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ERROR BOUND OF CERTAIN GAUSSIAN QUADRATURE RULES FOR TRIGONOMETRIC POLYNOMIALS

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Abstract. In this paper we give error bound for quadrature rules of Gaussian type for trigonometric polynomials with respect to the weight function $w(x) = 1 + \cos x$, $x \in (-\pi, \pi)$, for 2π-periodic integrand, analytic in a circular domain. Obtained theoretical bound is checked and illustrated on some numerical examples.

1. Introduction

The famous Gaussian quadrature rules which have maximal algebraic degree of exactness have been subject of study already almost two centuries. Also, during this period different ways of generalizations and extensions have been developed. A quite natural way of extension is construction of quadrature rules with maximal degree of exactness in some linear space instead of the space of algebraic polynomials. Quadrature rules with maximal trigonometric degree of exactness are examples of those extensions. There are several different approaches in construction of such quadrature rules (see [10] and references therein), but approach presented in [19] is a simulation of the development of Gaussian quadrature rules for the algebraic polynomials. We denote the system of trigonometric polynomials, i.e., the system $\{\cos kx, \sin kx : k = 0, 1, \ldots, n\}$, by $\mathcal{T}_n$, and by $w(x)$ a nonnegative and integrable weight function on the interval $[-\pi, \pi)$, vanishing there only on a set of a measure zero.

Definition 1.1. A quadrature rule of the following form

$$\int_{-\pi}^{\pi} f(x)w(x)\,\mathrm{d}x = \sum_{\nu=0}^{n} w_\nu f(x_\nu) + R_n(f),$$

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where $-\pi \leq x_0 < x_1 < \cdots < x_n < \pi$, has trigonometric degree of exactness equal to d if $R_n(f) = 0$ for all $f \in \mathcal{T}_d$ and there exists $g \in \mathcal{T}_{d+1}$ such that $R_n(g) \neq 0$.

Remark 1.1. Turetzkii [19] considered interval $[0, 2\pi)$ instead of $[-\pi, \pi)$, but, as it was proved in [10], dealing with a translation of the interval $[0, 2\pi)$, mentioned quadrature rule can be considered on any interval whose length is equal to 2π, i.e., on any interval of the form $[L, 2\pi + L)$, $L \in \mathbb{R}$. Thus, in what follows we always use $L = -\pi$, i.e., we considered quadrature rules on interval $[-\pi, \pi)$.

An interpolatory quadrature rule for trigonometric polynomials has the following form

$$\int_{-\pi}^{\pi} f(x)w(x)\,\mathrm{d}x = \sum_{\nu=0}^{2n} w_\nu f(x_\nu) + R_n(f), \tag{1.1}$$

where $-\pi \leq x_0 < x_1 < \cdots < x_{2n} < \pi$ and

$$w_k = \int_{-\pi}^{\pi} \ell_k(x)w(x)\,\mathrm{d}x, \quad k = 0, 1, \ldots, 2n, \tag{1.2}$$

where

$$\ell_k(x) = \prod_{\substack{j=0 \\ j\neq k}}^{2n} \frac{\sin\left(\frac{x-x_j}{2}\right)}{\sin\left(\frac{x_k-x_j}{2}\right)}.$$

