

ON THE OSCULATING SPACES OF SUBMANIFOLDS IN EUCLIDEAN SPACES

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ABSTRACT. This paper is a continuation of the papers [2], [3], [4], [5] and [6]. In this paper the osculating spaces of arbitrary order of a manifold embedded in Euclidean space are considered. A better estimation of their dimensions as well as the description of its basis are given.

1. INTRODUCTION

Let us consider an n -dimensional manifold M embedded in the Euclidean space R^m , where $m = n + k$. In [2] it is considered the following $k \times k$ matrix

$$(1.1) \quad P_{\alpha\beta}^{(1)} = \sum_{i=1}^n \sum_{j=1}^n (Y_i \cdot \nabla_{Y_j} N_\alpha)(Y_i \cdot \nabla_{Y_j} N_\beta),$$

where Y_1, \dots, Y_n are orthonormal tangent vectors at a chosen point on M and N_1, \dots, N_k are orthonormal vector fields which are orthogonal to the tangent space in a neighborhood of the chosen point. It is obvious that $P^{(1)}$ does not depend on the choice of the orthonormal base. The eigenvalues of $P^{(1)}$ are non-negative numbers and its eigenvectors are orthogonal. Let $k_1 = \text{rank}(P^{(1)})$ and let $\lambda_1^{(1)}, \dots, \lambda_{k_1}^{(1)}$ be the positive eigenvalues and for any eigenvector (p_1, \dots, p_k) we consider the vector $p_1 N_1 + \dots + p_k N_k$ and hence we obtain the following vectors $N_1^{(1)}, \dots, N_{k_1}^{(1)}$ as eigenvectors from the normal space. These vectors do not depend on the choice of the base N_α . The positive eigenvalues $\lambda_1^{(1)}, \dots, \lambda_{k_1}^{(1)}$ are called the *first normal curvatures* and the corresponding vectors $N_1^{(1)}, \dots, N_{k_1}^{(1)}$ are called the *first normal vectors*. Note that $k_1 \leq n^2$, because the right side of (1.1) is a sum of n^2 matrices of rank 1. Using the results in [1], in [6] it is proved that the vectors $Y_1, \dots, Y_n, N_1^{(1)}, \dots, N_{k_1}^{(1)}$, generate

Key words and phrases. osculating spaces, normal curvatures, normal vectors.
2010 Mathematics Subject Classification. 53A05, 53A07.
Received: June 15, 2012.

the osculating space (of the first order) at the considered point, which gives the geometrical interpretation of the first normal vectors.

In [2] are introduced the second normal curvature tensors and the second normal vectors as follows. Without loss of generality we assume that $N_1 = N_1^{(1)}, \dots, N_{k_1} = N_{k_1}^{(1)}$. Now it is $N_i \cdot N_j^{(1)} = 0$ for $i > k_1$ and $j \leq k_1$. We consider the $(k - k_1) \times (k - k_1)$ matrix

$$P_{\alpha\beta}^{(2)} = \sum_{i=1}^{n+k_1} \sum_{j=1}^n (Y_i \cdot \nabla_{Y_j} N_\alpha)(Y_i \cdot \nabla_{Y_j} N_\beta),$$

for $k_1 + 1 \leq \alpha, \beta \leq k$, and where we have denoted $Y_{n+1} = N_1, \dots, Y_{n+k_1} = N_{k_1}$. According to the choice of $N_1^{(1)}, \dots, N_{k_1}^{(1)}$ we get the following reduced form

$$(1.2) \quad P_{\alpha\beta}^{(2)} = \sum_{i=1}^{k_1} \sum_{j=1}^n (N_i^{(1)} \cdot \nabla_{Y_j} N_\alpha)(N_i^{(1)} \cdot \nabla_{Y_j} N_\beta).$$

