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A COMPANION OF OSTROWSKI'S INEQUALITY FOR MAPPINGS WHOSE FIRST DERIVATIVES ARE BOUNDED AND APPLICATIONS IN NUMERICAL INTEGRATION

MOHAMMAD WAJEEH ALOMARI

Abstract. A companion of Ostrowski's integral inequality for differentiable mappings whose first derivatives are bounded is proved. Applications to a composite quadrature rule and to probability density functions are considered.

1. Introduction

In 1938, Ostrowski established a very interesting inequality for differentiable mappings with bounded derivatives, as follows [4]:

Theorem 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality,*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \leq M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2} \right)^2}{(b-a)^2} \right] \tag{1.1}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

The following integral inequality which establishes a connection between the integral of the product of two functions and the product of the integrals of the two functions is well known in the literature as Grüss' inequality [8].

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Theorem 1.2. *Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions such that $\phi \le f(t) \le \Phi$ and $\gamma \le g(t) \le \Gamma$ for all $t \in [a, b]$, ϕ, Φ, γ and Γ are constants. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt\right| \le \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma). \tag{1.2}$$

Motivated by [3], Dragomir in [5] has proved the following companion of the Ostrowski inequality:

Theorem 1.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then we have the inequalities*

$$\left|\frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.3}$$
$$\le \begin{cases} \left[\frac{1}{8} + 2\left(\frac{x - \frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f'\|_\infty, & f' \in L_\infty[a,b], \\ \frac{2^{1/q}}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1} + \left(\frac{\frac{a+b}{2} - x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_{[a,b],p}, & \\ \qquad p > 1, \frac{1}{p} + \frac{1}{q} = 1, \text{ and } f' \in L_p[a,b] & \\ \left[\frac{1}{4} + \left|\frac{x - \frac{3a+b}{4}}{b-a}\right|\right]\|f'\|_{[a,b],1} & \end{cases}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Recently, Alomari [1] proved a companion inequality for differentiable mappings whose derivatives are bounded.

Theorem 1.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, and let $a, b \in I$ with $a < b$. If $f' \in L^1[a, b]$ and $\gamma \le f'(x) \le \Gamma$, for all $x \in [a, b]$, then the following inequality holds,*

$$\left|\frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le (b-a)\left[\frac{1}{16} + \left(\frac{x - \frac{3a+b}{4}}{b-a}\right)^2\right] \cdot (\Gamma - \gamma), \tag{1.4}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

In [6], Dragomir established some inequalities for this companion for mappings of bounded variation. In [7], Liu introduced some companions of an Ostrowski type inequality for functions whose second derivatives are absolutely continuous. Recently, Barnett, Dragomir and Gomma [2], have proved some companions for the Ostrowski inequality and the generalized trapezoid inequality.

In the present paper we shall derive a companion inequality of Ostrowski's type using Grüss' result and then discuss its applications for a composite quadrature rule and for probability density functions.

2. The Results

The following companion of Ostrowski's inequality holds.

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b). If $f' \in L^1[a, b]$ and $\gamma \leq f'(t) \leq \Gamma$, for all $t \in [a, b]$, then the inequality*

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{1}{8}(b-a)(\Gamma - \gamma), \tag{2.1}$$

holds for all $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. Let us define the mapping

$$p(x,t) = \begin{cases} t - a, & t \in [a, x], \\ t - \frac{a+b}{2}, & t \in (x, a+b-x], \\ t - b, & t \in (a+b-x, b] \end{cases}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Integrating by parts, we have

$$\frac{1}{b-a} \int_a^b p(x,t) f'(t)\, dt = \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt. \tag{2.2}$$

It is clear that for all $t \in [a, b]$ and $x \in \left[a, \frac{a+b}{2}\right]$, we have

$$x - \frac{a+b}{2} \leq p(x,t) \leq x - a.$$

Applying Theorem 1.2 to the mappings $p(x, \cdot)$ and $f'(\cdot)$, we obtain

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b p(x,t) f'(t)\, dt - \frac{1}{b-a} \int_a^b p(x,t)\, dt \cdot \frac{1}{b-a} \int_a^b f'(t)\, dt \right| \\ &\leq \frac{1}{4} \left\{ x - a - \left(x - \frac{a+b}{2} \right) \right\} (\Gamma - \gamma) = \frac{1}{8}(b-a)(\Gamma - \gamma), \end{aligned} \tag{2.3}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$. By a simple calculation we get

$$\int_a^b p(x,t)\, dt = 0, \qquad \text{and} \qquad \int_a^b f'(t)\, dt = \frac{f(b) - f(a)}{b-a}. \tag{2.4}$$

