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SUPER MEAN NUMBER OF A GRAPH

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Abstract. Let G be a graph and let $f : V(G) \to \{1, 2, \ldots, n\}$ be a function such that the label of the edge uv is $\frac{f(u)+f(v)}{2}$ or $\frac{f(u)+f(v)+1}{2}$ according as $f(u) + f(v)$ is even or odd and $f(V(G)) \cup \{f^*(e) : e \in E(G)\} \subseteq \{1, 2, \ldots, n\}$. If n is the smallest positive integer satisfying these conditions together with the condition that all the vertex and edge labels are distinct and there is no common vertex and edge labels, then n is called the super mean number of a graph G and it is denoted by $S_m(G)$. In this paper, we find the bounds for super mean number of some standard graphs.

1. Introduction

Throughout this paper, by a graph we mean a finite, undirected graph (simple graph) with $p \geq 2$ vertices. For notation and terminology, we follow [1].

Path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. $K_{1,m}$ is called a star and it is denoted by S_m. The union of m disjoint copies of a graph G is denoted by mG. The bistar $B_{m,n}$ is the graph obtained from K_2 by identifying the central vertices of $K_{1,m}$ and $K_{1,n}$ with the end vertices of K_2 respectively.

$\langle C_m, K_{1,n} \rangle$ is the graph obtained from C_m and $K_{1,n}$ by identifying any one of the vertices of C_m with the central vertex of $K_{1,n}$.

$\langle C_m * K_{1,n} \rangle$ is the graph obtained from C_m and $K_{1,n}$ by identifying any one of the vertices of C_m with a pendant vertex of $K_{1,n}$ (that is a non-central vertex of $K_{1,n}$).

The concept of super mean labeling was introduced by D. Ramya et al. [4]. They have studied in [4, 3, 2], the super mean labeling of some standard graphs. Further, some more results on super mean graphs are discussed in [6, 7].

Let $V(G)$ and $E(G)$ be the vertex set and edge set of a graph G, respectively, and $|V(G)| = p, |E(G)| = q$ (the order and the size of G, respectively). Let f :

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$V(G) \rightarrow \{1, 2, \ldots, p+q\}$ be injective. For a vertex labeling f, the induced edge labeling $f^*(e = uv)$ is defined by

$$f^*(e) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even,} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

Then, f is called super mean labeling if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$. A graph that admits a super mean labeling is called a super mean graph.

A super mean labeling of the graph $K_{2,4}$ is shown in Figure 1.

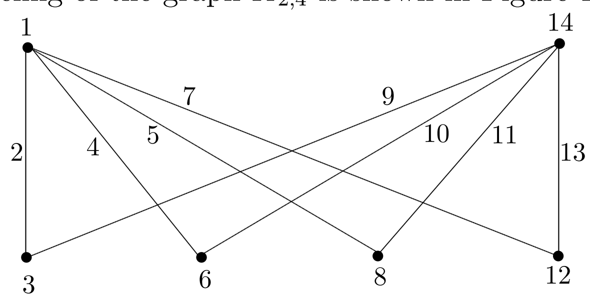


FIGURE 1.

The concept of mean number of a graph was introduced by M. Sundaram and R. Ponraj [5] and they have found the mean number of some standard graphs. Motivated by these work, we introduce the concept of super mean number of a graph.

Let $f : V(G) \rightarrow \{1, 2, \ldots, n\}$ be a function such that the label of the edge uv is $\frac{f(u)+f(v)}{2}$ or $\frac{f(u)+f(v)+1}{2}$ according as $f(u)+f(v)$ is even or odd and $f(V(G)) \cup \{f^*(e) : e \in E(G)\} \subseteq \{1, 2, \ldots, n\}$. If n is the smallest positive integer satisfying these conditions together with the condition that all the vertex and edge labels are distinct and there is no common vertex and edge labels, then n is called the super mean number of a graph G and it is denoted by $S_m(G)$.

For example, $S_m(K_{1,4}) = 10$ is shown in the following Figure 2.

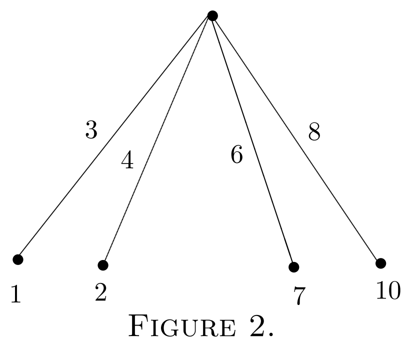


FIGURE 2.

It is observed that $S_m(G) \geq p+q$, where p is the order and q is the size of the graph G. Clearly, the equality holds for a super mean graph.

In this paper, we prove that $S_m(G) \leq 2^p - 2$ for any graph G. Also, we find an upper bound of the super mean number of the graphs $K_{1,n}, n \geq 7$, $tK_{1,n}$ for $n \geq 5, t > 1, B(p, n)$ for $p > n+1, n \geq 1, \langle C_m, K_{1,n} \rangle$ for $n \geq 5, m \geq 3$ and $\langle C_m * K_{1,n} \rangle$

for $n \geq 7, m \geq 3$. Further, we obtain the super mean number of the graphs K_p, $p \leq 4$, $K_{1,n}$ for $n \leq 6$, $tK_{1,4}, t > 1$, $B(p,n)$ for $p = n, n+1$ and any cycle C_n.

We use the following results in the subsequent theorems.

