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THE CONVEXITY GRAPH OF MINIMAL TOTAL DOMINATING FUNCTIONS OF A GRAPH

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Abstract. Let $G = (V, E)$ be a graph without isolated vertices. A function $f : V \to [0, 1]$ is a total dominating function if $\sum_{u \in N(v)} f(u) \geq 1$ for all $v \in V$. A total dominating function f is called a minimal total dominating function (MTDF) if any function $g : V \to [0, 1]$ with $g < f$ is not a total dominating function. If f is an MTDF of G, then $P_f = \{v \in V : f(v) > 0\}$ is the positive set of f and $B_f = \{v \in V : \sum_{u \in N(v)} f(u) = 1\}$ is the boundary set of f. The relation ρ defined on the set $\mathcal{F}$ of all MTDFs of G by $f \rho g$ if $P_f = P_g$ and $B_f = B_g$ is an equivalence relation which partitions $\mathcal{F}$ into a finite number of equivalence classes $X_1, X_2, \ldots, X_t$. The total convexity graph $\mathcal{C}_T(G)$ of G has $\{X_1, X_2, \ldots, X_t\}$ as its vertex set and X_i is adjacent to X_j if there exist $f \in X_i$ and $g \in X_j$ such that any convex combination of f and g is an MTDF of G. In this paper we determine the total convexity graphs of some standard graphs.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to Chartrand and Lesniak [1]. The order and size of G are denoted by n and m respectively.

A *dominating set* of a graph $G = (V, E)$ is a subset S of V such that every vertex of $V - S$ is adjacent to a vertex in S. A dominating set S is called a *minimal dominating set* if no proper subset of S is a dominating set. Domination and its variations in graphs are now well studied. A comprehensive study of the fundamentals of domination is given in the book by Haynes et al. [5]. For surveys of several advanced topics in domination, we refer to the book edited by Haynes et al. [6].

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Let $G = (V, E)$ be a graph without isolated vertices. A dominating set S of V is called a *total dominating set* of G if the induced subgraph $\langle S \rangle$ has no isolated vertices. A total dominating set S is called minimal total dominating set if no proper subset of S is a total dominating set.

A function $f : V \to [0, 1]$ is called a *total dominating function* (TDF) if $f(N(v)) = \sum_{u \in N(v)} f(u) \geq 1$ for all $v \in V$, where $N(v)$ is the neighborhood of v. A TDF f is called a *minimal total dominating function* ($MTDF$) if for all $g < f$, g is not a TDF. Let f be a total dominating function. The *boundary* of f is defined by $B_f = \{v \in V : f(N(v)) = \sum_{u \in N(v)} f(u) = 1\}$. The *positive set* of f is defined by $P_f = \{v \in V : f(v) > 0\}$.

Let A and B be two subsets of V. We say that B *totally dominates* A if every vertex in A is adjacent to at least one vertex in B and in this case we write $B \to_t A$.

Theorem 1.1. [3] *A TDF f is minimal if and only if $B_f \to_t P_f$.*

If f and g are two $TDFs$ of G, then its convex combination $h_\lambda = \lambda f + (1 - \lambda)g$, $0 < \lambda < 1$, is also a TDF of G. However the convex combination of two $MTDFs$ need not be an $MTDF$. The following theorem shows that either all convex combinations of f and g are $MTDFs$ or no convex combination of f and g is an $MTDF$.

Theorem 1.2. [3] *Let f and g be $MTDFs$. Then $h_\lambda = \lambda f + (1 - \lambda)g$ is an $MTDF$ if and only if $B_f \cap B_g \to_t P_f \cup P_g$.*

Definition 1.1. [4] A minimal total dominating function f is called *universal* ($UMTDF$) if the convex combination of f and any other $MTDF$ is also an $MTDF$.

Definition 1.2. Any vertex of degree 1 in a graph is called a leaf and the unique vertex which is adjacent to a leaf is called a support vertex.

