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WEAK AND STRONG CONVERGENCE OF COMMON FIXED POINTS FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we give necessary and sufficient condition for strong convergence of three-step iteration process with errors for approximating common fixed point for asymptotically quasi-nonexpansive type mappings and also prove weak convergence of three-step iteration process with errors for approximating common fixed point for said mappings in Banach spaces. The results presented in this paper extend and improve the corresponding results [1, 5, 6, 10, 11, 14] and many others.

1. Introduction

It is well known that the concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] who proved that every asymptotically nonexpansive self-mapping of nonempty closed bounded and convex subset of a uniformly convex Banach space has fixed point. In 1973, Petryshyn and Williamson [9] gave necessary and sufficient conditions for Mann iterative sequence ([7]) to converge to fixed points of quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [1] extended the results of Petryshyn and Williamson [9] and gave necessary and sufficient conditions for Ishikawa [3] iterative sequence to converge to fixed points for quasi-nonexpansive mappings.

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Liu [6] extended results of [1, 9] and gave necessary and sufficient conditions for Ishikawa iterative sequence with errors to converge to fixed point of asymptotically quasi-nonexpansive mappings. In 2002, Xu and Noor [14] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings.

In 2006, Quan [10] studied some necessary and sufficient conditions for three-step Ishikawa iterative sequences with error terms for uniformly quasi-Lipschitzian mappings to converge to fixed points. The results presented in [10] extend and improve the corresponding results of Liu [5, 6], Xu and Noor [14] and many others.

The purpose of this paper is to investigate some necessary and sufficient conditions for three-step iterative sequences with error terms for asymptotically quasi-nonexpansive type mappings to converge to common fixed points in Banach spaces. The results obtained in this paper extend and improve the corresponding results of [1, 5, 6, 10, 11, 14] and many others.

2. Preliminaries

Definition 2.1. Let E be a real Banach space, C be a nonempty convex subset of E and $F(T)$ denotes the set of fixed points of T. Let $T\colon C \to C$ be a mapping:

(1) T is said to be asymptotically nonexpansive if there exists a sequence $\{u_n\} \subset [0, \infty)$ with $u_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \;\leq\; (1+u_n)\,\|x-y\|\,, \tag{2.1}$$

for all $x, y \in C$ and $n \geq 1$.

(2) T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\} \subset [0, \infty)$ with $u_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - p\| \;\leq\; (1+u_n)\,\|x-p\|\,, \tag{2.2}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

(3) T is said to be asymptotically nonexpansive type [4], if

$$\limsup_{n\to\infty}\left\{\sup_{x,y\in C}\Big(\|T^n x - T^n y\| - \|x-y\|\Big)\right\} \;\leq\; 0. \tag{2.3}$$

(4) T is said to be asymptotically quasi-nonexpansive type [11], if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty}\left\{\sup_{x\in C,\ p\in F(T)}\Big(\|T^n x - p\| - \|x-p\|\Big)\right\} \;\leq\; 0. \tag{2.4}$$

Remark 2.1. It is easy to see that if $F(T)$ is nonempty, then asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive mapping and asymptotically nonexpansive type mapping all are the special cases of asymptotically quasi-nonexpansive type mappings.

Definition 2.2. Let E be a normed linear space, C be a nonempty convex subset of E, and $T\colon C \to C$ a given mapping. Then for arbitrary $x_1 \in C$, the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ defined by

$$\begin{aligned} z_n &= (1-\gamma_n-\nu_n)x_n + \gamma_n T_3^n x_n + \nu_n u_n, && n \geq 1, \\ y_n &= (1-\beta_n-\mu_n)x_n + \beta_n T_2^n z_n + \mu_n v_n, && n \geq 1, \\ x_{n+1} &= (1-\alpha_n-\lambda_n)x_n + \alpha_n T_1^n y_n + \lambda_n w_n, && n \geq 1, \end{aligned} \tag{2.5}$$

where $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$ are appropriate sequences in $[0,1]$, is called the three-step iterative sequence with error terms of T.