Theorem 1.1. [4] *A complete graph K_n is a super mean graph if $n \leq 3$.*

Theorem 1.2. [4] *K_n is not a super mean graph if $n > 3$.*

Theorem 1.3. [4] *The star $K_{1,n}$ is a super mean graph for $n \leq 3$.*

Theorem 1.4. [4] *$K_{1,n}$ is not a super mean graph for $n > 3$.*

Theorem 1.5. [2] *$nK_{1,4}, n > 1$, is a super mean graph.*

Theorem 1.6. [4] *The bistar, $B_{m,n}$ is a super mean graph for $m = n$ or $m = n+1$.*

Theorem 1.7. [7] *If G is a super mean graph then mG is also a super mean graph.*

Theorem 1.8. [4] *$C_{2n+1}, n \geq 1$, is a super mean graph.*

Theorem 1.9. [4] *C_4 is not a super mean graph.*

Theorem 1.10. [6] *C_{2n} is a super mean graph for $n \geq 3$.*

Based on the above theorems, we observe the following:

Observation 1.1. *$S_m(K_p) = \frac{p(p+1)}{2}$ if $p \leq 3$ and $S_m(K_p) \geq \frac{p(p+1)}{2} + 1$ if $p > 3$.*

A labeling of K_4 in Figure 3 shows that the above bound is attained for $p = 4$ and $S_m(K_4) = 11$.

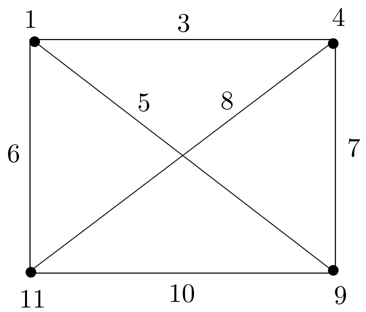


FIGURE 3.

Observation 1.2. *$S_m(K_{1,n}) = 2n+1$ for $n \leq 3$ and $S_m(K_{1,n}) \geq 2n+2$ $(n > 3)$.*

Observation 1.3. *$S_m(tK_{1,4}) = 9t$ for $t > 1$.*

Observation 1.4. *$S_m(B(p,n)) = 4n+3$ when $p = n$ and $S_m(B(p,n)) = 4n+5$ when $p = n+1$.*

Observation 1.5. *$S_m(C_{2n+1}) = 4n+2$ for $n \geq 1$ and $S_m(C_{2n}) = 4n$ for $n \geq 3$.*

Observation 1.6. *$S_m(C_4) = 9$, since C_4 is not a super mean graph and from a labeling of C_4 in Figure 4.*

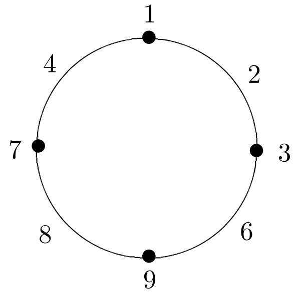


FIGURE 4.

Observation 1.7. *For any super mean graph G, $S_m(tG) = t(p+q), t > 1$ where p is the order and q is the size of the graph G.*

2. Super mean number of some standard graphs

The existence of the super mean number for any graph G is guaranteed by the following theorem.

Theorem 2.1. $S_m(G) \leq 2^p - 2$.

Proof. It is enough if we prove that $S_m(K_p) \leq 2^p - 2$. Let $v_1, v_2, \ldots, v_p$ be the vertices of K_p.

Define $f : V(K_p) \to \{1, 2, \ldots, n\}$ by $f(v_i) = 2^i - 1$ for $1 \leq i \leq p-1$ and $f(v_p) = 2^p - 2$. Clearly all the vertex labels are distinct. Let us prove the same for edge labels.

For $1 \leq i, j, s, t \leq p-1$, we consider the following two cases.

Suppose $f^*(v_iv_j) = f^*(v_sv_t)$; then $\frac{2^i-1+2^j-1}{2} = \frac{2^s-1+2^t-1}{2}$. This implies $2^i + 2^j = 2^s + 2^t$.

Case(i) Assume that the edges v_iv_j and v_sv_t have one vertex in common.

Take $i = s$ and $j \neq t$.

Since $j \neq t$, we have $2^j \neq 2^t$ then $2^i + 2^j \neq 2^s + 2^t$.

Hence if two edges have one vertex in common their edge values are distinct.

Case(ii) Assume that the edges v_iv_j and v_sv_t have no vertex in common. Then $i \neq s, i \neq t, j \neq s$ and $j \neq t$. Suppose $f^*(v_iv_j) = f^*(v_sv_t)$.

Without loss of generality, assume that i is the smallest integer.

Let $j = i + k_1, s = i + k_2, t = i + k_3, k_1, k_2, k_3 > 0$. Then $2^i + 2^j = 2^s + 2^t$ implies $2^i + 2^{i+k_1} = 2^{i+k_2} + 2^{i+k_3}$. Then $2^i(1 + 2^{k_1}) = 2^i(2^{k_2} + 2^{k_3})$, which gives $1 + 2^{k_1} = 2^{k_2} + 2^{k_3}$. This is a contradiction.

Case(iii) Suppose one of the vertices is v_p.

Subcase(i) Assume that v_iv_j and v_sv_t have the common vertex v_p.

Without loss of generality, assume that $i = s = p$ and $j \neq t$.

Suppose $f^*(v_pv_j) = f^*(v_pv_t)$. This implies $\frac{2^p-2+2^j-1+1}{2} = \frac{2^p-2+2^t-1+1}{2}$, that is $2^p + 2^j = 2^p + 2^t$. Then $2^j = 2^t$ which gives $j = t$. This is a contradiction.

Subcase(ii) Suppose v_iv_j and v_sv_t have a common vertex other than v_p.

For $1 \leq i, s, t \leq p-1$, take $i = s$ and $j = p$. Then $f^*(v_i v_p) = f^*(v_s v_t)$ implies $\frac{2^i-1+2^p-2+1}{2} = \frac{2^i-1+2^t-1}{2}$. Then we have $2^i + 2^p = 2^i + 2^t$, that is $2^p = 2^t$ which gives $p = t$. This is a contradiction.