Theorem 1.3. [3] *A graph G has a unique $MTDF$ if and only if every vertex of G is adjacent to a support vertex.*

Cockayne et al. [2] introduced the concept of convexity graph with respect to minimal dominating functions of a graph. If G is a graph without isolated vertices, then the definition of convexity graph can be extended to the set of all $MTDFs$ of G and is given in Reji Kumar [7]. Let $\mathcal{F}_T$ denote the set of all $MTDFs$ of G. The equivalence relation ρ defined on $\mathcal{F}_T$ by $f \rho g$ if f and g have the same positive set and same boundary set, gives a partition of $\mathcal{F}_T$ into a finite number of equivalence classes $Y_1, Y_2, \ldots, Y_r$. The *total convexity graph* $\mathcal{C}_T(G)$ is defined as follows: $V(\mathcal{C}_T(G)) = \{Y_1, Y_2, \ldots, Y_r\}$ and Y_i is adjacent to Y_j if there exist $f_i \in Y_i$ and $f_j \in Y_j$ such that any convex combination of f_i and f_j is a minimal total dominating function.

In this paper we determine the total convexity graphs of some standard graphs.

2. Total convexity graphs of some standard graphs

Each vertex of the total convexity graph of a graph is an equivalence class of MTDFs of G. Throughout we identify an equivalence class Y with an MTDF $f \in Y$. We start with the following simple observations.

Observation 2.1. *It follows from Theorem 1.3 that* $\mathcal{C}_T(G) = K_1$ *if and only if every vertex of* G *is adjacent to a support vertex.*

Observation 2.2. *If* v *is a support vertex of a graph* G*, then* $f(v) = 1$ *for all MTDFs* f *of* G *and hence* P_f *contains all support vertices.*

Observation 2.3. *If* f *and* g *are two MTDFs of a graph* G*, then it follows from Theorem 1.2 that* f *and* g *are adjacent in the total convexity graph* $\mathcal{C}_T(G)$ *if and only if* $B_f \cap B_g \rightarrow_t P_f \cup P_g$*. Hence if there exists a total dominating set* D *such that* $B_f \supseteq D$ *for all MTDFs* f*, then* $\mathcal{C}_T(G)$ *is complete and in this case every MTDF of* G *is a UMTDF.*

We now proceed to determine the total convexity graphs of some standard graphs. We first consider paths. For the path $P_n = (v_1, v_2, \ldots, v_n)$, we denote any *MTDF* f by $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_i = f(v_i)$.

Observation 2.4. *Any MTDF of* P_3 *is of the form* $(\lambda, 1, 1-\lambda)$ *where* $0 \leq \lambda \leq 1$*. Hence there exist exactly three equivalence classes of MTDFs, namely,* $f_1 = (1, 1, 0)$*,* $f_2 = (0, 1, 1)$ *and* $f_3 = (\lambda, 1, 1-\lambda)$*,* $0 < \lambda < 1$*. Hence* $\mathcal{C}_T(P_3) = K_3$*. Also the only MTDF of* P_4 *is* $(0, 1, 1, 0)$ *and hence* $\mathcal{C}_T(P_4) = K_1$*.*

Theorem 2.1. *For the path* P_5*, we have* $\mathcal{C}_T(P_5) = K_3$*.*

Proof. Let $P_5 = (v_1, v_2, v_3, v_4, v_5)$ and let $f = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ be any *MTDF* of P_5. Clearly $\lambda_2 = \lambda_4 = 1$, $\lambda_1 = f(v_1) \geq 1 - \lambda_3$ and $\lambda_5 = f(v_5) \geq 1 - \lambda_3$. Since f is minimal, $\lambda_1 = \lambda_5 = 1 - \lambda_3$. Hence $f = (1-\lambda_3, 1, \lambda_3, 1, 1-\lambda_3)$, where $0 \leq \lambda_3 \leq 1$. Further any function f of the above form is a *TDF* with $B_f = V - \{v_3\}$ and since B_f is a total dominating set of P_5, it follows that f is an *MTDF* of P_5. Hence it follows that $\mathcal{C}_T(P_5) = K_3$. □

Theorem 2.2. *For the path* P_6*, we have* $\mathcal{C}_T(P_6) = K_9$*.*

Proof. Let $P_6 = (v_1, v_2, v_3, v_4, v_5, v_6)$. Let $f = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$ be any *MTDF* of P_6. Clearly $\lambda_2 = \lambda_5 = 1$, $\lambda_3 = 1 - \lambda_1$ and $\lambda_4 = 1 - \lambda_6$. Hence $f = (\lambda_1, 1, 1-\lambda_1, 1-\lambda_6, 1, \lambda_6)$ and the boundary set B_f contains the total dominating set $\{v_1, v_2, v_5, v_6\}$. Thus any *MTDF* of P_6 is of the form $f = (\lambda_1, 1, 1-\lambda_1, 1-\lambda_6, 1, \lambda_6)$ where $0 \leq \lambda_1$, $\lambda_6 \leq 1$. Hence there exist exactly nine *MTDFs* for P_6 with distinct positive sets.

