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GENERALIZED DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULI

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Abstract. In this paper, we introduce and study the new sequence spaces $[V,\lambda,F,p,q,u]_0(\Delta_v^m)$, $[V,\lambda,F,p,q,u]_1(\Delta_v^m)$ and $[V,\lambda,F,p,q,u]_\infty(\Delta_v^m)$ which are generalized difference sequence spaces defined by a sequence of moduli in a locally convex Haussdorff topological linear space X whose topology is determined by a finite set Q of continuous seminorms q. We also study various algebraic and topological properties of these spaces, and some inclusion relations between these spaces. This study generalizes results of Atıci and Bektaş [11].

1. INTRODUCTION

Let ω be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x=(x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k\in\mathbb{N}=\{1,2,\dots\}$, the set of positive integers.

The difference sequence spaces were first introduced by Kızmaz [12]. The notion was further generalized by Et and Çolak [18]. Later Et and Esi [17] defined the sequence spaces

$$X(\Delta_v^m)=\{x=(x_k)\in w:\Delta_v^m x\in X\}$$

where $m\in\mathbb{N}$, $\Delta_v^0 x=(v_k x_k)$, $\Delta_v x=(v_k x_k - v_{k+1}x_{k+1})$, $\Delta_v^m x=(\Delta_v^{m-1}x_k-\Delta_v^{m-1}x_{k+1})$, and so that

$$\Delta_v^m x_k=\sum_{i=0}^{m}(-1)^i\binom{m}{i}v_{k+i}x_{k+i}.$$

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The notion of a modulus function was introduced by Nakano [13]. A modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

(i) $f(t)=0$ if and only if $t=0$,
(ii) $f(t+u)\leq f(t)+f(u)$, for all $t,u\geq 0$,
(iii) f is increasing,
(iv) f is continuous from the right at 0.

It follows from (ii) and (iv) that f must be continuous on $[0,\infty)$. Also from condition (ii), we have $f(nx)\leq n\cdot f(x)$ for all $n\in\mathbb{N}$. A modulus function may be bounded or unbounded. Ruckle [22] used the idea of a modulus function to construct some spaces of complex sequences. Later on some sequence spaces, defined by a modulus function or sequence of moduli, were introduced and studied by Et [16], Bektaş and Çolak [9], Atıci and Bektaş [11], Bataineh [1], Khan and Ahmad [21] and many others.

Throughout this paper, let $\lambda=(\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1}\leq\lambda_n+1$, $\lambda_1=1$. The generalized de la Vallée-Pousin mean is defined by

$$t_n(x)=\frac{1}{\lambda_n}\sum_{k\in I_n}x_k,$$

where $I_n=[n-\lambda_n+1,n]$ for $n=1,2,\ldots$.

Let $X,Y\subset\omega$. Then we shall write

$$M(X,Y)=\bigcap_{x\in X}x^{-1}*Y=\{a\in\omega: ax\in Y \text{ for all } x\in X\}\qquad ([14]).$$

The set $X^{\alpha}=M(X,\ell_1)$ is called Köthe-Toeplitz dual or the α-dual of X. If $X\subset Y$, then $Y^{\alpha}\subset X^{\alpha}$. It is clear that $X\subset(X^{\alpha})^{\alpha}=X^{\alpha\alpha}$. If $X=X^{\alpha\alpha}$, then X is called an α-space. In particular, an α-space is called a Köthe space or a perfect sequence space.

Definition 1.1. Let X be a sequence space. Then X is called:

(i) *solid (or normal)*, if $(\alpha_k x_k)\in X$ whenever $(x_k)\in X$ for all sequences (α_k) of scalar with $|\alpha_k|\leq 1$;
(ii) *monotone* provided X contains the canonical preimages of all its stepspaces;
(iii) *perfect* $X=X^{\alpha\alpha}$;
(iv) *symmetric* if $(x_k)\in X$ implies $\left(x_{\pi(k)}\right)\in X$, where $\pi(k)$ is a permutation of $\mathbb{N}$;
(v) *a sequence algebra* if $(x_k),(y_k)\in X$ implies $(x_k y_k)\in X$.

It is well known that if X is perfect, then X is normal [20].

