THE SPECTRUM OF NEIGHBORHOOD CORONA OF GRAPHS

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ABSTRACT. Given two graphs G_1 with vertices $\{v_1, v_2, \ldots, v_n\}$ and G_2, the neighbourhood corona, $G_1 \bigstar G_2$ is the graph obtained by taking n copies of G_2 and for each i, making all vertices in the ith copy of G_2 adjacent with the neighbours of v_i, $i = 1, 2, \ldots, n$. In this paper a complete description of the spectrum and eigenvectors of $G_1 \bigstar G_2$ is given when G_2 is regular, thus adding to the class of graphs whose spectrum is completely known.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of G is an $n \times n$ matrix denoted by $A(G) = [a_{ij}]$ and is defined as $a_{ij} = 1$ if v_i and v_j are adjacent in G, 0 otherwise. The spectrum of G is defined as $sp(G) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ where $\lambda_1 \geq \lambda_2 \geq \lambda_n$ are the eigenvalues $A(G)$. The Laplacian matrix of G, denoted by $L(G)$ is defined as $D(G) - A(G)$ where $D(G)$ is the diagonal degree matrix of G. The Laplacian spectrum of G is defined as

$$S(G) = \{\theta_1, \theta_2, \ldots, \theta_n\}$$

where $0 = \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_n$ are the eigenvalues of $L(G)$. We refer $\lambda_1(G)$ and $\theta_n(G)$ the spectral radius and Laplacian spectral radius, respectively. A plethora of papers have been available on works related to spectrum and Laplacian spectrum of a graph. See [2, 6, 7] and the references cited therein.

The corona of two graphs is defined in [4] and there have been some results on the corona of two graphs [3]. The complete information about the spectrum of the corona

of two graphs G, H in terms of the spectrum of G, H are given in [1]. A new variance of corona is defined in [5] and discussed its spectrum and the number of spanning trees.

In this paper we define another variation of corona of two graphs and discuss its spectrum, thus adding to the class of graphs whose spectrum is completely known. The discussion in subsequent sections are based upon the following definition.

Definition 1.1. Let G_1 and G_2 be two graphs on n_1 and n_2 vertices, m_1 and m_2 edges respectively. Then the *neighborhood corona*, $G_1 \bigstar G_2$ is the graph obtained by taking n_1 copies of G_2 and for each i, making all vertices in the ith copy of G_2 adjacent with the neighbors of v_i, $i = 1, 2, \ldots, n$.

The neighborhood corona $G_1 \bigstar G_2$ of G_1 and G_2 has $n_1 + n_1 n_2$ vertices and $m_1(2n_2 + 1) + n_1 m_2$ edges and when $G_2 = K_1$, $G_1 \bigstar G_2$ is the splitting graph defined in [8]. Note that in general this operation is not commutative.

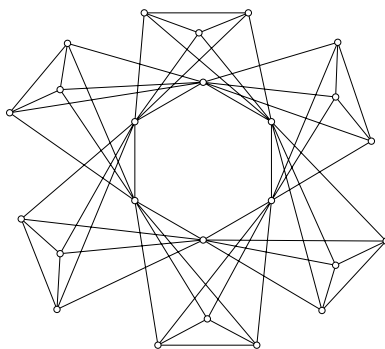*Example* 1.1. The following figure illustrates Definition 1.1.

FIGURE 1. $C_6 \bigstar K_2$

Throughout this paper we consider only simple graphs. In this paper, we give a complete description of the eigenvalues and the corresponding eigenvectors of the adjacency matrix of $G_1 \bigstar G_2$ when G_2 is regular.

Let $A = (a_{ij}), B$ be matrices. Then the Kronecker product of A and B is defined in [2] as the partition matrix $(a_{ij}B)$ and is denoted by $A \otimes B$. The row vector of size n with all entries equal to one is denoted by j_n and the identity matrix of order n is denoted by I_n.

2. THE SPECTRUM OF $G_1 \bigstar G_2$

In this section we obtain the spectrum of $G_1 \bigstar G_2$ when G_2 is regular.

Theorem 2.1. *Let G_1 be a graph with spectrum $\{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n_1}\}$ and G_2, a $t-$regular graph with spectrum $\{\eta_1 = t \geq \eta_2 \geq \ldots \geq \eta_{n_2}\}$. Then the spectrum $\sigma(G)$ of $G_1 \bigstar G_2$ consists of the numbers*

$$\frac{\mu_i + t + \sqrt{(\mu_i - t)^2 + 4n_2\mu_i^2}}{2}, \quad \frac{\mu_i + t - \sqrt{(\mu_i - t)^2 + 4n_2\mu_i^2}}{2}, \quad i = 1, 2, \ldots, n_1,$$

each with multiplicity one together with η_j, $j = 2, 3, \ldots, n_2$, each with multiplicity n_1.