For each of the *MTDFs* f, the positive set P_f uniquely determines the boundary set B_f and hence the total number of equivalence classes of *MTDFs* is nine. Since B_f contains the total dominating set $V - \{v_3, v_4\}$ for all *MTDFs* f, it follows that $\mathcal{C}_T(P_6) = K_9$. □

Theorem 2.3. *For the path P_7, we have $\mathcal{C}_T(P_7) = K_7$.*

Proof. Let $P_7 = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ and let $f = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)$ be any $MTDF$ of P_7. Clearly $\lambda_2 = \lambda_6 = 1$, $\lambda_3 = 1 - \lambda_1$, $\lambda_5 = 1 - \lambda_7$ and $\lambda_3 + \lambda_5 \geq 1$.

Now, $\lambda_3 + \lambda_5 = 2 - (\lambda_1 + \lambda_7) \geq 1$ and hence $\lambda_1 + \lambda_7 \leq 1$. Further since f is an $MTDF$, we have $\lambda_4 = 0$. Thus f is of the form $f = (\lambda_1, 1, 1 - \lambda_1, 0, 1 - \lambda_7, 1, \lambda_7)$ where $0 \leq \lambda_1, \lambda_7 \leq 1$ and $\lambda_1 + \lambda_7 \leq 1$. Further any function f of the above form is a TDF of P_7 and since B_f contains the total dominating set $V - \{v_4\}$, it follows that f is an $MTDF$ of P_7. Also if $0 < \lambda_1, \lambda_7 < 1$, then $P_f = V - \{v_4\}$ and $B_f = V$ or $V - \{v_4\}$ according as $\lambda_1 + \lambda_7 = 1$ or $\lambda_1 + \lambda_7 < 1$. Hence the number of equivalence classes of $MTDFs$ is seven. Since B_f contains the total dominating set $V - \{v_4\}$ for all $MTDFs$ f, $\mathcal{C}_T(P_7) = K_7$. □

Theorem 2.4. *For the path P_8, we have $\mathcal{C}_T(P_8) = K_9$.*

Proof. Let $P_8 = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8)$ and let $f = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)$ be any $MTDF$ of P_8. Clearly $\lambda_2 = 1, \lambda_3 = 1 - \lambda_1, \lambda_7 = 1$ and $\lambda_6 = 1 - \lambda_8$. Now since $f(N(v_4)) = \lambda_3 + \lambda_5 = 1 - \lambda_1 + \lambda_5 \geq 1$, we have $\lambda_5 \geq \lambda_1$. Also, $f(N(v_5)) = \lambda_4 + \lambda_6 = \lambda_4 + 1 - \lambda_8 \geq 1$ and hence $\lambda_4 \geq \lambda_8$. By minimality of f, it follows that $\lambda_5 = \lambda_1$ and $\lambda_4 = \lambda_8$. Hence $f = (\lambda_1, 1, 1 - \lambda_1, \lambda_8, \lambda_1, 1 - \lambda_8, 1, \lambda_8)$.

Further any function of the above form is a TDF and the corresponding boundary set B_f contains the total dominating set $\{v_1, v_2, v_4, v_5, v_7, v_8\}$. Thus an $MTDF$ of P_8 is determined by the two real numbers λ_1, λ_8 where $0 \leq \lambda_1, \lambda_8 \leq 1$. Hence the total number of positive sets is nine and each positive set P_f uniquely determines the boundary set B_f. Hence the number of equivalence classes of $MTDFs$ is nine and since B_f contains the total dominating set $\{v_1, v_2, v_4, v_5, v_7, v_8\}$ for all $MTDFs$ f, $\mathcal{C}_T(P_8) = K_9$. □

Theorem 2.5. *For the path P_9, we have $\mathcal{C}_T(P_9) = K_9 + (K_{12} \cup K_{12})$.*