We use the following inequality throughout this paper

$$|a_k+b_k|^{p_k}\leq D\left\{|a_k|^{p_k}+|b_k|^{p_k}\right\}\tag{1.1}$$

where a_k and b_k are complex numbers, $D = \max\left(1, 2^{G-1}\right)$ and $G = \sup_k p_k < \infty$ ([14]).

Lemma 1.1. [7] *Let q_1 and q_2 be seminorms on a linear space X. Then q_1 is stronger than q_2 if there exists a constant M such that $q_2(x) \leq M \cdot q_1(x)$ for all $x \in X$.*

2. Main results

In this section we introduce some new sequence spaces defined by a sequence of modulus functions. And we study various algebraic and topological properties of these spaces. Certain inclusion relations between these spaces will be discussed in this section.

Definition 2.1. Let $F = (f_k)$ be a sequence of moduli, q is a seminorm, $p = (p_k)$ be a sequence of strictly positive real numbers, $v = (v_k)$ be any fixed sequence of nonzero complex numbers and $u = (u_k)$ be a sequence of positive real numbers. By $\omega(X)$ we shall denote the space of all sequences defined over X. Now we define the following sequence spaces. Let $m \in \mathbb{N}$ be fixed, then

$$[V, \lambda, F, p, q, u]_1 (\Delta_v^m) = \{x \in \omega(X) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k\left(q\left(\Delta_v^m x_k - L\right)\right)\right]^{p_k} = 0, \exists L \in \mathbb{C}\},$$

$$[V, \lambda, F, p, q, u]_0 (\Delta_v^m) = \{x \in \omega(X) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k} = 0\},$$

$$[V, \lambda, F, p, q, u]_\infty (\Delta_v^m) = \{x \in \omega(X) : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k} < \infty\}.$$

Throughout the paper Z will denote any one of the notation $0, 1$ or ∞.

The above sequence spaces contain some unbounded sequences for $m \geq 1$. For example, let $X = \mathbb{C}$, $f_k(x) = x$ for all $k \in \mathbb{N}$, $q(x) = |x|$, $\lambda_n = n$ for all $n \in \mathbb{N}$, $v = (1, 1, \dots)$, $u = (1, 1, \dots)$ and $p_k = 1$ for all $k \in \mathbb{N}$, then $(k^m) \in [V, \lambda, F, p, q, u]_\infty (\Delta_v^m)$ but $(k^m) \notin \ell_\infty$.

In the case $p_k = 1$ for all $k \in \mathbb{N}$ we have $[V, \lambda, F, p, q, u]_Z (\Delta_v^m) = [V, \lambda, F, q, u]_Z (\Delta_v^m)$ and in the case $f_k(x) = x$ for every k we have $[V, \lambda, F, p, q, u]_Z (\Delta_v^m) = [V, \lambda, p, q, u]_Z (\Delta_v^m)$.

Theorem 2.1. *Let the sequence (p_k) be bounded. Then $[V, \lambda, F, p, q, u]_Z (\Delta_v^m)$ are linear spaces over the complex field $\mathbb{C}$.*

The proof is easy and thus omitted.

Theorem 2.2. *$[V, \lambda, F, p, q, u]_0 (\Delta_v^m)$ is a paranormed (need not to be totally paranormed) space with*

$$G_\Delta(x) = \sup_n \left(\frac{1}{\lambda_n} \sum_{k \in I_n} u_k [f_k(q(\Delta_v^m x_k))]^{p_k}\right)^{\frac{1}{M}}$$

where $M = \max(1, \sup_k p_k)$.

Proof. From Theorem 2.1, for each $x \in [V, \lambda, F, p, q, u]_0 (\Delta_v^m)$, $G_\Delta(x)$ exists. Clearly $G_\Delta(x) = G_\Delta(-x)$. It is trivial that $\Delta_v^m x_k = 0$ for $x = \theta$. Hence, we get $G_\Delta(\theta) = 0$. By Minkowski's inequality, we have $G_\Delta(x + y) \le G_\Delta(x) + G_\Delta(y)$. Let η be any fixed complex numbers. By definition of f_k for all k, we have $x \to \theta$ implies $G_\Delta(\eta x) \to 0$. Similarly we have x fixed and $\eta \to 0$ implies $G_\Delta(\eta x) \to 0$. Finally $x \to \theta$ and $\eta \to 0$ implies $G_\Delta(\eta x) \to 0$. This implies that the scalar multiplication is continuous. □