The maximal trigonometric degree of exactness of quadrature rule (1.1) is equal to $2n$. It is known (see [19, 10]) that (1.1) is Gaussian quadrature rule for trigonometric polynomials, i.e., exact for every trigonometric polynomial of degree $2n$, if and only if the nodes x_ν $(\in [-\pi, \pi))$, $\nu = 0, 1, \ldots, 2n$, are zeros of a trigonometric polynomial of semi-integer degree $n + 1/2$ which is orthogonal on $[-\pi, \pi)$ with respect to the weight function w to every trigonometric polynomial of a semi-integer degree less than or equal to $n - 1/2$. Notice that for any nonnegative integer n the linear space of all trigonometric polynomials of semi-integer degree less than or equal to $n + 1/2$ is the linear span of $\{\cos(k + 1/2)x, \sin(k + 1/2)x : k = 0, 1, \ldots, n\}$. Detailed study of such orthogonal systems, as well as of the corresponding Gaussian quadrature rules for trigonometric polynomials, can be found in [10, 11, 1, 12, 13]. However, the error estimates for such quadrature rules have not been considered so far as it was done for standard Gaussian quadrature rule (see e.g., [18, 3, 4, 5, 6, 16, 7, 14, 17, 15]). In this paper we estimate the reminder term $R_n(f)$ of Gaussian quadrature rule (1.1) for trigonometric polynomials with respect to even weight function $w(x) = 1 + \cos x$, $x \in (-\pi, \pi)$, when f is $2\pi-$periodic function, analytic in the domain $D = \{z \in \mathbb{C} : |z| \leq \rho\}$, where $\rho > 1$. Our approach is based on method of Stenger [18]. Section 2 is devoted to error bound and in Section 3 we give some numerical examples.

2. Error bound

In this section we consider the case of weight function $w(x) = 1+\cos x$, $x \in (-\pi, \pi)$. Then the quadrature rule (1.1) has the following form (see [10])

$$\int_{-\pi}^{\pi} f(x)(1+\cos x)\,\mathrm{d}x = \sum_{\nu=0}^{2n} w_\nu f(\tau_\nu) + R_n(f), \tag{2.1}$$

where $R_n(f) = 0$, $f \in \mathcal{T}_{2n}$.

Since the weight function $w(x) = 1 + \cos x$ is an even function on $(-\pi, \pi)$, we use the following result [10, Lemma 5.3].

Lemma 2.1. *Let w be an even weight function on $(-\pi, \pi)$. Let $x_\nu, \widetilde{w}_\nu$, $\nu = 1, 2, \ldots, n$, be nodes and weights of the n-point Gaussian quadrature rule with respect to the weight function $w(\arccos x)\sqrt{(1-x)/(1+x)}$, $x \in (-1, 1)$, constructed for the algebraic polynomials. Then, the weights w_ν, $\nu = 0, 1, \ldots, 2n$, and the nodes τ_ν, $\nu = 0, 1, \ldots, 2n$, of quadrature rule with maximal trigonometric degree of exactness with respect to the weight function w on $(-\pi, \pi)$ are given as follows:*

$$\begin{aligned}
&w_{2n-\nu} = w_\nu = \frac{\widetilde{w}_{\nu+1}}{1 - x_{\nu+1}}, \quad \nu = 0, 1, \ldots, n-1,\\
&w_n = \int_{-\pi}^{\pi} w(x)\,\mathrm{d}x - \sum_{\substack{\nu=0\\ \nu\neq n}}^{2n} w_\nu,\\
&\tau_{2n-\nu} = -\tau_\nu = \arccos x_{\nu+1}, \quad \nu = 0, 1, \ldots, n-1, \quad \tau_n = 0.
\end{aligned}$$

Now, we are ready to prove the following error bound.

Theorem 2.1. *Let f be $2\pi-$periodic function, analytic in the domain $D = \{z \in \mathbb{C} : |z| \le \rho\}$, where $\rho > 1$, and $C = \{z \in \mathbb{C} : |z| = \rho\}$. For the remainder term $R_n(f)$ in (2.1), the following estimate*

$$|R_n(f)| \le \frac{\pi(\rho+1)^2}{\rho(\rho^{2n+2}-1)} \max_{\xi \in C} |f(\xi)| \tag{2.2}$$

holds.