If $k_2 = \text{rank}(P^{(2)}) = 0$ at any point of the submanifold, then the manifold locally can be embedded in $n + k_1$ -dimensional affine subspace of R^m . If $k_2 > 0$, let $(\lambda_1, \dots, \lambda_{k-k_1})$ be an eigenvector of $P^{(2)}$, then we consider the vector $\lambda_1 N_{k_1+1} + \dots + \lambda_{k-k_1} N_k$ as an eigenvector. According to this identification, the eigenvectors of $P^{(2)}$ and the principal directions do not depend on the choice of the basis N_α . The positive eigenvalues $\lambda_1^{(2)}, \dots, \lambda_{k_2}^{(2)}$ are defined to be the *second normal curvatures* and the corresponding eigenvectors $N_1^{(2)}, \dots, N_{k_2}^{(2)}$ are defined to be the *second normal vectors*. In [6] it is proved that the vectors $Y_1, \dots, Y_n, N_1^{(1)}, \dots, N_{k_1}^{(1)}, N_1^{(2)}, \dots, N_{k_2}^{(2)}$ generate the osculating space of the second order at the considered point, which gives the geometrical interpretation of the second normal vectors.

Continuing this procedure, the normal curvatures and normal vectors of higher degree are introduced [2] and [6]. This procedure is finite, since the number m is finite. We give only the inductive step for the matrix $P^{(l+1)}$. Namely, $P^{(l+1)}$ is $(k - k_1 - \dots - k_l) \times (k - k_1 - \dots - k_l)$ matrix given by

$$(1.3) \quad P_{\alpha\beta}^{(l+1)} = \sum_{i=1}^{k_l} \sum_{j=1}^n (N_i^{(l)} \cdot \nabla_{Y_j} N_\alpha)(N_i^{(l)} \cdot \nabla_{Y_j} N_\beta),$$

and $\text{rank} P^{(l+1)} \leq n \cdot \text{rank} P^{(l)}$. By induction of l it follows that $\text{rank}(P^{(l)}) \leq n^{l+1}$.

The vectors

$$Y_1, \dots, Y_n, N_1^{(1)}, \dots, N_{k_1}^{(1)}, N_1^{(2)}, \dots, N_{k_2}^{(2)}, \dots, N_1^{(l)}, \dots, N_{k_l}^{(l)}$$

generate the osculating space of order l at the considered point.

2. MAIN RESULT

We saw in the introduction that

$$k_r = \text{rank}(P_{\alpha\beta}^{(r)}) \leq n^{r+1}.$$

In this paper we prove much better inequality

$$k_r = \text{rank}(P_{\alpha\beta}^{(r)}) \leq \binom{n+r}{r+1}.$$

First, let us consider the case $r = 1$. In this case the matrix

$$P_{\alpha\beta}^{(1)} = \sum_{i=1}^n \sum_{j=1}^n (Y_i \cdot \nabla_{Y_j} N_\alpha)(Y_i \cdot \nabla_{Y_j} N_\beta)$$

is a sum of n^2 ($1 \leq i, j \leq n$) matrices of type $a_i a_j$ of rank 1. But since the matrices for the pairs (i, j) and (j, i) are equal, we obtain that $P^{(1)}$ is a sum of $n + \binom{n}{2} = \binom{n+1}{2}$ matrices of rank equal to 1, and hence $k_1 \leq \binom{n+1}{2}$. Indeed, note that

$$Y_i \cdot \nabla_{Y_j} N_\alpha = -N_\alpha \cdot \nabla_{Y_j} Y_i = -N_\alpha \cdot (\nabla_{Y_i} Y_j - [Y_i, Y_j]) = -N_\alpha \cdot \nabla_{Y_i} Y_j = Y_j \cdot \nabla_{Y_i} N_\alpha,$$

where we used that the torsion tensor $T(Y_i, Y_j) = \nabla_{Y_i} Y_j - \nabla_{Y_j} Y_i - [Y_i, Y_j]$ is a zero tensor and we used that $[Y_i, Y_j]$ is a tangent vector. Hence we have equal summands for the pairs (i, j) and (j, i) .