Finally, combining (2.2)–(2.4), we obtain (2.1) as required. □

Corollary 2.1. *In the inequality* (2.1)*, choosing*

(a) $x=a$*, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{8}(b-a)(\Gamma-\gamma);\tag{2.5}$$

(b) $x=\frac{3a+b}{4}$*, we get*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{8}(b-a)(\Gamma-\gamma);\tag{2.6}$$

(c) $x=\frac{a+b}{2}$*, we get*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{8}(b-a)(\Gamma-\gamma).\tag{2.7}$$

An inequality of Ostrowski's type may be stated as follows:

Corollary 2.2. *Let f as in Theorem 2.1. Additionally, if f is symmetric about the x-axis, i.e., $f(a+b-x)=f(x)$, then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{8}(b-a)(\Gamma-\gamma),\tag{2.8}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$. For instance, choose $x=a$, we have

$$\left|f(a)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{8}(b-a)(\Gamma-\gamma).\tag{2.9}$$

3. A Composite Quadrature Formula

Let $I_n: a=x_0<x_1<\cdots<x_n=b$ be a division of the interval $[a,b]$ and $h_i=x_{i+1}-x_i$ $(i=0,1,2,\dots,n-1)$.

Consider the general quadrature formula

$$Q_n(I_n,f):=\frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i.\tag{3.1}$$

The following result holds.

Theorem 3.1. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, where $a,b\in I$ with $a<b$. If $f'\in L^1[a,b]$ and $\gamma\le f'(x)\le\Gamma$, for all $x\in[a,b]$. Then, we have*

$$\int_a^b f(t)\,dt=Q_n(I_n,f)+R_n(I_n,f),\tag{3.2}$$

where, $Q_n(I_n, f)$ *is defined by formula* (3.1)*, and the remainder satisfies the estimates*

$$|R_n(I_n, f)| \leq \frac{(\Gamma - \gamma)}{8} \cdot \sum_{i=0}^{n-1} h_i. \tag{3.3}$$

Proof. Applying inequality (2.1) on the intervals $[x_i, x_{i+1}]$, we may state that

$$R_i(I_i, f) = \int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{1}{2}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right] h_i.$$

Summing the above inequality over i from 0 to $n-1$, we get

$$\begin{aligned} R_n(I_n, f) &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right] h_i \\ &= \int_a^b f(t)\,dt - \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right] h_i. \end{aligned}$$

From (2.1) it follows that

$$\begin{aligned} |R_n(I_n, f)| &= \left|\int_a^b f(t)\,dt - \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right] h_i\right| \\ &\leq \frac{(\Gamma - \gamma)}{8} \cdot \sum_{i=0}^{n-1} h_i. \end{aligned}$$

which completes the proof. □

4. Applications to Probability Density Functions

Let X be a random variable taking values in the finite interval $[a, b]$, with the probability density function $f : [a, b] \to [0, 1]$ with the cumulative distribution function $F(x) = Pr(X \leq x) = \int_a^b f(t)dt$.

Theorem 4.1. *With the assumptions of Theorem 1.4, we have the inequality*

$$\left|\frac{1}{2}[F(x) + F(a + b - x)] - \frac{b - E(X)}{b - a}\right| \leq \frac{1}{8}(b - a)(\Gamma - \gamma),$$

for all $x \in \left[a, \frac{a+b}{2}\right]$*, where* $E(X)$ *is the expectation of* X*.*

Proof. In the proof of Theorem 1.4, let $f = F$, and taking into account that

$$E(X) = \int_a^b t dF(t) = b - \int_a^b F(t)\,dt.$$

We left the details to the interested reader. □

Corollary 4.1. *In Theorem 4.1, choosing* $x = \frac{3a+b}{4}$*, we get*

$$\left|\frac{1}{2}\left[F\left(\frac{3a + b}{4}\right) + F\left(\frac{a + 3b}{4}\right)\right] - \frac{b - E(X)}{b - a}\right| \leq \frac{1}{8}(b - a)(\Gamma - \gamma).$$

Corollary 4.2. *In Theorem 4.1, if F is symmetric about the x-axis, i.e., $F(a+b-x) = F(x)$, we have*

$$\left| F(x) - \frac{b - E(X)}{b-a} \right| \leq \frac{1}{8}(b-a)(\Gamma - \gamma),$$

for all $x \in [a, \frac{a+b}{2}]$.

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Department of Mathematics
Faculty of Science
Jerash Private University
26150 Jerash Jordan
E-mail address: mwomath@gmail.com