Proof. We start with application of the residue theorem on the contour integral

$$\frac{1}{2\pi \mathrm{i}} \frac{p(s)}{s^n} \oint_C \frac{f(\xi)\xi^n}{(\xi - s)p(\xi)}\,\mathrm{d}\xi,$$

where $s = \mathrm{e}^{\mathrm{i}x}$, $s_k = \mathrm{e}^{\mathrm{i}\tau_k}$, $k = 0, 1, \ldots, 2n$, and $p(s) = \prod_{k=0}^{2n}(s - s_k)$. Thus, we get

$$\begin{aligned}
\frac{1}{2\pi \mathrm{i}} \frac{p(s)}{s^n} \oint_C \frac{f(\xi)\xi^n}{(\xi - s)p(\xi)}\,\mathrm{d}\xi &= \frac{p(s)}{s^n}\left(\frac{f(s)s^n}{p(s)} + \sum_{k=0}^{2n} \frac{f(s_k)s_k^n}{(s_k - s)p'(s_k)}\right)\\
&= f(s) + \frac{p(s)}{s^n}\sum_{k=0}^{2n} \frac{f(s_k)s_k^n}{(s_k - s)p'(s_k)},
\end{aligned}$$

i.e.,

$$f(s)=\sum_{k=0}^{2n} f(s_k)\left(\frac{s_k}{s}\right)^n \frac{p(s)}{(s-s_k)p'(s_k)}+\frac{1}{2\pi\mathrm{i}}\frac{p(s)}{s^n}\oint_C \frac{f(\xi)\xi^n}{(\xi-s)p(\xi)}\,\mathrm{d}\xi. \tag{2.3}$$

Substituting $s=\mathrm{e}^{\mathrm{i}x}$ and $s_k=\mathrm{e}^{\mathrm{i}\tau_k}$, $k=0,1,\ldots,2n$, in

$$t(s)=\sum_{k=0}^{2n} f(s_k)\left(\frac{s_k}{s}\right)^n \frac{p(s)}{(s-s_k)p'(s_k)},$$

we obtain

$$\begin{aligned} t(x) &= \sum_{k=0}^{2n} f(\tau_k)\mathrm{e}^{\mathrm{i}n(\tau_k-x)}\prod_{\substack{j=0\\ j\neq k}}^{2n}\frac{\mathrm{e}^{\mathrm{i}x}-\mathrm{e}^{\mathrm{i}\tau_j}}{\mathrm{e}^{\mathrm{i}\tau_k}-\mathrm{e}^{\mathrm{i}\tau_j}}\\ &= \sum_{k=0}^{2n} f(\tau_k)\mathrm{e}^{\mathrm{i}n(\tau_k-x)}\mathrm{e}^{\mathrm{i}2n(x-\tau_k)/2}\prod_{\substack{j=0\\ j\neq k}}^{2n}\frac{2\sin\left(\frac{x-\tau_j}{2}\right)}{2\sin\left(\frac{\tau_k-\tau_j}{2}\right)}. \end{aligned}$$

Therefore,

$$t(x)=\sum_{k=0}^{2n} f(\tau_k)\prod_{\substack{j=0\\ j\neq k}}^{2n}\frac{\sin\left(\frac{x-\tau_j}{2}\right)}{\sin\left(\frac{\tau_k-\tau_j}{2}\right)},$$

i.e., $t(x)$ is the trigonometric interpolation polynomial for the function $f(x)$ (see [2, 8]). Since $s=\mathrm{e}^{\mathrm{i}x}$ and $x\in(-\pi,\pi)$, then $s\in C_1$, where C_1 is the unit circle with center at origin, $\mathrm{d}x=(\mathrm{i}s)^{-1}\,\mathrm{d}s$, and $w(x)=1+(\mathrm{e}^{\mathrm{i}x}+\mathrm{e}^{-\mathrm{i}x})/2$, i.e.,

$$w(s)=1+\frac{s+s^{-1}}{2}=\frac{(s+1)^2}{2s}.$$

Multiplying the both hand sides of (2.3) by $(\mathrm{i}s)^{-1}w(s)$ and integrating over the unit circle C_1, we have

$$\begin{aligned} \int_{C_1}\frac{f(s)w(s)}{\mathrm{i}s}\,\mathrm{d}s &= \int_{C_1}\sum_{k=0}^{2n} f(s_k)\left(\frac{s_k}{s}\right)^n \frac{p(s)}{(s-s_k)p'(s_k)}\frac{w(s)}{\mathrm{i}s}\,\mathrm{d}s\\ &\quad+\frac{1}{2\pi\mathrm{i}}\frac{1}{\mathrm{i}}\int_{C_1}\frac{p(s)w(s)}{s^{n+1}}\int_C\frac{f(\xi)\xi^n}{(\xi-s)p(\xi)}\,\mathrm{d}\xi\,\mathrm{d}s. \end{aligned} \tag{2.4}$$