We note that the usual modified Ishikawa and Mann iterations are special cases of the above three-step iterative scheme. If $\gamma_n = \nu_n = 0$ and $T_1 = T_2 = T$, then (2.5) reduces to the usual modified Ishikawa iterative scheme with errors,

$$\begin{aligned} y_n &= (1-\beta_n-\mu_n)x_n + \beta_n T^n x_n + \mu_n v_n, && n \geq 1, \\ x_{n+1} &= (1-\alpha_n-\lambda_n)x_n + \alpha_n T^n y_n + \lambda_n w_n, && n \geq 1, \end{aligned} \tag{2.6}$$

where $\{v_n\}$, $\{w_n\}$ are bounded sequences in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$ are appropriate sequences in $[0,1]$.

If $\beta_n = \mu_n = 0$, then (2.6) reduces to the usual modified Mann iterative scheme with errors,

$$x_1 \in C,$$

$$x_{n+1} = (1-\alpha_n-\lambda_n)x_n + \alpha_n T^n x_n + \lambda_n w_n, \quad n \geq 1, \tag{2.7}$$

where $\{w_n\}$ is a bounded sequence in C and $\{\alpha_n\}$, $\{\lambda_n\}$ are appropriate sequences in $[0,1]$.

We say that a Banach space E satisfies the *Opial's condition* [8] if for each sequence $\{x_n\}$ in E weakly convergent to a point x and for all $y \neq x$

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

The examples of Banach spaces which satisfy the Opial's condition are Hilbert spaces and all $L^p[0,2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition [8].

Let K be a nonempty closed convex subset of a Banach space E. Then $I - T$ is demiclosed at zero if, for any sequence $\{x_n\}$ in K, condition $x_n \to x$ weakly and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ implies $(I-T)x = 0$.

In the sequel, we shall need the following lemma:

Lemma 2.1. (see [13]) *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we prove weak and strong convergence theorems of three-step iteration scheme with errors for asymptotically quasi-nonexpansive type mappings in a real Banach space.

Theorem 3.1. *Let E be a real Banach space, C be a nonempty closed convex subset of E. Let $T_i\colon C \to C$, $(i = 1,2,3)$ be uniformly L-Lipschitzian asymptotically quasi-nonexpansive type mappings with $F = \cap_{i=1}^{3} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (2.5) with the restrictions $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Then $\{x_n\}$ converges to a common fixed point of the mappings T_1, T_2 and T_3 if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0,$$

where $d(x,F) = \inf_{p\in F} d(x,p)$.

Proof. The necessity is obvious. Thus we only prove the sufficiency. Let $p \in F$. It follows from (2.4) that

$$\limsup_{n\to\infty} \left\{ \sup_{x\in C,\ p\in F} \Big(\|T^n x - p\| - \|x - p\| \Big) \right\} \leq 0.$$

This implies that for any given $\varepsilon > 0$, there exists a positive integer n_0 such that for $n \geq n_0$ we have

$$\sup_{x\in C,\ p\in F} \Big(\|T^n x - p\| - \|x - p\| \Big) < \varepsilon. \tag{3.1}$$

Since $\{x_n\}, \{y_n\}, \{z_n\} \subset C$, we have

$$
\begin{aligned}
&\|T_3^n x_n - p\| - \|x_n - p\| < \varepsilon, \quad \forall p \in F, \quad \forall n \geq n_0 \\
&\|T_2^n z_n - p\| - \|z_n - p\| < \varepsilon, \quad \forall p \in F, \quad \forall n \geq n_0 \\
&\|T_1^n y_n - p\| - \|y_n - p\| < \varepsilon, \quad \forall p \in F, \quad \forall n \geq n_0.
\end{aligned} \tag{3.2}
$$