Subcase(iii) Suppose the edges $v_i v_j$ and $v_s v_t$ have no vertex in common.

For $1 \leq i, s, t \leq p-1$ and $j = p$ $f^*(v_i v_p) = f^*(v_s v_t)$ implies $\frac{2^i-1+2^p-2+1}{2} = \frac{2^s-1+2^t-1}{2}$ i.e. $2^i + 2^p = 2^s + 2^t$. Then, we get a contradiction as in Case (ii).

Let us prove now that for any $v \in V(G)$ and $e \in E(G)$, $f(v) \neq f^*(e)$.

Let us take any edge $v_i v_j$ in $K_p, i < j$, where $i \neq 1$ and $j \neq p$. Now, $f^*(v_i v_j) = \frac{2^i-1+2^j-1}{2} = \frac{2^i+2^j-2}{2} = 2^{i-1} + 2^{j-1} - 1 = 2^{i-1}(1 + 2^{j-i}) - 1 \neq 2^k - 1$ for any k. Also $f^*(v_i v_j) \neq 2^p - 2$. This implies that $f^*(v_i v_j) \notin f(V(G))$. If $j \neq p$ and $i = 1$, $f^*(v_i v_j) = \frac{1+2^j-1}{2} = 2^{j-1} \notin f(V(G))$. If $i \neq 1$ and $j = p$, $f^*(v_i v_p) = \frac{(2^i-1+2^p-2)+1}{2} = \frac{2^i+2^p-2}{2} = 2^{i-1} + 2^{p-1} - 1 \neq 2^k - 1$ for any k. Also $f^*(v_i v_p) \neq 2^p - 2$. This implies that $f^*(v_i v_p) \notin f(V(G))$. If $i = 1, j = p, f^*(v_1 v_p) = \frac{(1+2^p-2)+1}{2} = 2^{p-1} \notin f(V(G))$.

Thus, all the vertex and edge labels are distinct and no vertex and edge labels are equal.

Hence $S_m(K_p) \leq 2^p - 2$. □

Theorem 2.2. $S_m(K_{1,n}) = 2n + 2$ *for* $n = 4, 5, 6$.

Proof. Let $V(K_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$ and $E(K_{1,n}) = \{vv_i; 1 \leq i \leq n\}$.

Define f on $V(K_{1,4})$ and $V(K_{1,5})$ as follows:

$f(v) = 5, f(v_i) = i, 1 \leq i \leq 2, f(v_i) = 7+3(i-3), 3 \leq i \leq n-1$ and $f(v_n) = 2n+2$, for $n = 4, 5$.

The vertex labeling f on $V(K_{1,6})$ is defined by

$f(v) = 5, f(v_i) = i, 1 \leq i \leq 2, f(v_3) = 7, f(v_4) = 11, f(v_5) = 12$ and $f(v_6) = 14$.

Clearly, the vertex labels and the induced edge labels are distinct. Hence, $S_m(K_{1,n}) \leq 2n + 2$ for $n = 4, 5, 6$. Then by Observation 1.12, the result follows. □

Theorem 2.3. $S_m(K_{1,n}) \leq 4n - 10, n \geq 7$.

Proof. Let $V(K_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$ and $E(K_{1,n}) = \{vv_i; 1 \leq i \leq n\}$.

Define f on $V(K_{1,n})$ as follows:

$f(v) = 5, f(v_i) = i, 1 \leq i \leq 2, f(v_3) = 7,\ f(v_i) = 2i + 3, 4 \leq i \leq 5,\ f(v_i) = 15 + 4(i-6), 6 \leq i \leq n-1, f(v_n) = 4n - 10$.

Clearly, the vertex labels and the edge labels are distinct and no vertex and edge labels are equal. Hence, $S_m(K_{1,n}) \leq 4n - 10$ for $n \geq 7$. □

Theorem 2.4. $S_m(tK_{1,n}) \leq (2n + 1)t + 1$ *for* $n = 5, 6$ *and* $t > 1$.

Proof. Let $v_{0_j}, v_{i_j}, 1 \leq j \leq t, 1 \leq i \leq n$ be the vertices and $v_{0_j} v_{i_j}, 1 \leq j \leq t, 1 \leq i \leq n$ be the edges of $tK_{1,n}$.

Define f on $V(tK_{1,n}), n = 5, 6$ as follows:

When $t = 1$ and $n = 5$, define $f(v_{0_1}) = 5, f(v_{i_1}) = i,\ 1 \leq i \leq 2, f(v_{i_1}) = 7 + 3(i-3), 3 \leq i \leq 4, f(v_{5_1}) = 12$.

When $t = 1$ and $n = 6$, define $f(v_{0_1}) = 5, f(v_{i_1}) = i, 1 \le i \le 2, f(v_{3_1}) = 7, f(v_{4_1}) = 11, f(v_{5_1}) = 12$ and $f(v_{6_1}) = 14$.

For $t > 1$, label the vertices of $tK_{1,5}$ and $tK_{1,6}$ as follows :

$$f(v_{0_j}) = f(v_{0_1}) + (2n+1)(j-1), 2 \le j \le t,$$
$$f(v_{1_2}) = f(v_{1_1}) + 2n,$$
$$f(v_{1_j}) = f(v_{1_2}) + (2n+1)(j-2), 3 \le j \le t \text{ and}$$
$$f(v_{i_j}) = f(v_{i_1}) + (2n+1)(j-1), 2 \le j \le t, 2 \le i \le n.$$

Clearly, the vertex labels are distinct. Also, the vertex labeling f induces distinct edge labels and $f(E(G)) \subseteq \{1, 2, 3, \ldots, (2n+1)t+1\} - f(V(G))$. Hence $S_m(tK_{1,n}) \le (2n+1)t+1$. □

According to Theorem 2.4, in the following Figure 5, the labeling of $5K_{1,5}$ shows that $S_m(5K_{1,5}) \le 56$.