Proof. Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, the vertices in the ith copy of G_2 be $\{u_1^i, u_2^i, \ldots, u_{n_2}^i\}$ and $W_j = \{u_j^1, u_j^2, \ldots, u_j^{n_1}\}$. Then with the vertex partition $V \cup W_1 \cup W_2 \cup \cdots \cup W_{n_2}$, the adjacency matrix of $G_1 \bigstar G_2$ can be written in the block form

$$\begin{bmatrix} A(G_1) & j_{n_2} \otimes A(G_1) \\ (j_{n_2} \otimes A(G_1))^T & A(G_2) \otimes I_{n_1} \end{bmatrix}.$$

Let X_i be an eigenvector of $A(G_1)$ corresponding to the eigenvalue μ_i, $i = 1, 2, \ldots, n_1$.

Let $\delta_i = \dfrac{\mu_i + t + \sqrt{(\mu_i - t)^2 + 4n_2\mu_i^2}}{2}$ and $\hat{\delta}_i = \dfrac{\mu_i + t - \sqrt{(\mu_i - t)^2 + 4n_2\mu_i^2}}{2}$. Now we shall consider two cases.

Case 1. $\mu_i \neq 0$.

In this case it is easy to observe that $\delta_i \neq t$ and $\hat{\delta}_i \neq -t$.

Now $\Phi_i = \begin{bmatrix} \dfrac{\delta_i - t}{\mu_i} X_i \\ j_{n_2} \otimes X_i \end{bmatrix}$ is an eigenvector of $A(G_1 \bigstar G_2)$ with an eigenvalue δ_i. This is because

$$A(G_1 \bigstar G_2) \cdot \Phi_i = \begin{bmatrix} A(G_1) & j_{n_2} \otimes A(G_1) \\ (j_{n_2} \otimes A(G_1))^T & A(G_2) \otimes I_{n_1} \end{bmatrix} \begin{bmatrix} \dfrac{\delta_i - t}{\mu_i} X_i \\ j_{n_2} \otimes X_i \end{bmatrix}$$

$$= \begin{bmatrix} \left(\dfrac{\delta_i - t}{\mu_i}\mu_i + n_2\mu_i\right) X_i \\ \left(\dfrac{\delta_i - t}{\mu_i}\mu_i + t\right) j_{n_2} \otimes X_i \end{bmatrix}$$

$$= \begin{bmatrix} ((\delta_i - t) + n_2\mu_i) X_i \\ ((\delta_i - t) + t) j_{n_2} \otimes X_i \end{bmatrix}$$

$$= \delta_i \begin{bmatrix} \dfrac{\delta_i - t}{\mu_i} X_i \\ j_{n_2} \otimes X_i \end{bmatrix} = \delta_i \Phi_i.$$

Similarly, it can be proved that $\hat{\Phi}_i = \begin{bmatrix} \dfrac{\hat{\delta}_i - t}{\mu_i} X_i \\ j_{n_2} \otimes X_i \end{bmatrix}$ is an eigenvector of $A\left(G_1 \bigstar G_2\right)$ with an eigenvalue $\hat{\delta}_i$.

Case 2. $\mu_i = 0$.

In this case observe that $\delta_i = t$ and $\hat{\delta}_i = 0$.

Let $\Phi_i^0 = \begin{bmatrix} 0 \\ j_{n_2} \otimes X_i \end{bmatrix}$. Now Φ_i^0 is an eigenvector of $A\left(G_1 \bigstar G_2\right)$ with an eigenvalue

$t = \dfrac{\mu_i + t + \sqrt{\left(\mu_i - t\right)^2 + 4n_2\mu_i^2}}{2}$, $\mu_i = 0$. This is because

$$A\left(G_1 \bigstar G_2\right) \cdot \Phi_i^0 = \begin{bmatrix} A\left(G_1\right) & j_{n_2} \otimes A\left(G_1\right) \\ \left(j_{n_2} \otimes A\left(G_1\right)\right)^T & A\left(G_2\right) \otimes I_{n_1} \end{bmatrix} \begin{bmatrix} 0 \\ j_{n_2} \otimes X_i \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ t \cdot j_{n_2} \otimes X_i \end{bmatrix}$$

$$= t \begin{bmatrix} 0 \\ j_{n_2} \otimes X_i \end{bmatrix} = t \cdot \Phi_i^0.$$

Similarly, it can be proved that $\hat{\Phi}_i^0 = \begin{bmatrix} X_i \\ 0 \end{bmatrix}$ is an eigenvector of $A\left(G_1 \bigstar G_2\right)$ with

an eigenvalue $0 = \dfrac{\mu_i + t - \sqrt{\left(\mu_i - t\right)^2 + 4n_2\mu_i^2}}{2}$, $\mu_i = 0$.