Thus for each $n \geq 0$ and for any $p \in F$, using (2.5), and (3.2), we have

$$
\begin{aligned}
\|z_n - p\| &= \|(1-\gamma_n-\nu_n)x_n + \gamma_n T_3^n x_n + \nu_n u_n - p\| \\
&\leq (1-\gamma_n-\nu_n)\|x_n - p\| + \gamma_n \|T_3^n x_n - p\| \\
&\quad + \nu_n \|u_n - p\| \\
&\leq (1-\gamma_n-\nu_n)\|x_n - p\| + \gamma_n[\|x_n - p\| + \varepsilon] \\
&\quad + \nu_n \|u_n - p\| \\
&\leq \|x_n - p\| + \gamma_n \varepsilon + \nu_n \|u_n - p\|,
\end{aligned} \tag{3.3}
$$

using (2.5) and (3.3), we have

$$
\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n-\mu_n)x_n + \beta_n T_2^n z_n + \mu_n v_n - p\| \\
&\leq (1-\beta_n-\mu_n)\|x_n - p\| + \beta_n \|T_2^n z_n - p\| \\
&\quad + \mu_n \|v_n - p\| \\
&\leq (1-\beta_n-\mu_n)\|x_n - p\| + \beta_n[\|z_n - p\| + \varepsilon] \\
&\quad + \mu_n \|v_n - p\| \\
&\leq (1-\beta_n-\mu_n)\|x_n - p\| + \beta_n \|z_n - p\| + \beta_n \varepsilon \\
&\quad + \mu_n \|v_n - p\| \\
&\leq (1-\beta_n-\mu_n)\|x_n - p\| \\
&\quad + \beta_n \Big[\|x_n - p\| + \gamma_n \varepsilon + \nu_n \|u_n - p\| \Big] \\
&\quad + \beta_n \varepsilon + \mu_n \|v_n - p\| \\
&\leq \|x_n - p\| + \beta_n \varepsilon (1+\gamma_n) + \beta_n \nu_n \|u_n - p\| \\
&\quad + \mu_n \|v_n - p\| \\
&\leq \|x_n - p\| + 2\beta_n \varepsilon + \nu_n \|u_n - p\| + \mu_n \|v_n - p\|,
\end{aligned} \tag{3.4}
$$

again using (2.5) and (3.4), we have

$$\begin{aligned}
\|x_{n+1}-p\| &= \|(1-\alpha_n-\lambda_n)x_n+\alpha_n T_1^n y_n+\lambda_n w_n-p\| \\
&\leq (1-\alpha_n-\lambda_n)\|x_n-p\|+\alpha_n\|T_1^n y_n-p\| \\
&\quad +\lambda_n\|w_n-p\| \\
&\leq (1-\alpha_n-\lambda_n)\|x_n-p\|+\alpha_n[\|y_n-p\|+\varepsilon] \\
&\quad +\lambda_n\|w_n-p\| \\
&\leq (1-\alpha_n-\lambda_n)\|x_n-p\| \\
&\quad +\alpha_n\Big[\|x_n-p\|+2\beta_n\varepsilon+\nu_n\|u_n-p\|+\mu_n\|v_n-p\|\Big] \\
&\quad +\alpha_n\varepsilon+\lambda_n\|w_n-p\| \\
&\leq \|x_n-p\|+\alpha_n\varepsilon(1+2\beta_n)+\alpha_n\nu_n\|u_n-p\| \\
&\quad +\alpha_n\mu_n\|v_n-p\|+\lambda_n\|w_n-p\| \\
&\leq \|x_n-p\|+3\alpha_n\varepsilon+\alpha_n\|u_n-p\|+\alpha_n\|v_n-p\| \\
&\quad +\lambda_n\|w_n-p\| \\
&= \|x_n-p\|+H_n, \qquad (3.5)
\end{aligned}$$

where

$$H_n = 3\alpha_n\varepsilon+\alpha_n\|u_n-p\|+\alpha_n\|v_n-p\|+\lambda_n\|w_n-p\|.$$

Since by hypothesis $\sum_{n=1}^{\infty}\alpha_n<\infty$, $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded in C, it follows that $\sum_{n=1}^{\infty}H_n<\infty$. From (3.5) and Lemma 2.1, we have $\lim_{n\to\infty}\|x_n-p\|$ exists. Also from (3.5), we obtain

$$d(x_{n+1},F) \leq d(x_n,F)+H_n, \qquad (3.6)$$

for all $n\geq 1$. From Lemma 2.1 and (3.6), we know that $\lim_{n\to\infty}d(x_n,F)$ exists. Since $\liminf_{n\to\infty}d(x_n,F)=0$, we have that $\lim_{n\to\infty}d(x_n,F)=0$.