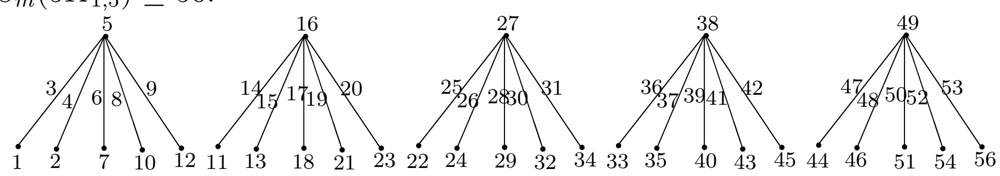


FIGURE 5. $5K_{1,5}$

Theorem 2.5. *When t is an even integer, $S_m(tK_{1,n}) \le t(2n+2) - 1$ for $n > 6$.*

Proof. Let $v_{0_j}, v_{i_j}, 1 \le i \le n, 1 \le j \le t$ be the vertices and $v_{0_j}v_{i_j}, 1 \le i \le n, 1 \le j \le t$ be the edges of $tK_{1,n}$.

Define f on $V(tK_{1,n})$ as follows:

$$f(v_{0_{2j+1}}) = (4n+4)j + 1, 0 \le j \le \frac{t}{2} - 1,$$
$$f(v_{0_{2j}}) = (4n+3)j + j - 1, 1 \le j \le \frac{t}{2},$$
$$f(v_{i_{2j+1}}) = (4n+4)j + 4i - 1, 0 \le j \le \frac{t}{2} - 1, 1 \le i \le n \text{ and}$$
$$f(v_{i_{2j}}) = (4n+4)(j-1) + 4i + 1, 1 \le j \le \frac{t}{2}, 1 \le i \le n.$$

It can be verified that all the vertex and edge labels are distinct and there is no common vertex and edge labels. Hence, $S_m(tK_{1,n}) \le t(2n+2) - 1$ for $n > 6$ and t is an even integer. □

Theorem 2.6. *When t is an odd integer, $S_m(tK_{1,n}) \le t(2n+2) + 3$ for $n > 6$.*

Proof. Let $V(tK_{1,n}) = \{v_{0_j}, v_{i_j} : 1 \le i \le n, 1 \le j \le t\}$ and $E(tK_{1,n}) = \{v_{0_j}v_{i_j} : 1 \le i \le n, 1 \le j \le t\}$. Let $t = 2k+1$ for some $k \in Z^+$.

Define f on $V(tK_{1,n})$ as follows:

For $1 \leq j \leq 2k$,

$$f(v_{0_{2j+1}}) = (4n+4)j+1, 0 \leq j \leq k-1,$$
$$f(v_{0_{2j}}) = (4n+3)j+j-1, 1 \leq j \leq k,$$
$$f(v_{i_{2j+1}}) = (4n+4)j+4i-1, 0 \leq j \leq k-1, 1 \leq i \leq n \text{ and}$$
$$f(v_{i_{2j}}) = (4n+4)(j-1)+4i+1, 1 \leq j \leq k, 1 \leq i \leq n.$$

When $j = 2k+1$,

$$f(v_{0_{2k+1}}) = 4+(2n+2)2k,$$
$$f(v_{i_{2k+1}}) = i+(2n+2)2k-1, 1 \leq i \leq 2,$$
$$f(v_{3_{2k+1}}) = 6+(2n+2)2k,$$
$$f(v_{i_{2k+1}}) = 2i+(2n+2)2k+2, 4 \leq i \leq 5,$$
$$f(v_{i_{2k+1}}) = 4i+(2n+2)2k-10, 6 \leq i \leq n-1 \text{ and}$$
$$f(v_{n_{2k+1}}) = (2n+2)2k+4n-11.$$

Clearly, the vertex labels and the induced edge labels are distinct and further $f(V(G)) \cap f(E(G)) = \emptyset$. Hence, $S_m(tK_{1,n}) \leq t(2n+2)+3$ for $n > 6$ and t is an odd integer. ☐

Theorem 2.7. *$S_m(B(p,n)) \leq 4p$ if $p > n+1$.*

Proof. Let $V(B(p,n)) = \{u, v, u_i, v_j : 1 \leq i \leq p, 1 \leq j \leq n\}$ and $E(B(p,n)) = \{uv, uu_i, vv_j : 1 \leq i \leq p, 1 \leq j \leq n\}$.

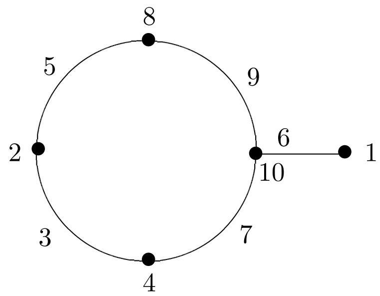
Define f on $V(B(p,n))$ as follows:

$f(u) = 3, f(u_i) = 4i-3, 1 \leq i \leq p, f(v) = 4p$ and $f(v_j) = 7+4(j-1), 1 \leq j \leq n$. It can be verified that the vertex and edge labels are distinct and $f(V(G)) \cap f(E(G)) = \emptyset$. Thus, $S_m(B(p,n)) \leq 4p$, $p > n+1$. ☐

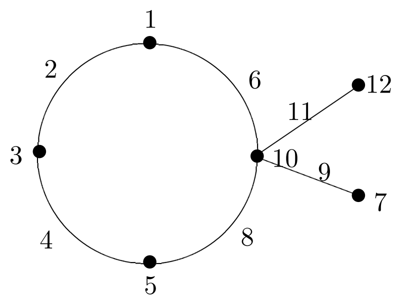
Theorem 2.8. *$\langle C_m, K_{1,n}\rangle$ is a super mean graph for $n \leq 4$ and $m \geq 3$.*

Proof. Let $V(\langle C_m, K_{1,n}\rangle) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n, v = u_1\}$ and $E(\langle C_m, K_{1,n}\rangle) = \{u_1u_2, u_2u_3, \ldots, u_mu_1, u_1v_i : 1 \leq i \leq n\}$.

