A companion of Ostrowski’s inequality for mappings whose first derivatives are bounded and applications in numerical integration

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Abstract. A companion of Ostrowski’s integral inequality for differentiable mappings whose first derivatives are bounded is proved. Applications to a composite quadrature rule and to probability density functions are considered.

1. Introduction

In 1938, Ostrowski established a very interesting inequality for differentiable mappings with bounded derivatives, as follows [4]:

Theorem 1.1. Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^\circ \), the interior of the interval \( I \), such that \( f' \in L[a,b] \), where \( a, b \in I \) with \( a < b \). If \( |f'(x)| \leq M \), then the following inequality,

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq M (b-a) \left[ \frac{1}{4} + \frac{(x-a+b)^2}{(b-a)^2} \right]
\]

holds for all \( x \in [a,b] \). The constant \( \frac{1}{4} \) is the best possible in the sense that it cannot be replaced by a smaller constant.

The following integral inequality which establishes a connection between the integral of the product of two functions and the product of the integrals of the two functions is well known in the literature as Grüss’ inequality [8].

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Theorem 1.2. Let \( f, g : [a, b] \to \mathbb{R} \) be two integrable functions such that \( \phi \leq f(t) \leq \Phi \) and \( \gamma \leq g(t) \leq \Gamma \) for all \( t \in [a, b] \), \( \phi, \Phi, \gamma \) and \( \Gamma \) are constants. Then we have

\[
\left| \frac{1}{b-a} \int_a^b f(t) g(t) \, dt - \frac{1}{b-a} \int_a^b f(t) \, dt \cdot \frac{1}{b-a} \int_a^b g(t) \, dt \right| \leq \frac{1}{4} (\Phi - \phi) (\Gamma - \gamma).
\]

Motivated by [3], Dragomir in [5] has proved the following companion of the Ostrowski inequality:

Theorem 1.3. Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be an absolutely continuous function on \( [a, b] \). Then we have the inequalities

\[
\left| \frac{f(x) + f(a + b - x)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left\{ \begin{array}{l}
\frac{1}{8} + 2 \left( \frac{x-3a+b}{b-a} \right)^2 (b-a) \|f'\|_\infty, \\
\left( \frac{x-a}{b-a} \right)^{q+1} + \left( \frac{a+b-x}{b-a} \right)^{q+1} (b-a)^{1/q} \|f'\|_{[a,b],p}, \\
\left[ \frac{1}{4} + \frac{x-3a+b}{b-a} \right] \|f'\|_{[a,b],1}
\end{array} \right. \quad \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \text{ and } f' \in L_p[a,b]
\]

for all \( x \in [a, \frac{a+b}{2}] \).

Recently, Alomari [1] proved a companion inequality for differentiable mappings whose derivatives are bounded.

Theorem 1.4. Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^c \), the interior of the interval \( I \), and let \( a, b \in I \) with \( a < b \). If \( f' \in L^1[a,b] \) and \( \gamma \leq f'(x) \leq \Gamma \), for all \( x \in [a, b] \), then the following inequality holds,

\[
\left| \frac{f(x) + f(a + b - x)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq (b-a) \left[ \frac{1}{16} + \left( \frac{x-3a+b}{b-a} \right)^2 \right] \cdot (\Gamma - \gamma),
\]

for all \( x \in [a, \frac{a+b}{2}] \).

In [6], Dragomir established some inequalities for this companion for mappings of bounded variation. In [7], Liu introduced some companions of an Ostrowski type inequality for functions whose second derivatives are absolutely continuous. Recently,
Barnett, Dragomir and Gomma [2], have proved some companions for the Ostrowski inequality and the generalized trapezoid inequality.

In the present paper we shall derive a companion inequality of Ostrowski’s type using Grüss’ result and then discuss its applications for a composite quadrature rule and for probability density functions.

2. The Results

The following companion of Ostrowski’s inequality holds.

**Theorem 2.1.** Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \((a, b)\). If \( f' \in L^1[a, b] \) and \( \gamma \leq f'(t) \leq \Gamma \), for all \( t \in [a, b] \), then the inequality

\[
|f(x) - f(a)| \leq \frac{1}{8} (b - a) (\Gamma - \gamma),
\]

holds for all \( x \in [a, a \frac{a+b}{2}] \).

**Proof.** Let us define the mapping

\[
p(x, t) = \begin{cases} 
  t - a, & t \in [a, x], \\
  t - a + \frac{b}{2}, & t \in (x, a + b - x], \\
  t - b, & t \in (a + b - x, b]
\end{cases}
\]

for all \( x \in [a, a \frac{a+b}{2}] \).