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. Let $\lim_{n\to\infty}\|x_n-p\|=r$. For any given $\varepsilon>0$, since $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded in C, there exists a constant $K>0$, such that for all $n\geq 1$, $\|x_n-p\|\leq K$, $\|u_n-p\|\leq K$, $\|v_n-p\|\leq K$, $\|w_n-p\|\leq K$ hold. Because $\sum_{n=1}^{\infty}\alpha_n<\infty$, $\sum_{n=1}^{\infty}\lambda_n<\infty$, there exists a positive integer n_1, such that for all $n\geq n_1$, we have

$$\sum_{i=n}^{\infty}\alpha_i<\frac{\varepsilon}{2(4K+3\varepsilon)}, \qquad \sum_{i=n}^{\infty}\lambda_i<\frac{\varepsilon}{4K}. \qquad (3.7)$$

From (2.5) and (3.4), it can be obtained that

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|\alpha_n(T_1^n y_n - x_n) + \lambda_n(w_n - x_n)\| \\
&\le \alpha_n \|T_1^n y_n - x_n\| + \lambda_n \|w_n - x_n\| \\
&\le \alpha_n \|T_1^n y_n - p\| + \alpha_n \|x_n - p\| + \lambda_n \|w_n - p\| + \lambda_n \|x_n - p\| \\
&\le \alpha_n[\|y_n - p\| + \varepsilon] + \alpha_n \|x_n - p\| + \lambda_n \|w_n - p\| + \lambda_n \|x_n - p\| \\
&\le \alpha_n\Big[\|x_n - p\| + 2\beta_n\varepsilon + \nu_n \|u_n - p\| + \mu_n \|v_n - p\|\Big] + \alpha_n\varepsilon \\
&\quad + \alpha_n \|x_n - p\| + \lambda_n \|w_n - p\| + \lambda_n \|x_n - p\| \\
&\le (2\alpha_n + \lambda_n)\|x_n - p\| + \alpha_n\varepsilon(1 + 2\beta_n) + \alpha_n\nu_n \|u_n - p\| \\
&\quad + \alpha_n\mu_n \|v_n - p\| + \lambda_n \|w_n - p\| \\
&\le 2\alpha_n \|x_n - p\| + 3\alpha_n\varepsilon + \lambda_n \|x_n - p\| + \alpha_n \|u_n - p\| \\
&\quad + \alpha_n \|v_n - p\| + \lambda_n \|w_n - p\| \\
&\le 4\alpha_n K + 3\alpha_n\varepsilon + 2\lambda_n K \\
&= (4K + 3\varepsilon)\alpha_n + 2\lambda_n K. \qquad (3.8)
\end{aligned}$$

Thus for all $n \ge n_1$ and $m \ge 1$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\le \sum_{i=1}^{m} \|x_{n+i} - x_{n+i-1}\| \\
&\le (4K + 3\varepsilon)\sum_{i=1}^{m} \alpha_{n+i-1} + 2K\sum_{i=1}^{m} \lambda_{n+i-1} \\
&< (4K + 3\varepsilon).\frac{\varepsilon}{2(4K + 3\varepsilon)} + 2K.\frac{\varepsilon}{4K} = \varepsilon. \qquad (3.9)
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Thus $\lim_{n\to\infty} x_n$ exists. Let $\lim_{n\to\infty} x_n = p$. We shall prove that p is a common fixed point, that is, $p \in F$.

