exists a point x_1 in X such that $Lx_0 = ABx_1$. By induction, we can define a sequence $\{x_n\}_{n \in N}$ such that $Lx_n = ABx_{n-1}$. By induction again,

$$\begin{aligned} F_{ABx_n,ABx_{n+1}}(t) = F_{Lx_{n-1},Lx_n}(t) &\geq F_{ABx_{n-1},ABx_n}\left(\frac{t}{k}\right) \\ &\geq \ldots \geq F_{ABx_0,ABx_1}\left(\frac{t}{k^n}\right), \end{aligned}$$

for $n = 1, 2, 3, \ldots$, which implies that

$$\begin{aligned} E_{\lambda,F}(ABx_n, ABx_{n+1}) &= \inf\{t > 0 : F_{ABx_n,ABx_{n+1}}(t) > 1 - \lambda\} \\ &\leq \inf\{t > 0 : F_{ABx_0,ABx_1}(\frac{t}{k^n}) > 1 - \lambda\} \\ &= k^n \inf\{t > 0 : F_{ABx_0,ABx_1}(t) > 1 - \lambda\} \\ &= k^n E_{\lambda,F}(ABx_0, ABx_1), \end{aligned}$$

for every $\lambda \in (0,1)$.

Now, we show that $\{ABx_n\}$ is a Cauchy sequence. For every $\mu \in (0,1)$, there exists $\gamma \in (0,1)$ such that, for $m \geq n$,

$$\begin{aligned} E_{\mu,F}(ABx_n, ABx_m) &\leq E_{\gamma,F}(ABx_{m-1}, ABx_m) + E_{\gamma,F}(ABx_{m-2}, ABx_{m-1}) + \dots \\ &\quad + E_{\gamma,F}(ABx_n, ABx_{n+1}) \\ &\leq E_{\gamma,F}(ABx_0, ABx_1) \sum_{i=n}^{m-1} k^i \to 0, \end{aligned}$$

as $m, n \to \infty$. Hence by Lemma 2.1, $\{ABx_n\}$ is a Cauchy sequence in X. Since X is complete, then there exists $z \in X$ such that $\lim_{n\to\infty} ABx_n = z$. So $Lx_n = ABx_{n-1}$ tends to z.

($\star$) Suppose that $AB(X)$ is a closed subset of X then there exists $v \in X$ such that $ABv = z$.

($\star$) Suppose that $L(X)$ is a closed subset of X. We have $z \in L(X) \subseteq AB(X)$ and so there exists $v \in X$ such that $z = ABv$.

Now we prove that $z = Lv = ABv$. Putting $x = v$ and $y = x_{2n+1}$ in (d), we get

$$F_{Lv, Lx_{2n+1}}(kt) \geq F_{ABv, ABx_{2n+1}}(t),$$

as $n \to \infty$, we have

$$F_{Lv,z}(kt) \geq F_{z,z}(t),$$

Hence, $F_{Lv,z}(kt) = 1$, for all $t > 0$, that is $Lv = z$. Therefore, $z = Lv = ABv$.

Also the pair (L, AB) is weakly compatible then $L(AB(v)) = AB(L(v))$, that is $Lz = ABz$. Putting $x = z$ and $y = x_{2n+1}$ in (d), we get

$$F_{Lz, Lx_{2n+1}}(kt) \geq F_{ABz, ABx_{2n+1}}(t),$$

as $n \to \infty$, we get

$$F_{ABz,z}(kt) \geq F_{ABz,z}(t).$$

Thus by Lemma 2.2, $ABz = z$. Therefore, $z = Lz = ABz$. Now we prove that $z = Bz$. Putting $x = x_{2n}$ and $y = Bz$ in (d), we get

$$F_{Lx_{2n}, L(Bz)}(kt) \geq F_{ABx_{2n}, AB(Bz)}(t),$$

as $n \to \infty$, we have

$$F_{z,Bz}(kt) \geq F_{z,Bz}(t).$$

Thus by Lemma 2.2, $z = Bz$. Therefore, $z = Az = Bz = Lz$. That is z is the common fixed point of the self maps A, B and L.

Uniqueness: Let w $(w \neq z)$ be another common fixed point of the self maps A, B and L. Putting $x = z$ and $y = w$ in (d), we get

$$F_{Lz,Lw}(kt) \geq F_{ABz,ABw}(t),$$

$$F_{z,w}(kt) \geq F_{z,w}(t).$$

Thus by Lemma 2.2, $z = w$ and so the uniqueness of the common fixed point. □

Taking $B = I$ (identity map) in Theorem 3.1, we get the following result:

Corollary 3.1. *Let A and L be self maps on a complete Menger space $(X, \mathcal{F}, \mathrm{T})$ and satisfy the following conditions:*

(a) $L(X) \subseteq A(X)$*;*

(b) $A(X)$ *or* $L(X)$ *is a closed subset of* X*;*

(c) *There exists* $k \in (0, 1)$ *such that*

$$F_{Lx,Ly}(kt) \geq F_{Ax,Ay}(t)$$

for all $x, y \in X$ *and* $t > 0$*;*

(d) *The pair* (L, A) *is weakly compatible.*

In addition assume that

$$E_{\lambda,F}(x, y) = \inf\{t > 0 : F_{x,y}(t) > 1 - \lambda\},$$

for each $\lambda \in (0, 1)$ *and* $x, y \in X$*.*
Then A and L have a unique common fixed point in X.

The following example illustrates Corollary 3.1.

Example 3.1. Let $X = [0, 20]$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$ define

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

for all $x, y \in X$. Clearly $(X, \mathcal{F}, \mathrm{T})$ is a complete Menger space, where T is a continuous t-norm. Now, we define A and $L : X \to X$ by

$$A(x) = \begin{cases} 0, & \text{if } x = 0; \\ 10 - x, & \text{if } 0 < x \leq 10; \\ x - 7, & \text{if } 10 < x \leq 20. \end{cases} \qquad L(x) = \begin{cases} 0, & \text{if } x = 0; \\ 3, & \text{if } 0 < x \leq 20. \end{cases}$$

Then A and L satisfy all the conditions of Corollary 3.1 for some $k \in (0, 1)$ and have a unique common fixed point $0 \in X$. It may be noted in this example that the mappings L and A commute at coincidence point $0 \in X$. So L and A are weakly compatible maps. To see the pair (L, A) is not compatible, let us consider a sequence $\{x_n\}$ defined as $x_n = 10 + \frac{1}{n}$, $n \geq 1$, then $x_n \to 10$ as $n \to \infty$. Then $\lim_{n\to\infty} Lx_n = 3, \lim_{n\to\infty} Ax_n = 3$ but $\lim_{n\to\infty} F_{LAx_n,ALx_n}(t) = \frac{t}{t+|3-7|} \neq 1$. Thus the pair (L, A) is not compatible. Also, all the mappings involved in this example are discontinuous even at the common fixed point $x = 0$.

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∗ Government Degree College,
Champawat, 262523, (Uttarakhand),
India.

† R. H. Government Postgraduate College,
Kashipur, 244713, U. S. Nagar, (Uttarakhand),
India.

Address for correspondence:
Sunny Chauhan
Near Nehru Training Centre,
H. No. 274, Nai Basti B-14, Bijnor, 246701, Uttar Pradesh,
India.
E-mail address: sun.gkv@gmail.com