Theorem 2.3. *Let $F = (f_k)$ and $G = (g_k)$ be two sequences of moduli. For any two sequences $p = (p_k)$ and $t = (t_k)$ of strictly positive real numbers and any two seminorms q_1, q_2 we have*

(i) $[V, \lambda, F, p, q, u]_Z (\Delta_v^m) \cap [V, \lambda, G, p, q, u]_Z (\Delta_v^m) \subset [V, \lambda, F + G, p, q, u]_Z (\Delta_v^m)$,
(ii) $[V, \lambda, F, p, q_1, u]_Z (\Delta_v^m) \cap [V, \lambda, F, p, q_2, u]_Z (\Delta_v^m) \subset [V, \lambda, F, p, q_1 + q_2, u]_Z (\Delta_v^m)$,
(iii) *if q_1 is stronger than q_2, then* $[V, \lambda, F, p, q_1, u]_Z (\Delta_v^m) \subset [V, \lambda, F, p, q_2, u]_Z (\Delta_v^m)$,
(iv) *if q_1 is equivalent to q_2, then* $[V, \lambda, F, p, q_1, u]_Z (\Delta_v^m) = [V, \lambda, F, p, q_2, u]_Z (\Delta_v^m)$,
(v) $[V, \lambda, F, p, q_1, u]_Z (\Delta_v^m) \cap [V, \lambda, F, t, q_2, u]_Z (\Delta_v^m) \neq \varnothing$.

Proof. We give the proof for $Z = \infty$ only. The other cases can be proved in a similar way.

(i) Let $x \in [V, \lambda, F, p, q, u]_\infty (\Delta_v^m) \cap [V, \lambda, G, p, q, u]_\infty (\Delta_v^m)$. Then we have

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k [(f_k + g_k)(q(\Delta_v^m x_k))]^{p_k} &\le D \frac{1}{\lambda_n} \sum_{k \in I_n} u_k [f_k(q(\Delta_v^m x_k))]^{p_k} \\ &+ D \frac{1}{\lambda_n} \sum_{k \in I_n} u_k [g_k(q(\Delta_v^m x_k))]^{p_k}. \end{aligned}$$

Thus $[V, \lambda, F, p, q, u]_\infty (\Delta_v^m) \cap [V, \lambda, G, p, q, u]_\infty (\Delta_v^m) \subset [V, \lambda, F + G, p, q, u]_\infty (\Delta_v^m)$.

(ii) It can be proved similar to (i).

(iii) Let $x \in [V, \lambda, F, p, q_1, u]_\infty (\Delta_v^m)$ and q_1 be stronger than q_2. Therefore we have $q_2(\Delta_v^m x_k) \le M\ q_1(\Delta_v^m x_k)$ for all $k \in I_n$ where $M > 0$. Since modulus function f_k for each k is non-decreasing, we have

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k [f_k(q_2(\Delta_v^m x_k))]^{p_k} &\le \frac{1}{\lambda_n} \sum_{k \in I_n} u_k [f_k(M q_1(\Delta_v^m x_k))]^{p_k} \\ &\le \mu^G \frac{1}{\lambda_n} \sum_{k \in I_n} u_k [f_k(q_1(\Delta_v^m x_k))]^{p_k} \\ &< \infty \end{aligned}$$

where $|M| \le \mu$ and $G = \sup_k p_k < \infty$. Thus $[V, \lambda, F, p, q_1, u]_\infty (\Delta_v^m) \subset [V, \lambda, F, p, q_2, u]_\infty (\Delta_v^m)$.

(iv) It can be proved using (iii).