Proof. Let $P_9 = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)$ and let $f = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9)$ be any $MTDF$ of P_9. Clearly $\lambda_2 = \lambda_8 = 1, \lambda_3 = 1 - \lambda_1$ and $\lambda_7 = 1 - \lambda_9$. Now, $f(N(v_4)) = \lambda_3 + \lambda_5 = 1 - \lambda_1 + \lambda_5 \geq 1$ and hence $\lambda_5 \geq \lambda_1$. Also $f(N(v_6)) = \lambda_5 + \lambda_7 = \lambda_5 + 1 - \lambda_9 \geq 1$ and hence $\lambda_5 \geq \lambda_9$. Hence $\lambda_5 \geq t_1 = max\{\lambda_1, \lambda_9\}$ and it follows from the minimality of f that $\lambda_5 = t_1$.

Now $f(N(v_5)) = \lambda_4 + \lambda_6 \geq 1$ and hence $\lambda_6 \geq 1 - \lambda_4$. Since f is minimal, it follows that $\lambda_6 = 1 - \lambda_4$. Hence $f = (\lambda_1, 1, 1 - \lambda_1, \lambda_4, t_1, 1 - \lambda_4, 1 - \lambda_9, 1, \lambda_9)$ where $0 \leq \lambda_1, \lambda_4, \lambda_9 \leq 1$ and $t_1 = max\{\lambda_1, \lambda_9\}$. Now any function f of above form is a TDF of P_9. Further

$B_f \supseteq \{v_1, v_2, v_5, v_8, v_9, v_6\}$ if $t_1 = \lambda_9$ and

$B_f \supseteq \{v_1, v_2, v_5, v_8, v_9, v_4\}$ if $t_1 = \lambda_1$.

Since B_f is a total dominating set of P_9, it follows that f is an $MTDF$ of P_9. Hence any $MTDF$ of P_9 is determined by the three real numbers λ_1, λ_4 and λ_9, where $0 \leq \lambda_1, \lambda_4, \lambda_9 \leq 1$ and hence the number of positive sets is twenty seven. Also

$v_3 \in B_f$ if $\lambda_4 = 0$ and $v_7 \in B_f$ if $\lambda_4 = 1$. Hence the possible boundary sets are given by
$B_1 = \{v_1, v_2, v_5, v_8, v_9, v_4, v_3\}$,
$B_2 = \{v_1, v_2, v_5, v_8, v_9, v_4, v_7\}$,
$B_3 = \{v_1, v_2, v_5, v_8, v_9, v_4\}$,
$B_4 = \{v_1, v_2, v_5, v_8, v_9, v_6, v_3\}$,
$B_5 = \{v_1, v_2, v_5, v_8, v_9, v_6, v_7\}$,
$B_6 = \{v_1, v_2, v_5, v_8, v_9, v_6\}$,
$B_7 = \{v_1, v_2, v_5, v_8, v_9, v_4, v_6, v_3\}$,
$B_8 = \{v_1, v_2, v_5, v_8, v_9, v_4, v_6, v_7\}$ and
$B_9 = \{v_1, v_2, v_5, v_8, v_9, v_4, v_6\}$.
Hence if $0 < \lambda_1 < 1, 0 < \lambda_9 < 1$ and $0 < \lambda_4 < 1$, then for the corresponding $MTDF\ f$ we have $P_f = V$ and the boundary set is any one of the sets B_3 or B_6 or B_9 according as $\lambda_1 > \lambda_9$, $\lambda_9 > \lambda_1$ or $\lambda_1 = \lambda_9$. Thus we get three equivalence classes of $MTDFs$, which are given below.
$f_1 = (\lambda_1, 1, 1 - \lambda_1, \lambda_4, \lambda_1, 1 - \lambda_4, 1 - \lambda_9, 1, \lambda_9)$, where $0 < \lambda_1, \lambda_4, \lambda_9 < 1$,
$P_{f_1} = V, B_{f_1} = B_3$;
$f_2 = (\lambda_1, 1, 1 - \lambda_1, \lambda_4, \lambda_9, 1 - \lambda_4, 1 - \lambda_9, 1, \lambda_9)$, where $0 < \lambda_1, \lambda_4, \lambda_9 < 1$,
$P_{f_2} = V, B_{f_2} = B_6$;
$f_3 = (\lambda_1, 1, 1 - \lambda_1, \lambda_4, \lambda_1, 1 - \lambda_4, 1 - \lambda_1, 1, \lambda_1)$, where $0 < \lambda_1, \lambda_4 < 1$,
$P_{f_3} = V, B_{f_3} = B_9$.