Integrating by parts, we have

\[
\frac{1}{b - a} \int_a^b p(x, t) f'(t) dt = \frac{f(x) + f(a + b - x)}{2} - \frac{1}{b - a} \int_a^b f(t) dt.
\]

It is clear that for all \( t \in [a, b] \) and \( x \in [a, a + \frac{a+b}{2}] \), we have

\[
x - \frac{a + b}{2} \leq p(x, t) \leq x - a.
\]

Applying Theorem 1.2 to the mappings \( p(x, \cdot) \) and \( f'() \), we obtain

\[
\left| \frac{1}{b - a} \int_a^b p(x, t) f'(t) dt - \frac{1}{b - a} \int_a^b p(x, t) dt \cdot \frac{1}{b - a} \int_a^b f'(t) dt \right| \leq \frac{1}{4} \left( x - a - \left( x - \frac{a + b}{2} \right) \right) (\Gamma - \gamma) = \frac{1}{8} (b - a) (\Gamma - \gamma),
\]

for all \( x \in [a, a + \frac{a+b}{2}] \). By a simple calculation we get

\[
\int_a^b p(x, t) dt = 0, \quad \text{and} \quad \int_a^b f'(t) dt = \frac{f(b) - f(a)}{b - a}.
\]

Finally, combining (2.2)–(2.4), we obtain (2.1) as required. \( \square \)
Corollary 2.1. In the inequality (2.1), choosing 

(a) $x = a$, we get

$$\left| f(a) + f(b) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{8} (b-a) (\Gamma - \gamma);$$  

(2.5) 

(b) $x = \frac{3a+b}{4}$, we get

$$\left| f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{8} (b-a) (\Gamma - \gamma);$$  

(2.6) 

(c) $x = \frac{a+b}{2}$, we get

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{8} (b-a) (\Gamma - \gamma).$$  

(2.7) 

An inequality of Ostrowski’s type may be stated as follows:

Corollary 2.2. Let $f$ as in Theorem 2.1. Additionally, if $f$ is symmetric about the $x$-axis, i.e., $f(a + b - x) = f(x)$, then we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{8} (b-a) (\Gamma - \gamma),$$  

(2.8) 

for all $x \in [a, \frac{a+b}{2}]$. For instance, choose $x = a$, we have

$$\left| f(a) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{8} (b-a) (\Gamma - \gamma).$$  

(2.9) 

3. A Composite Quadrature Formula

Let $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a division of the interval $[a, b]$ and $h_i = x_{i+1} - x_i$ ($i = 0, 1, 2, \ldots, n-1$).

Consider the general quadrature formula

$$Q_n (I_n, f) := \frac{1}{2} \sum_{i=0}^{n-1} \left[ f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right) \right] h_i.$$  

(3.1) 

The following result holds.

Theorem 3.1. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, the interior of the interval $I$, where $a, b \in I$ with $a < b$. If $f' \in L^1[a, b]$ and $\gamma \leq f'(x) \leq \Gamma$, for all $x \in [a, b]$. Then, we have

$$\int_a^b f(t) \, dt = Q_n (I_n, f) + R_n (I_n, f),$$  

(3.2)
where, \(Q_n(I_n, f)\) is defined by formula (3.1), and the remainder satisfies the estimates

\[
|R_n(I_n, f)| \leq \frac{\Gamma - \gamma}{8} \cdot \sum_{i=0}^{n-1} h_i.
\]  

(3.3)

**Proof.** Applying inequality (2.1) on the intervals \([x_i, x_{i+1}]\), we may state that

\[
R_i(I_i, f) = \int_{x_i}^{x_{i+1}} f(t) \, dt - \frac{1}{2} \left[ f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right] h_i.
\]

Summing the above inequality over \(i\) from 0 to \(n - 1\), we get

\[
R_n(I_n, f) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(t) \, dt - \frac{1}{2} \sum_{i=0}^{n-1} \left[ f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right] h_i.
\]

From (2.1) it follows that

\[
|R_n(I_n, f)| = \left| \int_a^b f(t) \, dt - \frac{1}{2} \sum_{i=0}^{n-1} \left[ f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right] h_i \right| \leq \frac{\Gamma - \gamma}{8} \cdot \sum_{i=0}^{n-1} h_i.
\]

which completes the proof. \(\square\)

4. **Applications to Probability Density Functions**

Let \(X\) be a random variable taking values in the finite interval \([a, b]\), with the probability density function \(f : [a, b] \rightarrow [0, 1]\) with the cumulative distribution function

\[
F(x) = Pr(X \leq x) = \int_a^x f(t) \, dt.
\]

**Theorem 4.1.** With the assumptions of Theorem 1.4, we have the inequality

\[
\left| \frac{1}{2} \left[ F(x) + F(a + b - x) \right] - \frac{b - E(X)}{b - a} \right| \leq \frac{1}{8} (b - a) (\Gamma - \gamma),
\]

for all \(x \in [a, \frac{a+b}{2}]\), where \(E(X)\) is the expectation of \(X\).

**Proof.** In the proof of Theorem 1.4, let \(f = F\), and taking into account that

\[
E(X) = \int_a^b t \, dF(t) = b - \int_a^b F(t) \, dt.
\]

We left the details to the interested reader. \(\square\)

**Corollary 4.1.** In Theorem 4.1, choosing \(x = \frac{3a+b}{4}\), we get

\[
\left| \frac{1}{2} \left[ F\left(\frac{3a+b}{4}\right) + F\left(\frac{a+3b}{4}\right) \right] - \frac{b - E(X)}{b - a} \right| \leq \frac{1}{8} (b - a) (\Gamma - \gamma).
\]
Corollary 4.2. In Theorem 4.1, if $F$ is symmetric about the $x$-axis, i.e., $F(a + b - x) = F(x)$, we have

$$
\left| F(x) - \frac{b - E(X)}{b - a} \right| \leq \frac{1}{8} (b - a) (\Gamma - \gamma),
$$

for all $x \in [a, \frac{a+b}{2}]$.

References