(v) Since each the above classes of sequences is linear space, the zero element belongs to these spaces. Thus the intersection is non-empty. □

Theorem 2.4. *Let X stand for $[V,\lambda,F,q,u]_Z$ and $m \geq 1$. Then $X\left(\Delta_v^{m-1}\right) \subset X\left(\Delta_v^m\right)$ and inclusions are strict. In general $X\left(\Delta_v^i\right) \subset X\left(\Delta_v^m\right)$ for all $i = 1,2,\ldots,m-1$ and the inclusions are strict.*

Proof. We give the proof for $[V,\lambda,F,q,u]_\infty(\Delta_v^m)$ only. In a similar way we proceed for $[V,\lambda,F,q,u]_1(\Delta_v^m)$ and $[V,\lambda,F,q,u]_0(\Delta_v^m)$. Let $x \in [V,\lambda,F,q,u]_\infty\left(\Delta_v^{m-1}\right)$. Then we have

$$\sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} u_k \left[f_k\left(q\left(\Delta_v^{m-1}x_k\right)\right)\right] < \infty.$$

Since f_k is a modulus for each k and so non-decreasing, we have

$$\begin{aligned}\frac{1}{\lambda_n} \sum_{k\in I_n} u_k \left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right] &= \frac{1}{\lambda_n} \sum_{k\in I_n} u_k \left[f_k\left(q\left(\Delta_v^{m-1}x_k - \Delta_v^{m-1}x_{k+1}\right)\right)\right] \\ &\leq \frac{1}{\lambda_n} \sum_{k\in I_n} u_k \left[f_k\left(q\left(\Delta_v^{m-1}x_k\right)\right)\right] \\ &+ \frac{1}{\lambda_n} \sum_{k\in I_n} u_k \left[f_k\left(q\left(\Delta_v^{m-1}x_{k+1}\right)\right)\right].\end{aligned}$$

Thus $[V,\lambda,F,q,u]_\infty\left(\Delta_v^{m-1}\right) \subset [V,\lambda,F,q,u]_\infty\left(\Delta_v^m\right)$. Proceeding in this way one will have $[V,\lambda,F,q,u]_\infty\left(\Delta_v^i\right) \subset [V,\lambda,F,q,u]_\infty\left(\Delta_v^m\right)$ for $i = 1,2,\ldots,m-1$. The sequence $x = (k^m)$, for example, belongs to $[V,\lambda,F,q,u]_\infty\left(\Delta_v^m\right)$, but does not belong to $[V,\lambda,F,q,u]_\infty\left(\Delta_v^{m-1}\right)$ for $f_k(u) = u$, $q(x) = |x|$, $u_k = 1$, $v_k = 1$ $(\forall k \in \mathbb{N})$. Therefore the inclusions are strict. □

Theorem 2.5. *Let $0 < p_k \leq t_k$ and (t_k/p_k) be bounded. Then $[V,\lambda,F,t,q,u]_Z\left(\Delta_v^m\right) \subset [V,\lambda,F,p,q,u]_Z\left(\Delta_v^m\right)$ where $Z = 0$, 1 or ∞.*

Proof. We shall prove only $Z = 0$. Let $x \in [V,\lambda,F,t,q,u]_0\left(\Delta_v^m\right)$. Write $w_k = \left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{t_k}$ and $\mu_k = p_k/t_k$, so that $0 < \mu \leq \mu_k \leq 1$ for each k.

We define the sequences (z_k) and (s_k) as follows:

Let $z_k = w_k$ and $s_k = 0$ if $w_k \geq 1$, and let $z_k = 0$ and $s_k = w_k$ if $w_k < 1$. Then it is clear that for all $k \in \mathbb{N}$, we have $w_k = z_k + s_k$, $w_k^{\mu_k} = z_k^{\mu_k} + s_k^{\mu_k}$. Now it follows that $z_k^{\mu_k} \leq z_k \leq w_k$ and $s_k^{\mu_k} \leq s_k^{\mu}$. Therefore

$$\lambda_n^{-1} \sum_{k\in I_n} u_k w_k^{\mu_k} \leq \lambda_n^{-1} \sum_{k\in I_n} u_k w_k + (\lambda_n^{-1} \sum_{k\in I_n} u_k s_k)^{\mu}.$$

Hence $x \in [V,\lambda,F,p,q,u]_0\left(\Delta_v^m\right)$. □

Theorem 2.6. *If*

$$\sup_k u_k \left[f_k(t)\right]^{p_k} < \infty, \text{ for all } t > 0 \tag{2.1}$$

we have

$$[V,\lambda,F,p,q,u]_1\left(\Delta_v^m\right) \subset [V,\lambda,F,p,q,u]_\infty\left(\Delta_v^m\right).$$