Thus we get $2n_1$ eigenvalues $\dfrac{\mu_i + t \pm \sqrt{\left(\mu_i - t\right)^2 + 4n_2\mu_i^2}}{2}$ of $A\left(G_1 \bigstar G_2\right)$ with eigenvectors described earlier.

Now let Y_j be an eigenvector of $A(G_2)$ with an eigenvalue η_j, $i = 2, 3, \ldots, n_2$. Since G_2 is regular, by Perron-Frobenius theory that Y_j is orthogonal to the all one vector. Let $e_{n_1}^i$ be an $n_1 \times 1$ column vector with i^{th} entry equal to one and all other entries equal to zero. Now $\Psi_j^i = \begin{bmatrix} 0 \\ Y_j \otimes e_{n_1}^i \end{bmatrix}$ is an eigenvector of $A\left(G_1 \bigstar G_2\right)$ with an eigenvalue η_j for each $i = 1, 2, \ldots, n_1$. This is because

$$A\left(G_1 \bigstar G_2\right) \cdot \Psi_j^i = \begin{bmatrix} A\left(G_1\right) & j_{n_2} \otimes A\left(G_1\right) \\ \left(j_{n_2} \otimes A\left(G_1\right)\right)^T & A\left(G_2\right) \otimes I_{n_1} \end{bmatrix} \begin{bmatrix} 0 \\ Y_j \otimes e_{n_1}^i \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \eta_j \cdot Y_j \otimes e_{n_1}^i \end{bmatrix}$$

$$= \eta_j \Psi_j^i.$$

The multiplicity of η_j follows from the observation that Ψ_j^i is an eigenvector for each $i = 1, 2, \ldots, n_1$. Thus we have listed $n_1 n_2 + n_2$ eigenvectors for $A\left(G_1 \bigstar G_2\right)$ and

by the very construction, they are all linearly independent and as $A(G_1 \bigstar G_2)$ has a basis consisting of linearly independent eigenvectors, the theorem follows. $\qquad \square$

3. THE LAPLACIAN SPECTRUM OF $G_1 \bigstar G_2$

In this section we obtain the Laplacian spectrum of $G_1 \bigstar G_2$.

Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, the vertices in the ith copy of G_2 be $\{u_1^i, u_2^i, \ldots, u_{n_2}^i\}$ and $W_j = \{u_j^1, u_j^2, \ldots, u_j^{n_1}\}$. Then with the vertex partition $V \bigcup W_1 \bigcup W_2 \bigcup \cdots \bigcup W_{n_2}$, we have the following degree relations in $G_1 \bigstar G_2$:

$$\deg_{G_1 \bigstar G_2} v_i = \deg_{G_1} v_i (n_2 + 1)$$

$$\deg_{G_1 \bigstar G_2} u_j^i = \deg_{G_2} u_j + \deg_{G_1} v_i, \quad i = 1, 2, \ldots, n_1; \quad j = 1, 2, \ldots, n_2.$$

Thus the degree diagonal matrix $D(G_1 \bigstar G_2)$ can be written in the block form

$$\begin{bmatrix} (n_2 + 1) D(G_1) & j_{n_2} \otimes 0 \\ (j_{n_2} \otimes 0)^T & D(G_2) \otimes I_{n_1} + I_{n_2} \otimes D(G_1) \end{bmatrix}.$$

Let $L(G_1)$ and $L(G_2)$ respectively denote the Laplacian matrices of G_1 and G_2, then the Laplacian matrix, $L(G_1 \bigstar G_2)$ of $G_1 \bigstar G_2$ is the block matrix

$$\begin{bmatrix} (n_2 + 1) L(G_1) & -j_{n_2} \otimes A(G_1) \\ (-j_{n_2} \otimes A(G_1))^T & L(G_2) \otimes I_{n_1} + I_{n_2} \otimes D(G_1) \end{bmatrix}.$$

In what follows, we give a complete description of the Laplacian eigenvalues and eigenvectors of $G_1 \bigstar G_2$ for an r-regular graph G_1 and any graph G_2. Note that if G is an r-regular graph with adjacency spectrum $\{r = \lambda_1 \geq \lambda_2, \ldots, \lambda_n\}$, then its Laplacian spectrum is $\{0 = \theta_1 \leq \theta_2 = r - \lambda_2 \leq \ldots \leq \theta_n = r - \lambda_n\}$.