Since

$$\begin{aligned} &\int_{C_1}\sum_{k=0}^{2n} f(s_k)\left(\frac{s_k}{s}\right)^n \frac{p(s)}{(s-s_k)p'(s_k)}\frac{w(s)}{\mathrm{i}s}\,\mathrm{d}s\\ &=\sum_{k=0}^{2n} f(\tau_k)\int_{-\pi}^{\pi}\prod_{\substack{j=0\\ j\neq k}}^{2n}\frac{\sin\left(\frac{x-\tau_j}{2}\right)}{\sin\left(\frac{\tau_k-\tau_j}{2}\right)}w(x)\,\mathrm{d}x, \end{aligned}$$

due to (1.2), we get

$$\int_{C_1} \sum_{k=0}^{2n} f(s_k) \left(\frac{s_k}{s}\right)^n \frac{p(s)}{(s-s_k)p'(s_k)} \frac{w(s)}{\mathrm{i}s} \,\mathrm{d}s = \sum_{k=0}^{2n} \omega_k f(\tau_k).$$

Since

$$\int_{C_1} \frac{f(s)w(s)}{\mathrm{i}s} \,\mathrm{d}s = \int_{-\pi}^{\pi} f(x)w(x) \,\mathrm{d}x,$$

from (2.4), we get

$$R_n(f) = \frac{1}{2\pi\mathrm{i}} \frac{1}{\mathrm{i}} \int_{C_1} \frac{p(s)w(s)}{s^{n+1}} \int_C \frac{f(\xi)\xi^n}{(\xi - s)p(\xi)} \,\mathrm{d}\xi \,\mathrm{d}s.$$

Changing the order of integrations we obtain

$$R_n(f) = \frac{1}{2\pi\mathrm{i}} \frac{1}{\mathrm{i}} \int_C \frac{f(\xi)\xi^n}{p(\xi)} \int_{C_1} \frac{p(s)w(s)}{s^{n+1}(\xi - s)} \,\mathrm{d}s \,\mathrm{d}\xi. \tag{2.5}$$

Thus, we have to estimate the right hand side of (2.5). We first consider the integral

$$\int_{C_1} \frac{p(s)w(s)}{s^{n+1}(\xi - s)} \,\mathrm{d}s.$$

For $|s/\xi| < 1$, i.e., $|\xi| > |s| = 1$, one has

$$\begin{aligned}
\int_{C_1} \frac{p(s)w(s)}{s^{n+1}(\xi - s)} \,\mathrm{d}s &= \int_{C_1} \frac{p(s)w(s)}{s^{n+1}\xi \left(1 - \frac{s}{\xi}\right)} \,\mathrm{d}s \\
&= \int_{C_1} \frac{p(s)w(s)}{s^{n+1}\xi} \sum_{k=0}^{\infty} \frac{s^k}{\xi^k} \,\mathrm{d}s \\
&= \int_{C_1} p(s)w(s) \sum_{k=0}^{\infty} \frac{s^{k-n-1}}{\xi^{k+1}} \,\mathrm{d}s \\
&= \sum_{k=0}^{\infty} \frac{1}{\xi^{k+1}} \int_{C_1} p(s)w(s)s^{k-n-1} \,\mathrm{d}s.
\end{aligned}$$