Since $\lim_{n\to\infty} x_n = p$, for all $\varepsilon_1 > 0$, there exists a positive integer n_2 such that when $n \ge n_2$, we have

$$\|x_n - p\| < \frac{\varepsilon_1}{2(L+1)}. \qquad (3.10)$$

Moreover, $\lim_{n\to\infty} d(x_n, F) = 0$ implies that there exists a positive integer $n_3 \ge n_2$, such that when $n \ge n_3$, we have

$$d(x_n, F) < \frac{\varepsilon_1}{2(L+1)}, \qquad d(x_{n_3}, F) < \frac{\varepsilon_1}{2(L+1)}. \qquad (3.11)$$

Thus there exists a $p^* \in F$, such that

$$\|x_{n_3} - p^*\| = d(x_{n_3}, p^*) < \frac{\varepsilon_1}{2(L+1)}. \tag{3.12}$$

It follows from (3.10), (3.12) and for $i = 1, 2, 3$ that

$$\begin{aligned} \|T_i p - p\| &= \|T_i p - p^* + p^* - x_{n_3} + x_{n_3} - p\| \\ &\leq \|T_i p - p^*\| + \|x_{n_3} - p^*\| + \|x_{n_3} - p\| \\ &\leq L\|p - p^*\| + \|x_{n_3} - p^*\| + \|x_{n_3} - p\| \\ &\leq L\|x_{n_3} - p\| + L\|x_{n_3} - p^*\| + \|x_{n_3} - p^*\| \\ &\quad + \|x_{n_3} - p\| \\ &\leq (L+1)\|x_{n_3} - p\| + (L+1)\|x_{n_3} - p^*\| \\ &< (L+1).\frac{\varepsilon_1}{2(L+1)} + (L+1).\frac{\varepsilon_1}{2(L+1)} = \varepsilon_1. \end{aligned} \tag{3.13}$$

By the arbitrariness of $\varepsilon_1 > 0$, we have $T_i p = p$ for $i = 1, 2, 3$, that is, p is a common fixed point of the mappings T_1, T_2 and T_3. This completes the proof. □

Theorem 3.2. *Let E be a real Banach space, C be a nonempty closed convex subset of E. Let $T_i\colon C \to C$, $(i = 1, 2, 3)$ be uniformly L-Lipschitzian asymptotically quasi-nonexpansive type mappings with $F = \cap_{i=1}^3 F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (2.5) with the restrictions $\sum_{n=1}^\infty \alpha_n < \infty$ and $\sum_{n=1}^\infty \lambda_n < \infty$. Then $\{x_n\}$ converges to a common fixed point p of the mappings T_1, T_2 and T_3 if and only if there exists some infinite subsequence of $\{x_n\}$ which converges to p.*

Proof. The proof of Theorem 3.2 follows from Lemma 2.1 and Theorem 3.1. □

Theorem 3.3. *Let E be a real Banach space, C be a nonempty closed convex subset of E. Let $T_i\colon C \to C$, $(i = 1, 2, 3)$ be uniformly L-Lipschitzian asymptotically quasi-nonexpansive type mappings with $F = \cap_{i=1}^3 F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (2.5) with the restrictions $\sum_{n=1}^\infty \alpha_n < \infty$ and $\sum_{n=1}^\infty \lambda_n < \infty$. Suppose that the mappings T_1, T_2 and T_3 satisfy the following conditions:*

(i) $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0, \lim_{n\to\infty} \|x_n - T_2 x_n\| = 0, \lim_{n\to\infty} \|x_n - T_3 x_n\| = 0$;

(ii) *there exists a constant $A > 0$ such that*

$$\Big\{ \|x_n - T_1 x_n\| + \|x_n - T_2 x_n\| + \|x_n - T_3 x_n\| \Big\} \geq A d(x_n, F), \quad \forall n \geq 1.$$

Then $\{x_n\}$ converges strongly to a common fixed point of the mappings T_1, T_2 and T_3.