Theorem 3.1. *Let G_1 be an r-regular graph with adjacency spectrum $\{r = \mu_1 \geq \mu_2, \ldots, \mu_n\}$, $S(G_1) = \{\theta_1, \theta_2, \ldots, \theta_{n_1}\}$ and G_2 be a graph with $S(G_2) = \{\tau_1, \tau_2, \ldots, \tau_{n_2}\}$. Then the Laplacian spectrum of $G_1 \bigstar G_2$ consists of the numbers*

$$\frac{(n_2 + 1) r + \theta_i + \sqrt{((n_2 + 1) r + \theta_i)^2 + 4\theta_i (n_2\theta_i - r(2n_2 + 1))}}{2},$$

$$\frac{(n_2 + 1) r + \theta_i - \sqrt{((n_2 + 1) r + \theta_i)^2 + 4\theta_i (n_2\theta_i - r(2n_2 + 1))}}{2}, \quad i = 1, 2, \ldots, n_1$$

together with $(\tau_j + r)$ of multiplicity n_1 for each $j = 2, 3, \ldots, n_2$.

Proof. Let X_i be an eigenvector with an eigenvalue μ_i of $A(G_1)$. As G_1 is r-regular, X_i is an eigenvector of $L(G_1)$ with an eigenvalue $\theta_i = r - \mu_i$.

Let

$$\sigma_i = \frac{(n_2 + 1)\, r + \theta_i + \sqrt{((n_2 + 1)\, r + \theta_i)^2 + 4\theta_i\, (n_2\theta_i - r\,(2n_2 + 1))}}{2}$$

and

$$\hat{\sigma}_i = \frac{(n_2 + 1)\, r + \theta_i - \sqrt{((n_2 + 1)\, r + \theta_i)^2 + 4\theta_i\, (n_2\theta_i - r\,(2n_2 + 1))}}{2}.$$

Now we shall consider two cases.

Case 1. $\mu_i \neq 0$ *or in other words* $\theta_i \neq r$.

We observe that $\sigma_i, \hat{\sigma}_i = r$ only when $\theta_i = r$ and hence $\sigma_i, \hat{\sigma}_i \neq r$ when $\mu_i \neq 0$. Now in this case, $\Theta_i = \begin{bmatrix} \dfrac{\sigma_i - r}{\theta_i - r} X_i \\ j_{n_2 \times 1} \otimes X_i \end{bmatrix}$ is an eigenvector of $L(G_1 \bigstar G_2)$ with an eigenvalue σ_i.

This is because

$$L(G_1 \bigstar G_2)\Theta_i$$

$$= \begin{bmatrix} (n_2 + 1)\, L\,(G_1) & -j_{n_2} \otimes A\,(G_1) \\ (-j_{n_2} \otimes A\,(G_1))^T & L\,(G_2) \otimes I_{n_1} + rI_{n_2} \otimes I_{n_1} \end{bmatrix} \begin{bmatrix} \dfrac{\sigma_i - r}{\theta_i - r} X_i \\ j_{n_2 \times 1} \otimes X_i \end{bmatrix}$$

$$= \begin{bmatrix} \left((n_2 + 1)\dfrac{\sigma_i - r}{\theta_i - r}\,(r - \mu_i) - n_2\mu_i\right) X_i \\ \left(-\left(\dfrac{\sigma_i - r}{\theta_i - r}\right) + r\right) X_i \end{bmatrix} \quad \because L\,(G_2) \perp\ j_{n_2 \times 1}$$

$$= \sigma_i \begin{bmatrix} \dfrac{\sigma_i - r}{\theta_i - r} X_i \\ j_{n_2 \times 1} \otimes X_i \end{bmatrix}$$

Similarly, it can be proved that $\hat{\Theta}_i = \begin{bmatrix} \dfrac{\hat{\sigma}_i - r}{\theta_i - r} X_i \\ j_{n_2 \times 1} \otimes X_i \end{bmatrix}$ is an eigenvector of $L(G_1 \bigstar G_2)$ with an eigenvalue $\hat{\sigma}_i$.