For $m = 4$, the super mean labeling of the graphs $\langle C_4, K_{1,1}\rangle$, $\langle C_4, K_{1,2}\rangle$, $\langle C_4, K_{1,3}\rangle$ and $\langle C_4, K_{1,4}\rangle$ are shown in Figure 6.



$\langle C_4, K_{1,1}\rangle$



$\langle C_4, K_{1,2}\rangle$

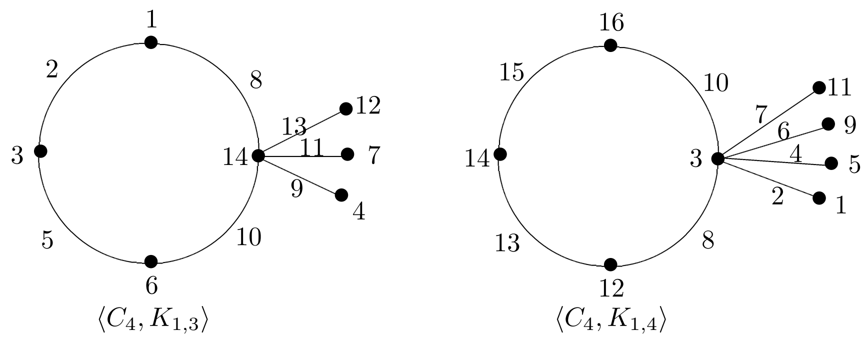


FIGURE 6.

Case(i) $n = 1, 2$.

Define f on $V(\langle C_m, K_{1,n}\rangle), n = 1, 2,\ m \geq 3$ and $m \neq 4$ as follows:

Subcase(i) When m is odd.

Let $m = 2k + 1, k \in Z^+$.

$$\begin{aligned} f(v_i) &= i, 1 \leq i \leq n, \\ f(u_1) &= 2n + 1, \\ f(u_j) &= 2n + 4j - 5, 2 \leq j \leq k + 1 \text{ and} \\ f(u_{k+1+j}) &= 2n + 4k - 4j + 6, 1 \leq j \leq k. \end{aligned}$$

Subcase(ii) When m is even.

Let $m = 2k, k \in Z^+$.

$$\begin{aligned} f(v_i) &= i, 1 \leq i \leq n, \\ f(u_1) &= 2n + 1, \\ f(u_j) &= 2n + 4j - 5, 2 \leq j \leq k, \\ f(u_{k+j}) &= 2n + 4k - 3(j - 1), 1 \leq j \leq 2 \text{ and} \\ f(u_{k+2+j}) &= 2n + 4k - 4j - 2, 1 \leq j \leq k - 2. \end{aligned}$$

Clearly, f induces distinct edge labels and it can be verified that f induces a super mean labeling and hence $\langle C_m, K_{1,n}\rangle$, $n = 1, 2, m \geq 3$ and $m \neq 4$ is a super mean graph.

Case (ii) $n = 3, 4$.

Define f on $V(\langle C_m, K_{1,n}\rangle), n = 3, 4, m \geq 3$ and $m \neq 4$ as follows:

Label the vertices of $K_{1,n}, n = 3, 4$ as

$f(v_i) = i, 1 \leq i \leq 2, f(v_3) = 7$ and $f(v_4) = 11$ in the case of $n = 4$.

Label the vertices of C_m as follows:

Subcase(i) When m is odd.

Let $m = 2k + 1$ for some $k \in Z^+$.

$$\begin{aligned} f(u_1) &= 5, \\ f(u_j) &= 2n + 4j - 1, 2 \le j \le k, \\ f(u_{k+j}) &= 2n + 4k - 4j + 6, 1 \le j \le k \text{ and} \\ f(u_{2k+1}) &= 2n + 4. \end{aligned}$$

Subcase(ii) When m is even.
Let $m = 2k$ for some $k \in Z^+$.

$$\begin{aligned} f(u_1) &= 5, \\ f(u_j) &= 2n + 4i - 1, 2 \le j \le k - 1, \\ f(u_{k-1+j}) &= 2n + 4k - 3(j - 1), 1 \le j \le 2, \\ f(u_{k+1+j}) &= 2n + 4k - 4j - 2, 1 \le j \le k - 2 \text{ and} \\ f(u_{2_k}) &= 2n + 4. \end{aligned}$$

Clearly, f induces distinct edge labels and it is easy to check that f generates a super mean labeling and hence $\langle C_m, K_{1,n}\rangle$, $n = 3, 4, m \ge 3$ and $m \ne 4$ is a super mean graph. Thus, $\langle C_m, K_{1,n}\rangle$ is a super mean graph for $n \le 4$ and $m \ge 3$. □

For example, the super mean labelings of $\langle C_8, K_{1,3}\rangle$ and $\langle C_9, K_{1,4}\rangle$ are shown in the following Figure 7.

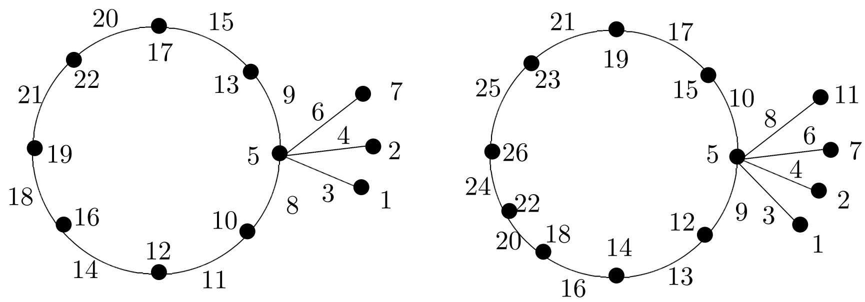


FIGURE 7.