If $0 < \lambda_1 < 1, 0 < \lambda_9 < 1$ and $\lambda_4 = 0$, then $P_f = V - \{v_4\}$ and the boundary set is any one of the sets B_1 or B_4 or B_7 according as $\lambda_1 > \lambda_9$, $\lambda_9 > \lambda_1$ or $\lambda_9 = \lambda_1$. Thus we get three equivalence classes of $MTDFs$, which are given below.
$f_4 = (\lambda_1, 1, 1 - \lambda_1, 0, \lambda_1, 1, 1 - \lambda_9, 1, \lambda_9)$, where $0 < \lambda_1, \lambda_9 < 1$,
$P_{f_4} = V - \{v_4\}, B_{f_4} = B_1$;
$f_5 = (\lambda_1, 1, 1 - \lambda_1, 0, \lambda_9, 1, 1 - \lambda_9, 1, \lambda_9)$, where $0 < \lambda_1, \lambda_9 < 1$,
$P_{f_5} = V - \{v_4\}, B_{f_5} = B_4$;
$f_6 = (\lambda_1, 1, 1 - \lambda_1, 0, \lambda_1, 1, 1 - \lambda_1, 1, \lambda_1)$, where $0 < \lambda_1 < 1$,
$P_{f_6} = V - \{v_4\}, B_{f_6} = B_7$.

Similarly if $0 < \lambda_1 < 1, 0 < \lambda_9 < 1$ and $\lambda_4 = 1$, then $P_f = V - \{v_6\}$ and the boundary set is any one of the sets B_2 or B_5 or B_8 according as $\lambda_1 > \lambda_9$, $\lambda_9 > \lambda_1$ or $\lambda_9 = \lambda_1$. Thus we get three equivalence classes of $MTDFs$, which are given below.
$f_7 = (\lambda_1, 1, 1 - \lambda_1, 1, \lambda_1, 0, 1 - \lambda_9, 1, \lambda_9)$, where $0 < \lambda_1, \lambda_9 < 1$,
$P_{f_7} = V - \{v_6\}, B_{f_7} = B_2$;
$f_8 = (\lambda_1, 1, 1 - \lambda_1, 1, \lambda_9, 0, 1 - \lambda_9, 1, \lambda_9)$, where $0 < \lambda_1, \lambda_9 < 1$,
$P_{f_8} = V - \{v_6\}, B_{f_8} = B_5$;
$f_9 = (\lambda_1, 1, 1 - \lambda_1, 1, \lambda_1, 0, 1 - \lambda_1, 1, \lambda_1)$, where $0 < \lambda_1 < 1$,
$P_{f_9} = V - \{v_6\}, B_{f_9} = B_8$.
In all other cases, the positive set P_f uniquely determines the boundary set B_f. The $MTDFs$ with various positive sets and the corresponding boundary sets are listed below.