Proof. From conditions (i) and (ii), we have $\lim_{n\to\infty} d(x_n, F) = 0$, it follows as in the proof of Theorem 3.1, that $\{x_n\}$ must converges strongly to a common fixed point of the mappings T_1, T_2 and T_3. This completes the proof. □

Theorem 3.4. *Let E be a real Banach space satisfying Opial's condition and C be a weakly compact subset of E. Let $T_i\colon C \to C$, $(i = 1, 2, 3)$ be uniformly L-Lipschitzian asymptotically quasi-nonexpansive type mappings. Let $\{x_n\}$ be the sequence defined by (2.5) with the restrictions $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Suppose that T_1, T_2 and T_3 have a common fixed point, $I - T_i$ for $i = 1, 2, 3$ is demiclosed at zero and $\{x_n\}$ is an approximating common fixed point sequence for T_i for $i = 1, 2, 3$, that is, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, for $i = 1, 2, 3$. Then $\{x_n\}$ converges weakly to a common fixed point of the mappings T_1, T_2 and T_3.*

Proof. First, we show that $\omega_w(x_n) \subset F = \cap_{i=1}^{3} F(T_i)$. Let $x_{n_k} \to x$ weakly. By assumption, we have $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2, 3$. Since $I - T_i$ for $i = 1, 2, 3$ is demiclosed at zero, $x \in F = \cap_{i=1}^{3} F(T_i)$. By Opial's condition, $\{x_n\}$ possesses only one weak limit point, that is, $\{x_n\}$ converges weakly to a common fixed point of the mappings T_1, T_2 and T_3. This completes the proof. □

Example 3.1. Let E be the real line with the usual norm $|\cdot|$ and $K = [0, 1]$. Define $T_1, T_2, T_3\colon K \to K$ by

$$T_1 x = \sin x, \quad x \in [0, 1],$$

$$T_2 x = x/3, \quad x \in [0, 1],$$

$$T_3 x = x/2, \quad x \in [0, 1],$$

for $x \in K$. Obviously $T_1(0) = 0$, $T_2(0) = 0$ and $T_3(0) = 0$, that is, 0 is a common fixed point of T_1, T_2 and T_3, that is, $F = F(T_1) \cap F(T_2) \cap F(T_3) = \{0\}$. Now we check that T_1, T_2 and T_3 are asymptotically quasi-nonexpansive type mappings. In fact, if $x \in [0, 1]$ and $p = 0 \in [0, 1]$, then

$$|T_1 x - p| = |T_1 x - 0| = |\sin x - 0| = |\sin x| \le |x| = |x - 0| = |x - p|,$$

that is

$$|T_1 x - p| \le |x - p|.$$

That is, T_1 is quasi-nonexpansive. It follows that T_1 is uniformly quasi-1 Lipschitzian and asymptotically quasi-nonexpansive with $k_n = 1$ for each $n \ge 1$ and hence it is asymptotically quasi-nonexpansive type mapping since

$$|T_1x - p| - |x - p| \le 0, \ \ \forall p \in F(T_1), \ \forall x \in K.$$

Therefore we have

$$\limsup_{n\to\infty}\left\{\sup_{x\in K,\ p\in F(T_1)}\{|T_1x-p|-|x-p|\}\right\}\le 0.$$

This implies that T_1 is an asymptotically quasi-nonexpansive type mapping. Similarly for the mappings T_2 and T_3, we have

$$|T_2x - p| = |T_2x - 0| = |x/3 - 0| = 1/3|x| \le |x| = |x - 0| = |x - p|,$$

that is

$$|T_2x - p| \le |x - p|,$$

and

$$|T_2x - p| - |x - p| \le 0, \ \ \forall p \in F(T_2), \ \forall x \in K.$$

Therefore we have

$$\limsup_{n\to\infty}\left\{\sup_{x\in K,\ p\in F(T_2)}\{|T_2x-p|-|x-p|\}\right\}\le 0.$$