Corollary 2.1. *$S_m(\langle C_m, K_{1,n}\rangle) = 2m + 2n$ for $n \le 4$ and $m \ge 3$.*

Theorem 2.9. *$S_m(\langle C_m, K_{1,n}\rangle) \le 2m + 4n - 7$ for $n \ge 5, m \ge 3$ and $m \ne 4$ and $S_m(\langle C_4, K_{1,n}\rangle) \le 4n + 2$ for $n \ge 5$.*

Proof. Let $V(\langle C_m, K_{1,n}\rangle) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n, v = u_1\}$ and $E(\langle C_m, K_{1,n}\rangle) = \{u_1u_2, u_2u_3, \ldots, u_mu_1, u_1v_i : 1 \le i \le n\}$.
Define f on $V(\langle C_m, K_{1,n}\rangle)$ as follows:

For $n \geq 5$, label the vertices of $K_{1,n}$ as follows:

$$\begin{aligned} f(v_1) &= 2m+2, \\ f(v_i) &= 2m+6+2(i-2), 2 \leq i \leq 4, \\ f(v_i) &= 2m+14+4(i-5), 5 \leq i \leq n-1, \\ f(v_n) &= 2m+4n-7 \text{ and} \\ f(v=u_1) &= 2m. \end{aligned}$$

Label the vertices of C_m as follows:
Case(i) When m is odd.
Let $m = 2k+1, k \in Z^+$.
The vertex labeling f is given by

$$\begin{aligned} f(u_1) &= 2m, \\ f(u_j) &= 2m-4j+5, 2 \leq j \leq k+1, \\ f(u_{k+2}) &= 1 \text{ and} \\ f(u_{k+2+j}) &= 6+4(j-1), 1 \leq j \leq k-1. \end{aligned}$$

Case(ii) When m is even.
Let $m = 2k$ for some $k \in Z^+$.

$$\begin{aligned} f(u_1) &= 2m, \\ f(u_j) &= 2m-4j+3, 2 \leq j \leq k, \\ f(u_{k+1}) &= 1, \\ f(u_{k+1+j}) &= 6+4(j-1), 1 \leq j \leq k-2 \text{ and} \\ f(u_{2_k}) &= 2m-3. \end{aligned}$$

Clearly, the vertex labels and the induced edge labels are distinct and $f(E(G)) \subseteq \{1,2,3,\ldots,2m+4n-7\} - f(V(G))$.

Hence, $S_m(\langle C_m, K_{1,n}\rangle) \leq 2m+4n-7, n \geq 5, m \geq 3$ and $m \neq 4$.

Define f on $V(\langle C_4, K_{1,n}\rangle), n \geq 5$ as follows:

$f(v_1) = 11, f(v_i) = 15+2(i-2), 2 \leq i \leq 4, f(v_i) = 23+4(i-5), 5 \leq i \leq n-1, f(v_n) = 4n+2, f(v=u_1) = 9, f(u_2) = 3, f(u_3) = 1$ and $f(u_4) = 7$. Clearly, the vertex labels and the edge labels are distinct and no vertex and edge labels are equal. Hence, $S_m(\langle C_4, K_{1,n}\rangle) \leq 4n+2$ for $n \geq 5$. □

Theorem 2.10. *$\langle C_m * K_{1,n}\rangle$ is a super mean graph for $n \leq 6$ and $m \geq 3$.*

Proof. Let $V(\langle C_m * K_{1,n}\rangle) = \{u_1, u_2, \ldots, u_m, v_1 = u_1, v_2, \ldots, v_n, v\}$ and $E(\langle C_m * K_{1,n}\rangle) = \{u_1u_2, u_2u_3, \ldots, u_mu_1, u_1v, vv_i : 1 \leq i \leq n-1\}$.

For $m = 4$, the super mean labelings of the graphs $\langle C_4, K_{1,n}\rangle$, $n = 1, 2, 3, 4, 5, 6$ are shown in Figure 8.