$f_{10} = (0, 1, 1, \lambda_4, \lambda_9, 1 - \lambda_4, 1 - \lambda_9, 1, \lambda_9)$, where $0 < \lambda_4, \lambda_9 < 1$,
$P_{f_{10}} = V - \{v_1\}, B_{f_{10}} = B_6$;
$f_{11} = (1, 1, 0, \lambda_4, 1, 1 - \lambda_4, 1 - \lambda_9, 1, \lambda_9)$, where $0 < \lambda_4, \lambda_9 < 1$,
$P_{f_{11}} = V - \{v_3\}, B_{f_{11}} = B_3$;
$f_{12} = (\lambda_1, 1, 1 - \lambda_1, \lambda_4, \lambda_1, 1 - \lambda_4, 1, 1, 0)$, where $0 < \lambda_1, \lambda_4 < 1$,
$P_{f_{12}} = V - \{v_9\}, B_{f_{12}} = B_3$;
$f_{13} = (\lambda_1, 1, 1 - \lambda_1, \lambda_4, 1, 1 - \lambda_4, 0, 1, 1)$, where $0 < \lambda_1, \lambda_4 < 1$,
$P_{f_{13}} = V - \{v_7\}, B_{f_{13}} = B_6$;
$f_{14} = (0, 1, 1, 0, \lambda_9, 1, 1 - \lambda_9, 1, \lambda_9)$, where $0 < \lambda_9 < 1$,
$P_{f_{14}} = V - \{v_1, v_4\}, B_{f_{14}} = B_4$;
$f_{15} = (0, 1, 1, 1, \lambda_9, 0, 1 - \lambda_9, 1, \lambda_9)$, where $0 < \lambda_9 < 1$,
$P_{f_{15}} = V - \{v_1, v_6\}, B_{f_{15}} = B_5$;
$f_{16} = (0, 1, 1, \lambda_4, 0, 1 - \lambda_4, 1, 1, 0)$, where $0 < \lambda_4 < 1$,
$P_{f_{16}} = V - \{v_1, v_5, v_9\}, B_{f_{16}} = B_9$;
$f_{17} = (0, 1, 1, \lambda_4, 1, 1 - \lambda_4, 0, 1, 1)$, where $0 < \lambda_4 < 1$,
$P_{f_{17}} = V - \{v_1, v_7\}, B_{f_{17}} = B_6$;
$f_{18} = (1, 1, 0, 0, 1, 1, 1 - \lambda_9, 1, \lambda_9)$, where $0 < \lambda_9 < 1$,
$P_{f_{18}} = V - \{v_3, v_4\}, B_{f_{18}} = B_1$;
$f_{19} = (1, 1, 0, 1, 1, 0, 1 - \lambda_9, 1, \lambda_9)$, where $0 < \lambda_9 < 1$,
$P_{f_{19}} = V - \{v_3, v_6\}, B_{f_{19}} = B_2$;
$f_{20} = (1, 1, 0, \lambda_4, 1, 1 - \lambda_4, 1, 1, 0)$, where $0 < \lambda_4 < 1$,
$P_{f_{20}} = V - \{v_3, v_9\}, B_{f_{20}} = B_3$;
$f_{21} = (1, 1, 0, \lambda_4, 1, 1 - \lambda_4, 0, 1, 1)$, where $0 < \lambda_4 < 1$,
$P_{f_{21}} = V - \{v_3, v_7\}, B_{f_{21}} = B_9$;
$f_{22} = (\lambda_1, 1, 1 - \lambda_1, 0, \lambda_1, 1, 1, 1, 0)$, where $0 < \lambda_1 < 1$,
$P_{f_{22}} = V - \{v_4, v_9\}, B_{f_{22}} = B_1$;
$f_{23} = (\lambda_1, 1, 1 - \lambda_1, 0, 1, 1, 0, 1, 1)$, where $0 < \lambda_1 < 1$,
$P_{f_{23}} = V - \{v_4, v_7\}, B_{f_{23}} = B_4$;
$f_{24} = (\lambda_1, 1, 1 - \lambda_1, 1, \lambda_1, 0, 1, 1, 0)$, where $0 < \lambda_1 < 1$,
$P_{f_{24}} = V - \{v_6, v_9\}, B_{f_{24}} = B_2$;
$f_{25} = (\lambda_1, 1, 1 - \lambda_1, 1, 1, 0, 0, 1, 1)$, where $0 < \lambda_1 < 1$,
$P_{f_{25}} = V - \{v_6, v_7\}, B_{f_{25}} = B_5$;
$f_{26} = (0, 1, 1, 0, 0, 1, 1, 1, 0), P_{f_{26}} = V - \{v_1, v_4, v_9, v_5\}, B_{f_{26}} = B_7$;
$f_{27} = (0, 1, 1, 0, 1, 1, 0, 1, 1), P_{f_{27}} = V - \{v_1, v_4, v_7\}, B_{f_{27}} = B_4$;
$f_{28} = (0, 1, 1, 1, 0, 0, 1, 1, 0), P_{f_{28}} = V - \{v_1, v_6, v_9, v_5\}, B_{f_{28}} = B_8$;
$f_{29} = (0, 1, 1, 1, 1, 0, 0, 1, 1), P_{f_{29}} = V - \{v_1, v_6, v_7\}, B_{f_{29}} = B_5$;
$f_{30} = (1, 1, 0, 0, 1, 1, 1, 1, 0), P_{f_{30}} = V - \{v_3, v_4, v_9\}, B_{f_{30}} = B_1$;
$f_{31} = (1, 1, 0, 0, 1, 1, 0, 1, 1), P_{f_{31}} = V - \{v_3, v_4, v_7\}, B_{f_{31}} = B_7$;
$f_{32} = (1, 1, 0, 1, 1, 0, 1, 1, 0), P_{f_{32}} = V - \{v_3, v_6, v_9\}, B_{f_{32}} = B_2$;
$f_{33} = (1, 1, 0, 1, 1, 0, 0, 1, 1), P_{f_{33}} = V - \{v_3, v_6, v_7\}, B_{f_{33}} = B_8$.