Proof. Let $x \in [V, \lambda, F, p, q, u]_1 (\Delta_v^m)$. By using the definition of modulus function, we have

$$\begin{aligned} &\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k \left(q \left(\Delta_v^m x_k \right) \right) \right]^{p_k} \\ \leq & D \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k \left(q \left(\Delta_v^m x_k - L \right) \right) \right]^{p_k} + D \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k \left(q \left(L \right) \right) \right]^{p_k} \\ \leq & D \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k \left(q \left(\Delta_v^m x_k - L \right) \right) \right]^{p_k} + D \sup_k u_k \left[f_k \left(q \left(L \right) \right) \right]^{p_k} \end{aligned}$$

where $D = \max\left(1, 2^{G-1}\right)$. Thus we get the result by (2.1). □

Theorem 2.7. *Let $0 < \inf p_k \leq \sup p_k < \infty$. Then the following statements are equivalent:*

(i) $[V, \lambda, p, q, u]_\infty (\Delta_v^m) \subseteq [V, \lambda, F, p, q, u]_\infty (\Delta_v^m)$,
(ii) $[V, \lambda, p, q, u]_0 (\Delta_v^m) \subseteq [V, \lambda, F, p, q, u]_\infty (\Delta_v^m)$,
(iii) $\sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k \left(t \right) \right]^{p_k} < \infty$ *for all* $t > 0$.

Proof. It is trivial that (i) implies (ii). Let (ii) hold and suppose that (iii) does not hold. Then for some $t > 0$

$$\sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k \left(t \right) \right]^{p_k} = \infty$$

and therefore there exists an increasing sequence (n_i) of positive integers such that

$$\frac{1}{\lambda_{n_i}} \sum_{k \in I_{n_i}} u_k \left[f_k \left(i^{-1} \right) \right]^{p_k} > i, \; i = 1, 2, \ldots. \tag{2.2}$$

Define $x = (x_k)$ such that

$$\Delta_v^m x_k = \begin{cases} i^{-1}, & k \in I_{n_i}, \; i = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Then $x \in [V, \lambda, p, q, u]_0 (\Delta_v^m)$, but by (2.2), $x \notin [V, \lambda, F, p, q, u]_\infty (\Delta_v^m)$ which contradicts (ii). Hence (iii) must hold.

Let (iii) hold and $x \in [V, \lambda, p, q, u]_\infty (\Delta_v^m)$. Suppose that $x \notin [V, \lambda, F, p, q, u]_\infty (\Delta_v^m)$. Then we have

$$\sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k \left(q \left(\Delta_v^m x_k \right) \right) \right]^{p_k} = \infty. \tag{2.3}$$

Let $q\left(\Delta_v^m x_k\right) = t$ for each k. Then by (2.3)

$$\sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k \left(t \right) \right]^{p_k} = \infty,$$

which contradicts (iii). Hence (i) must hold. □

Theorem 2.8. *Let* $1 < p_k \leq \sup p_k < \infty$. *Then if*

$$\inf_n \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k\left(t\right)\right]^{p_k} > 0 \text{ for all } t > 0, \tag{2.4}$$

we have

$$[V, \lambda, F, p, q, u]_0 \left(\Delta_v^m\right) \subseteq [V, \lambda, p, q, u]_0 \left(\Delta_v^m\right). \tag{2.5}$$

Proof. Let (2.4) hold and suppose that $x \in [V, \lambda, F, p, q, u]_0 \left(\Delta_v^m\right)$, but $x \notin [V, \lambda, p, q, u]_0 \left(\Delta_v^m\right)$. Then

$$\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k} \to 0, \text{ as } n \to \infty. \tag{2.6}$$

For given $\epsilon > 0$ there exist n' such that $q\left(\Delta_v^m x_k\right) \geq \epsilon$ and $k \in I_{n'}$. Therefore

$$\left[f_k\left(\epsilon\right)\right]^{p_k} \leq \left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k}$$

and by (2.6), we have

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k\left(\epsilon\right)\right]^{p_k} = 0.$$

This contradicts (2.4). Hence (2.5) must hold. □

Theorem 2.9. *Let* $1 \leq p_k \leq \sup p_k < \infty$. *If*

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k\left(t\right)\right]^{p_k} = \infty \text{ for all } t > 0 \tag{2.7}$$

then we have $[V, \lambda, F, p, q, u]_\infty \left(\Delta_v^m\right) \subseteq [V, \lambda, p, q, u]_0 \left(\Delta_v^m\right)$.