Since $w(\arccos x)\sqrt{(1-x)/(1+x)} = \sqrt{1-x^2}$ is Chebyshev weight function of the second kind, we know that nodes of the n-point Gaussian quadrature rule for this weight function, constructed for algebraic polynomials, are given by (see [9])

$$x_k = \cos \frac{k\pi}{n+1}, \quad k = 1, 2, \ldots, n.$$

According to Lemma 2.1, the nodes of the quadrature rule of Gaussian type (2.1) with respect of the even weight function $w(x) = 1 + \cos x$, $x \in (-\pi, \pi)$, are given as follows:

$$\tau_{2n-\nu} = -\tau_\nu = \arccos x_{\nu+1}, \;\; \nu = 0, 1, \ldots, n-1; \quad \tau_n = 0.$$

It is easy to see that in this case polynomial $p(s)$ is given by

$$p(s) = -1 + s - s^2 + \cdots - s^{2n} + s^{2n+1},$$

i.e.,

$$p(s) = \frac{s^{2n+2} - 1}{s+1}, \quad s \neq -1.$$

Indeed, from $p(s) = 0$ we obtain $s_k = \mathrm{e}^{\mathrm{i}\frac{2k\pi}{2n+2}} = \mathrm{e}^{\mathrm{i}\frac{k\pi}{n+1}}$, $k = 0, 1\ldots, 2n+1$, $k \neq n+1$ (notice that $k \neq n+1$ because $s \neq -1$). By using elementary transformations we get

$$s_{2n+1-\nu} = \mathrm{e}^{\mathrm{i}\frac{2n+1-\nu}{n+1}\pi} = \mathrm{e}^{2\mathrm{i}\pi}\mathrm{e}^{-\mathrm{i}\frac{\nu+1}{n+1}\pi} = \mathrm{e}^{\mathrm{i}\tau_\nu}, \quad \nu = 0, 1, \ldots, n-1,$$

$s_0 = \mathrm{e}^0 = \mathrm{e}^{\mathrm{i}\tau_n}$ and $s_{\nu+1} = \mathrm{e}^{\mathrm{i}\tau_{2n-\nu}}$, $\nu = 0, 1, \ldots, n-1$, where $\tau_{2n-\nu} = -\tau_\nu = \frac{\nu+1}{n+1}\pi$, $\nu = 0, 1, \ldots, n-1$, and $\tau_n = 0$.

Further, we have

$$\begin{aligned}
\int_{C_1} \frac{p(s)w(s)}{s^{n+1}(\xi - s)}\,\mathrm{d}s =& \sum_{k=0}^{\infty} \frac{1}{\xi^{k+1}} \int_{C_1} \frac{s^{2n+2}-1}{s+1}\frac{(s+1)^2}{2s} s^{k-n-1}\,\mathrm{d}s \\
=& \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{\xi^{k+1}} \left(\int_{C_1} \frac{\mathrm{d}s}{s^{-n-1-k}} + \int_{C_1} \frac{\mathrm{d}s}{s^{-n-k}} \right) \\
& - \frac{1}{2}\sum_{\substack{k=0 \\ k\neq n}}^{\infty} \frac{1}{\xi^{k+1}} \int_{C_1} \frac{\mathrm{d}s}{s^{n+1-k}} - \frac{1}{2}\frac{1}{\xi^{n+1}} \int_{C_1} \frac{\mathrm{d}s}{s} \\
& - \frac{1}{2}\sum_{\substack{k=0 \\ k\neq n+1}}^{\infty} \frac{1}{\xi^{k+1}} \int_{C_1} \frac{\mathrm{d}s}{s^{n+2-k}} - \frac{1}{2}\frac{1}{\xi^{n+2}} \int_{C_1} \frac{\mathrm{d}s}{s}.
\end{aligned}$$