Moreover, according to the definition of $P_{\alpha\beta}^{(1)}$ we see that the space generated by the eigenvectors of $P^{(1)}$ coincides with the space generated by the vectors

$$\sum_{\alpha=1}^k N_\alpha (Y_i \cdot \nabla_{Y_j} N_\alpha) = - \sum_{\alpha=1}^k N_\alpha (N_\alpha \cdot \nabla_{Y_j} Y_i),$$

i.e. the projection of the vectors $\nabla_{Y_j} Y_i$ on the normal space of the tangent space. Hence the first osculating space generated by $Y_1, \dots, Y_n, N_1^{(1)}, \dots, N_{k_1}^{(1)}$ is the same with the space generated by the vectors

$$Y_1, \dots, Y_n, \left\{ \nabla_{Y_i} Y_j \right\} \quad (i \leq j).$$

Further we will consider the case $r = 2$.

According to the definition

$$P_{\alpha\beta}^{(2)} = \sum_{i=1}^{k_1} \sum_{j=1}^n (N_i^{(1)} \cdot \nabla_{Y_j} N_\alpha)(N_i^{(1)} \cdot \nabla_{Y_j} N_\beta)$$

we obtain that the space of eigenvectors of $P^{(2)}$ coincides with the space generated by the vectors of type

$$(2.1) \quad \sum_{\alpha=k_1+1}^k N_\alpha (N_\alpha \cdot \nabla_{Y_j} N_i^{(1)})$$

for $1 \leq j \leq n$ and $1 \leq i \leq k_1$. Note that the vectors $N_1^{(1)}, \dots, N_{k_1}^{(1)}$ generate the same space as the space of projections of $\nabla_{Y_j} Y_i$ on the normal space, i.e. the space of vectors

$$\nabla_{Y_j} Y_i - \sum_{p=1}^n Y_p (Y_p \cdot \nabla_{Y_j} Y_i).$$

By replacing these vectors instead of $N_i^{(1)}$ in (2.1) we obtain that the space of eigenvectors of $P^{(2)}$ coincides with the space generated by the projection of the vectors

$$\nabla_{Y_j} \nabla_{Y_i} Y_p \quad (1 \leq i, j, p \leq n)$$

over the space orthogonal to $Y_1, \dots, Y_n, N_1^{(1)}, \dots, N_{k_1}^{(1)}$. Similarly as we proved that $\nabla_{Y_j} Y_i$ and $\nabla_{Y_i} Y_j$ differ only for a vector in the tangent space, we shall prove that the vectors

$$\nabla_{Y_j} \nabla_{Y_i} Y_p, \nabla_{Y_j} \nabla_{Y_p} Y_i, \nabla_{Y_i} \nabla_{Y_j} Y_p, \nabla_{Y_i} \nabla_{Y_p} Y_j, \nabla_{Y_p} \nabla_{Y_j} Y_i, \nabla_{Y_p} \nabla_{Y_i} Y_j,$$

differ only for a vector belonging to the space generated by $Y_1, \dots, Y_n, N_1^{(1)}, \dots, N_{k_1}^{(1)}$. Indeed, it is sufficient to prove that the differences

$$\nabla_{Y_j} \nabla_{Y_i} Y_p - \nabla_{Y_i} \nabla_{Y_j} Y_p \quad \text{and} \quad \nabla_{Y_j} \nabla_{Y_i} Y_p - \nabla_{Y_j} \nabla_{Y_p} Y_i$$

belong to that space. Namely, using that

$$R(Y_j, Y_i)Y_p = \nabla_{Y_j} \nabla_{Y_i} Y_p - \nabla_{Y_i} \nabla_{Y_j} Y_p - \nabla_{[Y_j, Y_i]} Y_p$$

belongs to the tangent space, and $\nabla_{[Y_j, Y_i]} Y_p$ belongs to the space generated by $Y_1, \dots, Y_n, N_1^{(1)}, \dots, N_{k_1}^{(1)}$, we obtain that $\nabla_{Y_j} \nabla_{Y_i} Y_p - \nabla_{Y_i} \nabla_{Y_j} Y_p$ belongs to the space generated by $Y_1, \dots, Y_n, N_1^{(1)}, \dots, N_{k_1}^{(1)}$. The second difference $\nabla_{Y_j} \nabla_{Y_i} Y_p - \nabla_{Y_j} \nabla_{Y_p} Y_i$ belongs to the space generated by $Y_1, \dots, Y_n, N_1^{(1)}, \dots, N_{k_1}^{(1)}$ because

$$\nabla_{Y_j} \nabla_{Y_i} Y_p - \nabla_{Y_j} \nabla_{Y_p} Y_i = \nabla_{Y_j} ([Y_i, Y_p])$$

and $[Y_i, Y_p]$ is a vector from the tangent space.