Hence there exist exactly thirty three equivalence classes of $MTDFs$ for P_9. Thus $|V(\mathcal{C}_T(P_9)| = 33$.

$$\text{Now let } V_1 = \{f \in V(\mathcal{C}_T(P_9) : B_f = B_1 \text{ or } B_2 \text{ or } B_3\},$$
$$V_2 = \{f \in V(\mathcal{C}_T(P_9) : B_f = B_4 \text{ or } B_5 \text{ or } B_6\} \text{ and}$$
$$V_3 = \{f \in V(\mathcal{C}_T(P_9) : B_f = B_7 \text{ or } B_8 \text{ or } B_9\}.$$

Then $|V_1| = 12, |V_2| = 12$ and $|V_3| = 9$.

For any $f \in V_1$, the boundary set B_f contains the total dominating set B_3 and hence $\langle V_1\rangle = K_{12}$. Similarly $\langle V_2\rangle = K_{12}$ and $\langle V_3\rangle = K_9$. Also $B_3 \cap B_9 = B_3$ and $B_6 \cap B_9 = B_6$. Hence every vertex of V_1 is adjacent to every vertex of V_3 and every vertex of V_2 is adjacent to every vertex of V_3. Also if $f \in V_1$ and $g \in V_2$, then $B_f \cap B_g = \{v_1, v_2, v_5, v_8, v_9\}$. However $P_f \cup P_g$ contains v_5 and v_5 is not adjacent to any vertex in $B_f \cap B_g$. Hence f and g are not adjacent. Thus $\mathcal{C}_T(P_9) = K_9 + (K_{12} \cup K_{12})$. □

Theorem 2.6. *Let G be the total convexity graph of the path P_{12}. Then $V(G)$ can be partitioned into nine subsets $V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8$ and V_9 such that $|V_1| = 9, |V_2| = |V_3| = |V_5| = |V_6| = 16, |V_4| = |V_7| = |V_8| = |V_9| = 12$, $\langle V_1 \cup V_2 \cup V_4 \cup V_8\rangle = K_{49}$, $\langle V_1 \cup V_3 \cup V_4 \cup V_9\rangle = K_{49}$, $\langle V_1 \cup V_5 \cup V_7 \cup V_8\rangle = K_{49}$ and $\langle V_1 \cup V_6 \cup V_7 \cup V_9\rangle = K_{49}$.*

Proof. Let $P_{12} = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12})$ and let $f = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11}, \lambda_{12})$ be any $MTDF$ of P_{12}. Clearly $\lambda_2 = \lambda_{11} = 1$, $\lambda_3 = 1 - \lambda_1$ and $\lambda_{10} = 1 - \lambda_{12}$. Now $f(N(v_5)) = \lambda_4 + \lambda_6 \geq 1$ and hence $\lambda_6 \geq 1 - \lambda_4$. Since f is minimal, it follows that $\lambda_6 = 1 - \lambda_4$. Now $f(N(v_7)) = 1 - \lambda_4 + \lambda_8 \geq 1$ and hence $\lambda_8 \geq \lambda_4$. Also $f(N(v_9)) = \lambda_8 + 1 - \lambda_{12} \geq 1$ and hence $\lambda_8 \geq \lambda_{12}$. Hence $\lambda_8 \geq t_2 = max(\lambda_4, \lambda_{12})$ and it follows from the minimality of f that $\lambda_8 = t_2$.