Case 2. $\mu_i = 0$ *or in other words* $\theta_i = r$

In this case by applying similar arguments we used in the above discussion it can be showed that $\begin{bmatrix} X_i \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ j_{n_2 \times 1} \otimes X_i \end{bmatrix}$ respectively are the eigenvectors with eigenvalues

$$\sigma_i = \frac{(n_2 + 1)\, r + \theta_i + \sqrt{((n_2 + 1)\, r + \theta_i)^2 + 4\theta_i\, (n_2\theta_i - r\,(2n_2 + 1))}}{2} = (n_2 + 1)r$$

and

$$\hat{\sigma}_i = \frac{(n_2 + 1)\, r + \theta_i - \sqrt{((n_2 + 1)\, r + \theta_i)^2 + 4\theta_i\, (n_2 \theta_i - r\, (2n_2 + 1))}}{2} = r.$$

Now let $Z_1, Z_2, \ldots, Z_{n_2}$ be the set of eigenvectors of $L(G_2)$. Then, as shown in the proof of Theorem 2.1, it can be shown that $\Omega_j^i = \begin{bmatrix} 0 \\ Z_j \otimes e_{n_1}^i \end{bmatrix}$ is an eigenvector of $L(G_1 \bigstar G_2)$ with an eigenvalue $\tau_j + r$ of multiplicity n_1 for $j = 2, 3, \ldots, n_2$. Thus we get $n_1 n_2 + n_2$ eigenvectors for $L(G_1 \bigstar G_2)$ and by the very construction, they are all linearly independent. Thus, the theorem is proved. $\qquad\square$

3.1. **An application.** As an application of the above results, in this section we obtain an expression for the number of spanning trees of $G_1 \bigstar G_2$ for an r-regular graph G_1 with $S(G_1) = \{0 = \theta_1, \theta_2, \ldots, \theta_{n_1}\}$ and G_2 with $S(G_2) = \{\tau_1, \tau_2, \ldots, \tau_{n_2}\}$. From the Matrix-Tree theorem [2] the number of spanning trees of G_1 is

$$t\,(G_1) = \frac{\theta_2 \theta_3 \ldots \theta_{n_1}}{n_1}.$$

Theorem 3.2. *Let G_1 be a connected r-regular graph with $S(G_1) = \{0 = \theta_1, \theta_2, \ldots, \theta_{n_1}\}$ and G_2 be any arbitrary graph with $S(G_2) = \{\tau_1, \tau_2, \ldots, \tau_{n_2}\}$. Let $t(G_1)$ be the number of spanning trees of G_1. Then*

$$t\,(G_1 \bigstar G_2) = rt\,(G_1) \prod_{i=2}^{n_1} ((2n_2 + 1)\, r - n_2 \theta_i) \prod_{j=2}^{n_2} (\tau_j + r)^{n_1}.$$

Proof. Using the notations in Theorem 3.1, we have $\sigma_i \hat{\sigma}_i = \theta_i\,((2n_2 + 1)\, r - n_2 \theta_i)$, $\sigma_1 = (n_2 + 1)\, r;\, \hat{\sigma}_1 = 0$ for $i = 2, 3, \ldots, n_1$. Thus

$$t\,(G_1 \bigstar G_2) = \frac{(n_2 + 1)\, r \times \prod\limits_{i=2}^{n_1} \theta_i\,((2n_2 + 1)\, r - n_2 \theta_i) \times \prod\limits_{j=2}^{n_2} (\tau_j + r)^{n_1}}{n_1\,(n_2 + 1)}$$

$$= rt\,(G_1) \prod_{i=2}^{n_1} ((2n_2 + 1)\, r - n_2 \theta_i) \prod_{j=2}^{n_2} (\tau_j + r)^{n_1}.$$

$\qquad\square$

Corollary 3.1.

$$t(K_{n_1} \bigstar K_{n_2})$$
$$= n_1^{n_1 - 2}\, (n_1 + n_2 - 1)^{n_1(n_2 - 1)}\, (n_1 - 1)\,((2n_2 + 1)\,(n_1 - 1) - n_1 n_2)^{n_1 - 1}.$$

Proof. The proof follows from the fact that $\theta_i = n_1$, $i = 2, 3, \ldots, n_1$; $\tau_j = n_2$, $j = 2, 3, \ldots, n_2$; $r = n_1 - 1$ and $t(K_{n_1}) = n_1^{n_1 - 2}$. $\qquad\square$

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