Since $\int_{C_1} s^{-1}\,\mathrm{d}s = 2\pi\mathrm{i}$, and all other integrals in the sums on the right hand side of the previous equality are equal to zero, we get

$$\int_{C_1} \frac{p(s)w(s)}{s^{n+1}(\xi - s)}\,\mathrm{d}s = \frac{1}{2}\left(-\frac{2\pi\mathrm{i}}{\xi^{n+1}} - \frac{2\pi\mathrm{i}}{\xi^{n+2}} \right) = -\pi\mathrm{i}\frac{\xi+1}{\xi^{n+2}},$$

which, together with (2.5), gives

$$\begin{aligned}
R_n(f) &= \frac{1}{2\pi\mathrm{i}}\frac{1}{\mathrm{i}} \int_C \frac{f(\xi)\xi^n}{p(\xi)}(-\pi\mathrm{i})\frac{\xi+1}{\xi^{n+2}}\,\mathrm{d}\xi \\
&= -\frac{1}{2\mathrm{i}} \int_C \frac{f(\xi)(\xi+1)}{\frac{\xi^{2n+2}-1}{\xi+1}\xi^2}\,\mathrm{d}\xi \\
&= -\frac{1}{2\mathrm{i}} \int_C \frac{f(\xi)(\xi+1)^2}{(\xi^{2n+2}-1)\xi^2}\,\mathrm{d}\xi.
\end{aligned}$$

Therefore,

$$|R_n(f)| = \left| -\frac{1}{2\mathrm{i}} \int_C \frac{f(\xi)(\xi+1)^2}{(\xi^{2n+2}-1)\xi^2}\,\mathrm{d}\xi \right|,$$

and hence

$$|R_n(f)| \leq \frac{\ell(C)}{2} \max_{\xi \in C} |f(\xi)| \max_{\xi \in C} \frac{|\xi+1|^2}{|\xi^{2n+2}-1||\xi|^2},$$

where $\ell(C) = 2\rho\pi$ is the length of the circle C of radius $\rho > 1$. It is easy to see that $|\xi+1|^2$ attains its maximum on C at $\xi = \rho$. Since $\xi = \rho$ is one of the points at which $|\xi^{2n+2} - 1|$ attains its minimum on C, we conclude that $|\xi+1|/|\xi^{2n+2}-1|$ attains its maximum on C at $\xi = \rho$, i.e., that

$$|R_n(f)| \leq \frac{2\rho\pi}{2} \frac{|\rho+1|^2}{|\rho^{2n+2}-1||\rho|^2} \max_{\xi \in C} |f(\xi)| = \frac{\pi(\rho+1)^2}{\rho(\rho^{2n+2}-1)} \max_{\xi \in C} |f(\xi)|.$$

□

3. Numerical examples

In this section we check our theoretical error bound on some numerical examples.

Let us notice that for given function f it is interesting to consider the error bound as follows

$$|R_n(f)| \leq \inf_{1<r<\rho} \left(\frac{\pi(r+1)^2}{r(r^{2n+2}-1)} \left(\max_{\psi \in C_r} |f(\psi)| \right) \right), \tag{3.1}$$

where $C_r = \{\psi : |\psi| = r\}$, and ρ is the maximal possible value such that f is analytic in C_ρ.

Example 3.1. We are going to consider integration of the function $f(x) = 1/(\mathrm{e}^{\mathrm{i}x}+4/3)$, over the interval $(-\pi, \pi)$ with the weight function $w(x) = 1 + \cos x$. One can easily calculate that

$$\int_{-\pi}^{\pi} \frac{1+\cos x}{\mathrm{e}^{\mathrm{i}x}+4/3} \mathrm{d}x = \frac{15}{16}\pi.$$

Here $f(z) = 1/(z + 4/3)$, and according to conditions of Theorem 2.1, we can choose $\rho \approx 4/3$, for the maximum possible allowed value of ρ. Obtained theoretical error bound indicates that the error term, as function of n, should behave like $|R_n| \approx c/1.33^{2n+1}$, where c is independent on n and dependent on ρ. In Table 1 the absolute actual errors (the numbers in parentheses denote decimal exponents) are given. Since we check asymptotic behaviour of $|R_n(f)|$ the quotients of successive error terms are represented, too. According to theoretical result we must have $|R_n/R_{n-10}| \approx (3/4)^{20} \approx 3.2(-3)$, which is exactly demonstrated in Table 1.