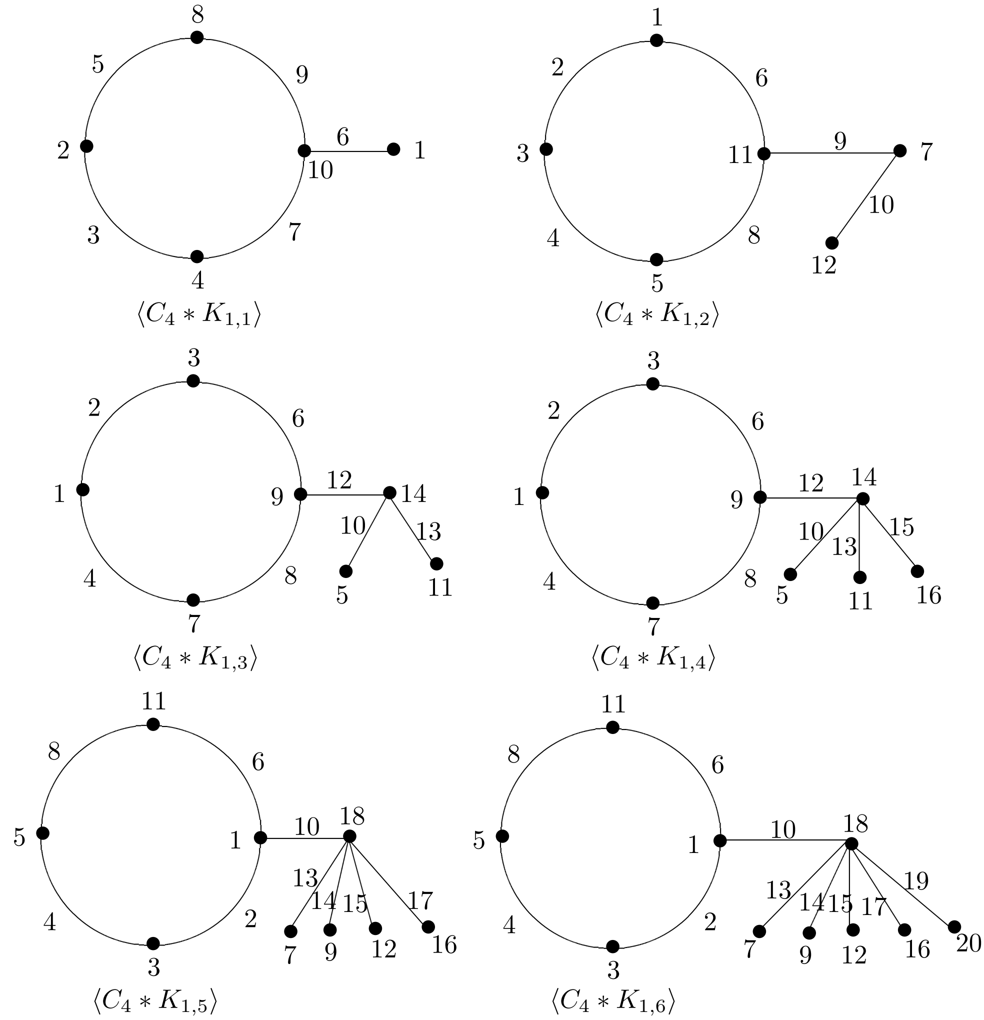


FIGURE 8.

Case (i) $n = 1, 2, 3$.

Define f on $V(\langle C_m * K_{1,n}\rangle), n = 1, 2, 3,\ m \geq 3$ and $m \neq 4$ as follows:

Subcase(i) When m is odd, say $m = 2k + 1, k \in Z^+$.

$$\begin{aligned} f(v) &= 2n - 1, \\ f(u_1 = v_1) &= 2n + 1, \\ f(v_{n+1-i}) &= i, 1 \leq i \leq n - 1, n = 2, 3, \\ f(u_j) &= 2n + 4j - 5, 2 \leq j \leq k + 1 \text{ and} \\ f(u_{k+1+j}) &= 2n + 4k - 4j + 6, 1 \leq j \leq k. \end{aligned}$$

Subcase(ii) When m is even, say $m = 2k, k \in Z^+$.

$$\begin{aligned} f(v) &= 2n-1, \\ f(u_1 = v_1) &= 2n+1, \\ f(v_{n+1-i}) &= i, 1 \leq i \leq n-1, n = 2,3 \\ f(u_j) &= 2n+4j-5, 2 \leq j \leq k, \\ f(u_{k+j}) &= 2n+4k-3(j-1), 1 \leq j \leq 2 \text{ and} \\ f(u_{k+2+j}) &= 2n+4k-4j-2, 1 \leq j \leq k-2. \end{aligned}$$

Clearly, the vertex labeling f induces distinct edge labels and it is easy to check that f is a super mean labeling. Hence, $\langle C_m * K_{1,n} \rangle, n = 1,2,3, m \geq 3$ and $m \neq 4$ is a super mean graph.

Case (ii) $n = 4,5,6$.

Define f on $V(\langle C_m * K_{1,n} \rangle), n = 4,5,6,\ m \geq 3$ and $m \neq 4$ as follows:

Label the vertices of $K_{1,n}, n = 4,5,6$ as given below:

For $n = 4,5$,

$$\begin{aligned} f(v) &= 5, \\ f(v_1 = u_1) &= 2n+3 \text{ and} \\ f(v_{n+1-i}) &= \begin{cases} i, & 1 \leq i \leq 2 \\ 7+3(i-3), & 3 \leq i \leq n-1. \end{cases} \end{aligned}$$

For $n = 6$,

$f(v) = 5, f(v_1 = u_1) = 15, f(v_2) = 12, f(v_3) = 11, f(v_4) = 7, f(v_5) = 2$ and $f(v_6) = 1$.

Label the vertices of C_m as follows:

Subcase(i) When m is odd, take $m = 2k+1, k \in Z^+$.

$$\begin{aligned} f(u_1) &= 2n+3, f(u_2) = 2n+1, \\ f(u_j) &= 2n+4j-6, 3 \leq j \leq k+2 \text{ and} \\ f(u_{k+2+j}) &= 2n+4k-4j+3, 1 \leq j \leq k-1. \end{aligned}$$

Subcase(ii) When m is even, take $m = 2k, k \in Z^+$

$$\begin{aligned} f(u_1) &= 2n+3, \\ f(u_2) &= 2n+1, \\ f(u_j) &= 2n+4j-6, 3 \leq j \leq k, \\ f(u_{k+j}) &= 2n+4k-3+3(j-1), 1 \leq j \leq 2 \text{ and} \\ f(u_{k+2+j}) &= 2n+4k-4j-1, 1 \leq j \leq k-2. \end{aligned}$$

Clearly, the edge labels are distinct. It can be easily verified that f is a super mean labeling. Hence, $\langle C_m * K_{1,n} \rangle, n = 4,5,6, m \geq 3$ and $m \neq 4$ is a super mean graph.

Thus, $\langle C_m * K_{1,n} \rangle$ for $n \leq 6, m \geq 3$ is a super mean graph. □

For example, the super mean labelings of $\langle C_{10} * K_{1,3}\rangle$ and $\langle C_9 * K_{1,6}\rangle$ are shown in the Figure 9.