Similarly

$$|T_3x - p| = |T_3x - 0| = |x/2 - 0| = 1/2|x| \le |x| = |x - 0| = |x - p|,$$

that is

$$|T_3x - p| \le |x - p|,$$

and

$$|T_3x - p| - |x - p| \le 0, \ \ \forall p \in F(T_3), \ \forall x \in K.$$

Therefore we have

$$\limsup_{n\to\infty}\left\{\sup_{x\in K,\ p\in F(T_3)}\{|T_3x-p|-|x-p|\}\right\}\le 0.$$

Thus we see that T_2 and T_3 are also asymptotically quasi-nonexpansive type mappings.

Remark 3.1. The main result of this paper can be extended to a finite family of asymptotically quasi-nonexpansive type mappings $\{T_i : 1 \le i \le N\}$ by introducing the following iteration scheme:
Let $T_1, T_2, \ldots, T_N : C \to C$ be N asymptotically quasi-nonexpansive type mappings. Let $x_1 \in C$ be a given point. Then the sequence $\{x_n\}$ defined by

$$
\begin{aligned}
x_{n+1} &= (1-a_{n_1}-b_{n_1})x_n + a_{n_1}T_1^n y_{n_1} + b_{n_1}u_{n_1}, \\
y_{n_1} &= (1-a_{n_2}-b_{n_2})x_n + a_{n_2}T_2^n y_{n_2} + b_{n_2}u_{n_2}, \\
&\vdots \\
y_{n_{(N-2)}} &= (1-a_{n_{(N-1)}}-b_{n_{(N-1)}})x_n + a_{n_{(N-1)}}T_{N-1}^n y_{n_{(N-1)}} + b_{n_{(N-1)}}u_{n_{(N-1)}}, \\
y_{n_{(N-1)}} &= (1-a_{n_N}-b_{n_N})x_n + a_{n_N}T_N^n x_n + b_{n_N}u_{n_N}, \quad n \geq 1,
\end{aligned} \tag{3.14}
$$

is called N-step iterative sequence with errors of $T_1, T_2, \ldots, T_N$, where $\{u_{n_i}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, N$, are N bounded sequences in C, and $\{a_{n_i}\}_{n=1}^{\infty}$, $\{b_{n_i}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, N$, are N appropriate sequences in $[0, 1]$.

Remark 3.2. Theorem 3.1 extends, improves and unifies the corresponding results of [1, 5, 6, 9, 11, 12]. Especially Theorem 3.1 extends, improves and unifies Theorem 1 and 2 in [6], Theorem 1 in [5] and Theorem 3.2 in [12] in the following ways:

(1) The asymptotically quasi-nonexpansive mapping in [5], [6] and [12] is extended to more general asymptotically quasi-nonexpansive type mapping.

(2) The usual Ishikawa iteration scheme in [5], the usual modified Ishikawa iteration scheme with errors in [6] and the usual modified Ishikawa iteration scheme with errors for two mappings are extended to the three-step iteration scheme with errors for three mappings.

Remark 3.3. Theorem 3.2 extends, improves and unifies Theorem 3 in [6] and Theorem 3.3 extends, improves and unifies Theorem 3 in [5] in the following aspects:

(1) The asymptotically quasi-nonexpansive mapping in [5] and [6] is extended to more general asymptotically quasi-nonexpansive type mapping.

(2) The usual Ishikawa iteration scheme in [5] and the usual modified Ishikawa iteration scheme with errors in [6] are extended to the three-step iteration scheme with errors for three mappings.

Remark 3.4. Our results also extend the corresponding results of Quan [10] to the case of more general class of uniformly quasi-Lipschitzian mapping considered in this paper.

Remark 3.5. Our results also extend the corresponding results of Xu and Noor [14] to the case of more general class of asymptotically nonexpansive mapping considered in this paper.

Remark 3.6. Theorem 3.4 extends and improves Theorem 2.6 and 2.7 of Sahu and Jung [11] to the case of modified three-step iteration scheme with errors considered in this paper.

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