Proof. Suppose that (2.7) holds and let $x \in [V, \lambda, F, p, q, u]_\infty \left(\Delta_v^m\right)$. Then for each n

$$\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k} \leq K < \infty \tag{2.8}$$

for some $K > 0$. Suppose that $x \notin [V, \lambda, p, q, u]_0 \left(\Delta_v^m\right)$. Then for given $\epsilon_0 > 0$ there exists an integer n' such that $q\left(\Delta_v^m x_k\right) \geq \epsilon_0$ for $k \in I_{n'}$. Therefore

$$\left[f_k\left(\epsilon_0\right)\right]^{p_k} \leq \left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k}$$

and hence by (2.8) for each k we get

$$\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k\left(\epsilon_0\right)\right]^{p_k} \leq K < \infty$$

for some $K > 0$. This contradicts (2.7), i.e., $x \in [V, \lambda, p, q, u]_0 \left(\Delta_v^m\right)$. □

Theorem 2.10. *Let* $1 \leq p_k \leq \sup p_k < \infty$. *If*

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[f_k\left(t\right)\right]^{p_k} = 0 \text{ for all } t > 0 \tag{2.9}$$

then $[V, \lambda, p, q, u]_\infty \left(\Delta_v^m\right) \subseteq [V, \lambda, F, p, q, u]_0 \left(\Delta_v^m\right)$.

Proof. Suppose that (2.9) holds and $x \in [V, \lambda, p, q, u]_\infty (\Delta_v^m)$. Then

$$q(\Delta_v^m x_k) \leq K < \infty$$

for every k and for some $K > 0$. Therefore

$$[f_k (q (\Delta_v^m x_k))]^{p_k} \leq [f_k (K)]^{p_k}$$

and hence, by (2.9)

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} u_k [f_k (q (\Delta_v^m x_k))]^{p_k} \leq \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} u_k [f_k (K)]^{p_k} = 0.$$

Thus $x \in [V, \lambda, F, p, q, u]_0 (\Delta_v^m)$. ☐

Theorem 2.11. *The sequence spaces* $[V, \lambda, F, p, q, u]_Z (\Delta_v^m)$ *are not solid for* $m \geq 1$.

Proof. If we take $u_k = 1$ for all $k \in \mathbb{N}$, the proof can be shown like in [11]. ☐

From the above theorem we may give the following corollary.

Corollary 2.1. *The sequence spaces* $[V, \lambda, F, p, q, u]_Z (\Delta_v^m)$ *are not perfect for* $m \geq 1$.

Theorem 2.12. *The sequence spaces* $[V, \lambda, F, p, q, u]_1 (\Delta_v^m)$ *and* $[V, \lambda, F, p, q, u]_\infty (\Delta_v^m)$ *are not symmetric for* $m \geq 1$.

Proof. Under the restrictions on X, p, f_k, q, u, v and λ as given in the proof of Theorem 2.11, consider the sequence $x = (k^m)$, then $x \in [V, \lambda, F, p, q, u]_\infty (\Delta_v^m)$. Let (y_k) be a rearrangement of (x_k), which is defined as follows:

$$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, \dots\}.$$

Then $(y_k) \notin [V, \lambda, F, p, q, u]_\infty (\Delta_v^m)$. ☐

Theorem 2.13. *The space* $[V, \lambda, F, p, q, u]_0 (\Delta_v^m)$ *is not symmetric for* $m \geq 2$.

Theorem 2.14. *The sequence spaces* $[V, \lambda, F, p, q, u]_Z (\Delta_v^m)$ *are not sequence algebras.*

Proof. Under the restrictions on X, p, f_k, q, u, v and λ as given in the proof of Theorem 2.11, consider the sequence $x = (k^{m-2})$ and $y = (k^{m-2})$, then $x, y \in [V, \lambda, F, p, q, u]_Z (\Delta_v^m)$ but $x \cdot y \notin [V, \lambda, F, p, q, u]_Z (\Delta_v^m)$. The other cases can be proved on considering similar examples. ☐

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