n	10	20	30	40
$\lvert R_n\rvert$	0.35(−3)	0.11(−5)	0.35(−8)	0.11(−10)
$\lvert R_n/R_{n-10}\rvert$		3.2(−3)	3.2(−3)	3.2(−3)

Table 1. Actual error and the quotients of successive error terms in integration of $f(x) = 1/(\mathrm{e}^{\mathrm{i}x} + 4/3)$ on $(-\pi, \pi)$ with the weight function $w(x) = 1 + \cos x$.

Now we are going to consider error bound (3.1). It is easy to see that in our case

$$\max_{\psi\in C_r}|f(\psi)| = \frac{1}{4/3-r},$$

and hence

$$|R_n(f)| \leq \inf_{1<r<\rho}\left(\frac{\pi(r+1)^2}{r(r^{2n+2}-1)(4/3-r)}\right).$$

Thus, we obtain the error bounds given in Table 2.

n	20	30	40
$\lvert R_n\rvert$	6.28(−3)	2.94(−5)	1.25(−7)
r	1.30	1.31	1.32

TABLE 2. Error bound (3.1) for $f(x) = 1/(\mathrm{e}^{\mathrm{i}x} + 4/3)$ on $(-\pi, \pi)$ with the weight function $w(x) = 1 + \cos x$ and corresponding approximate value r for which infimum is attained.

Example 3.2. Let now consider the following integral

$$\int_{-\pi}^{\pi}\frac{\mathrm{e}^{\mathrm{i}x}}{\mathrm{e}^{\mathrm{i}x}+5/4\mathrm{i}}(1+\cos x)\,\mathrm{d}x = -\frac{4\pi}{5}\mathrm{i}.$$

Here we have that $w(x) = 1 + \cos x$ and $f(z) = z/(z + 5/4\,\mathrm{i})$. According to Theorem 2.1 function f must be analytic in domain D, hence all values $1 < \rho < 5/4$ are admissible, i.e., the maximum allowed value of ρ is $\rho \approx 5/4$. In this case the error term, as function of n, should behave like $|R_n| \approx c/1.25^{2n+1}$ and for the quotients of successive error terms must hold $|R_n/R_{n-10}| \approx (4/5)^{20} \approx 1.1(-2)$, which is exactly shown in the Table 3 (the numbers in parentheses denote decimal exponents).

n	5	15	25	35
$\lvert R_n\rvert$	0.48	0.51(−2)	0.59(−4)	0.68(−6)
$\lvert R_n/R_{n-10}\rvert$		1.1(−2)	1.1(−2)	1.1(−2)

TABLE 3. Actual error and the quotients of successive error terms in integration of $f(x) = \mathrm{e}^{\mathrm{i}x}/(\mathrm{e}^{\mathrm{i}x}+5/4\mathrm{i})$ on $(-\pi, \pi)$ with the weight function $w(x) = 1 + \cos x$.

Now we consider error bound (3.1). Since

$$\max_{\psi\in C_r}|f(\psi)| = \frac{r}{5/4-r},$$

one has

$$|R_n(f)| \leq \inf_{1<r<\rho}\left(\frac{\pi(r+1)^2}{(r^{2n+2}-1)(5/4-r)}\right).$$

Thus, we obtain the error bounds given in Table 4.

n	15	25	35
$\lvert R_n\rvert$	8.62(−1)	1.62(−2)	2.60(−4)
r	1.211	1.226	1.233

TABLE 4. Error bound (3.1) for $f(x) = \mathrm{e}^{\mathrm{i}x}/(\mathrm{e}^{\mathrm{i}x}+5/4\mathrm{i})$ on $(-\pi, \pi)$ with the weight function $w(x) = 1 + \cos x$ and corresponding approximate value r for which infimum is attained.

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