So, without loss of generality we can consider those triples (i, j, p) such that $1 \leq i \leq j \leq p$. But, such triples there are exactly $\frac{n(n+1)(n+2)}{3!}$. Hence, $k_2 \leq \binom{n+2}{3}$. Moreover, according to this discussion we obtain that the second osculating space, i.e. generated by $Y_1, \dots, Y_n, N_1^{(1)}, \dots, N_{k_1}^{(1)}, N_1^{(2)}, \dots, N_{k_2}^{(2)}$ coincides with the space generated by the vectors

$$Y_1, \dots, Y_n, \left\{ \nabla_{Y_i} Y_j \right\} \quad (i \leq j), \quad \left\{ \nabla_{Y_i} \nabla_{Y_j} Y_p \right\} \quad (i \leq j \leq p).$$

We can continue this consideration and using an inductive step analogously to the step from $r = 1$ to $r = 2$, we obtain the main theorem.

Theorem 2.1. (a) $k_r = \dim P^{(r)} \leq \binom{n+r}{r+1}$;

(b) *The space generated by the eigenvectors of $P^{(r)}$ coincides with the vector space generated by the projections of the vectors*

$$\nabla_{Y_{i_1}} \nabla_{Y_{i_2}} \cdots \nabla_{Y_{i_{r-1}}} Y_{i_r} \quad (i_1 \leq i_2 \leq \cdots \leq i_r)$$

to the orthogonal complement of the space generated by the vectors

$$Y_1, \dots, Y_n, N_1^{(1)}, \dots, N_{k_1}^{(1)}, \dots, N_1^{(r-1)}, \dots, N_{k_{r-1}}^{(r-1)};$$

(c) The r -dimensional osculating space, i.e. the space generated by

$$Y_1, \dots, Y_n, N_1^{(1)}, \dots, N_{k_1}^{(1)}, \dots, N_1^{(r)}, \dots, N_{k_r}^{(r)}$$

coincides with the space generated by

$$Y_1, \dots, Y_n, \left\{ \nabla_{Y_i} Y_j \right\} \quad (i \leq j), \left\{ \nabla_{Y_i} \nabla_{Y_j} Y_p \right\} \quad (i \leq j \leq p), \dots, \\ \left\{ \nabla_{Y_{i_1}} \nabla_{Y_{i_2}} \cdots \nabla_{Y_{i_{r-1}}} Y_{i_r} \right\} \quad (i_1 \leq i_2 \leq \dots \leq i_r),$$

where Y_1, \dots, Y_n is an arbitrary orthonormal basis of the tangent space.

At the end we will prove that there does not exist a better estimation of k_r than $k_r = \dim P^{(r)} \leq \binom{n+r}{r+1}$.

Let r be a given positive integer. We choose m sufficiently large number such that $m \geq n + \binom{n+1}{2} + \binom{n+2}{3} + \dots + \binom{n+r}{r+1}$. Then we define an n -dimensional surface in R^m parameterized by u_1, \dots, u_n by

$$f(u_1, \dots, u_n) = (u_1, \dots, u_n, a_{11}u_1^2, a_{12}u_1u_2, \dots, a_{nn}u_n^2, \\ a_{111}u_1^3, a_{112}u_1^2u_2, \dots, a_{nnn}u_n^3, \dots, a_{11\dots 1}u_1^r, a_{11\dots 12}u_1^{r-1}u_2, \dots, a_{nn\dots n}u_n^r, 0, \dots, 0),$$

where all of the coefficients $a_{i_1 i_2}, a_{i_1 i_2 i_3}, \dots, a_{i_1 i_2 \dots i_r}$ ($i_1 \leq i_2 \leq \dots \leq i_r$) are nonzero coefficients. In this case (at the coordinate origin) we have

$$k_1 = \binom{n+1}{2}, \quad k_2 = \binom{n+2}{3}, \quad \dots, \quad k_r = \binom{n+r}{r+1}.$$

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