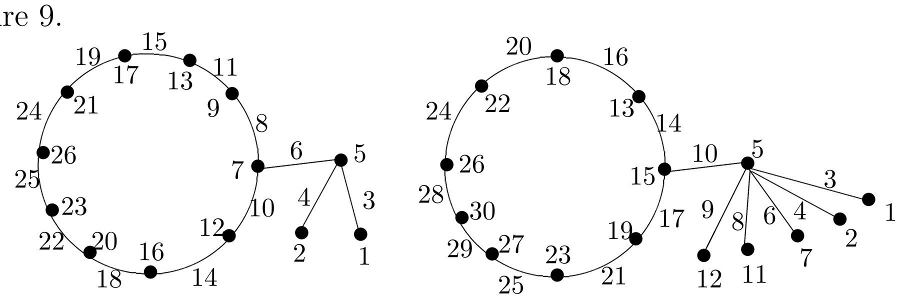


FIGURE 9.

Corollary 2.2. *$S_m(\langle C_m * K_{1,n}\rangle) = 2m + 2n$ for $n \le 6$ and $m \ge 3$.*

Theorem 2.11. *$S_m(\langle C_m * K_{1,n}\rangle) \le 2m + 4n - 11$ for $n \ge 7, m \ge 3$ and $m \ne 4$ and $S_m(\langle C_4 * K_{1,n}\rangle) \le 4n - 2$ for $n \ge 7$.*

Proof. Let $V(\langle C_m * K_{1,n}\rangle) = \{v_i : 1 \le i \le n, v, u_1 = v_1, u_2, \ldots, u_m\}$ and $E(\langle C_m * K_{1,n}\rangle = \{u_1u_2, u_2u_3, \ldots, u_mu_1, u_1v, vv_i, 1 \le i \le n-1\}$.

Define f on $V(\langle C_m * K_{1,n}\rangle)$ as follows:

For $n \ge 7$ label the vertices of $K_{1,n}$ by

$$\begin{aligned} f(v_1 = u_1) &= 2m, \\ f(v_2) &= 2m+1, \\ f(v_3) &= 2m+6, \\ f(v_i) &= 2m+10+2(i-4), 4 \le i \le 6, \\ f(v_i) &= 2m+18+4(i-7), 7 \le i \le n-1 \text{ and} \\ f(v_n) &= 2m+4n-11. \end{aligned}$$

Now label the vertices of C_m as follows:

Case (i) When m is odd, say $m = 2k+1, k \in Z^+$.

$$\begin{aligned} f(u_1) &= 2m, \\ f(u_j) &= 2m-4j+5, 2 \le j \le k+1, \\ f(u_{k+2}) &= 1 \text{ and} \\ f(u_{k+2+j}) &= 6+4(j-1), 1 \le j \le k-1. \end{aligned}$$

Case (ii) When m is even, say $m = 2k, k \in Z^+$.

$$\begin{aligned} f(u_1) &= 2m, \\ f(u_j) &= 2m-4j+3, 2 \le j \le k, \\ f(u_{k+1}) &= 1, \\ f(u_{k+1+j}) &= 6+4(j-1), 1 \le j \le k-2 \text{ and} \\ f(u_{2k}) &= 2m-3. \end{aligned}$$

Clearly, the vertex labels and the induced edge labels are distinct and $f(V(G)) \cap f(E(G)) = \emptyset$. Hence, $S_m(\langle C_m * K_{1,n}\rangle) \leq 2m + 4n - 11$ for $n \geq 7, m \geq 3$ and $m \neq 4$.

Define f on $V(\langle C_4 * K_{1,n}\rangle), n \geq 7$ as follows:

$f(v_1 = u_1) = 9, f(v_2) = 10, f(v_3) = 15, f(v_i) = 19 + 2(i-4), 4 \leq i \leq 6, f(v_i) = 27 + 4(i-7), 7 \leq i \leq n-1, f(v_n) = 4n-2, f(u_2) = 3, f(u_3) = 1$ and $f(u_4) = 7$. Clearly, the vertex labels and edge labels are distinct and $f(E(G)) \subseteq \{1, 2, 3, \ldots, 4n-2\} - f(V(G))$. Hence, $S_m(\langle C_4 * K_{1,n}\rangle) \leq 4n - 2$ for $n \geq 7$. □

Theorem 2.12. *For any graph G, if k is a super mean number of the graph G, then $S_m(tG) \leq kt$.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_p\}$ and $V(tG) = V(G) \cup \{u_{1_i}, u_{2_i}, \ldots, u_{p_i} : 2 \leq i \leq t\}$. Let f be the vertex labeling of G which yields k as the super mean number.

Define g on $V(tG)$ by

$g(u_j) = f(u_j)$ for $1 \leq j \leq p$ and

$g(u_{j_i}) = f(u_j) + (i-1)k, 1 \leq j \leq p, 2 \leq i \leq t.$

Clearly, g induces distinct edge labels and hence $S_m(tG) \leq kt$. □

3. Conclusion

We proved that the graph $\langle C_m, K_{1,n}\rangle$ for $n \leq 4, m \geq 3$ and the graph $\langle C_m * K_{1,n}\rangle$ for $n \leq 6, m \geq 3$ are super mean graphs. We found an upper bound of the super mean number of the graphs $K_{1,n}, n \geq 7, tK_{1,n}$ for $n \geq 5, t > 1, B(p,n)$ for $p > n+1, n \geq 1, \langle C_m, K_{1,n}\rangle$ for $n \geq 5, m \geq 3$ and $\langle C_m * K_{1,n}\rangle$ for $n \geq 7, m \geq 3$. It is also established that $S_m(G) \leq 2^p - 2$ for any graph G. Further, we obtained the super mean number of the graphs K_p for $p \leq 4, K_{1,n}$ for $n \leq 6, tK_{1,4}$ for $t > 1, B(p,n)$ for $p = n, n+1$ and C_n.

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