Now $f(N(v_8)) = \lambda_7 + \lambda_9 \geq 1$ and hence $\lambda_7 \geq 1 - \lambda_9$. Since f is minimal, it follows that $\lambda_7 = 1 - \lambda_9$. Now $f(N(v_6)) = \lambda_5 + 1 - \lambda_9 \geq 1$ and hence $\lambda_5 \geq \lambda_9$. Also $f(N(v_4)) = 1 - \lambda_1 + \lambda_5 \geq 1$ and hence $\lambda_5 \geq \lambda_1$. Hence $\lambda_5 \geq t_1 = max\{\lambda_1, \lambda_9\}$ and it follows from minimality of f that $\lambda_5 = t_1$. Hence $f = (\lambda_1, 1, 1 - \lambda_1, \lambda_4, t_1, 1 - \lambda_4, 1 - \lambda_9, t_2, \lambda_9, 1 - \lambda_{12}, 1, \lambda_{12})$, where $0 \leq \lambda_1, \lambda_4, \lambda_9, \lambda_{12} \leq 1$. Also any function f of the above form is a TDF of P_{12}. Further

$B_f \supseteq S \cup \{v_4, v_7\}$ if $t_1 = \lambda_1$ and $t_2 = \lambda_4$,
$B_f \supseteq S \cup \{v_4, v_9\}$ if $t_1 = \lambda_1$ and $t_2 = \lambda_{12}$,
$B_f \supseteq S \cup \{v_6, v_7\}$ if $t_1 = \lambda_9$ and $t_2 = \lambda_4$
and $B_f \supseteq S \cup \{v_6, v_9\}$ if $t_1 = \lambda_9$ and $t_2 = \lambda_{12}$, where $S = \{v_1, v_2, v_5, v_8, v_{11}, v_{12}\}$.

Also $v_3 \in B_f$ when $\lambda_4 = 0$ and $v_{10} \in B_f$ when $\lambda_9 = 0$. Since B_f is a total dominating set of P_{12}, it follows that f is an $MTDF$ of P_{12}. Hence any $MTDF$ of P_{12} is determined by the four real numbers $\lambda_1, \lambda_4, \lambda_9$ and λ_{12}, where $0 \leq \lambda_1, \lambda_4, \lambda_9, \lambda_{12} \leq 1$ and hence the total number of positive sets is eighty one.

If $0 < \lambda_1, \lambda_4, \lambda_9, \lambda_{12} < 1$, then for the corresponding $MTDFs$ we have $P_f = V$ and there exist nine possible boundary sets $B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8$ and B_9 depending on the values of t_1 and t_2 as given below.

If $t_1 = \lambda_1$ and $t_2 = \lambda_4$, then $B_1 = S \cup \{v_4, v_7\}$.
If $t_1 = \lambda_1$ and $t_2 = \lambda_{12}$, then $B_2 = S \cup \{v_4, v_9\}$.
If $t_1 = \lambda_1$ and $t_2 = \lambda_4 = \lambda_{12}$, then $B_3 = S \cup \{v_4, v_7, v_9\}$.
If $t_1 = \lambda_9$ and $t_2 = \lambda_4$, then $B_4 = S \cup \{v_6, v_7\}$.
If $t_1 = \lambda_9$ and $t_2 = \lambda_{12}$, then $B_5 = S \cup \{v_6, v_9\}$.
If $t_1 = \lambda_9$ and $t_2 = \lambda_4 = \lambda_{12}$, then $B_6 = S \cup \{v_6, v_7, v_9\}$.
If $t_1 = \lambda_1 = \lambda_9$ and $t_2 = \lambda_4$, then $B_7 = S \cup \{v_4, v_6, v_7\}$.
If $t_1 = \lambda_1 = \lambda_9$ and $t_2 = \lambda_{12}$, then $B_8 = S \cup \{v_4, v_6, v_9\}$.
If $t_1 = \lambda_1 = \lambda_9$ and $t_2 = \lambda_4 = \lambda_{12}$, then $B_9 = S \cup \{v_4, v_6, v_7, v_9\}$. Thus we get nine equivalence classes of $MTDFs$ with $P_f = V$.

If $0 < \lambda_4, \lambda_{12} < 1$ and at least one of the values λ_1, λ_9 is equal to zero or one, then for the corresponding $MTDFs$ we have eight positive sets and there exist exactly three possible boundary sets for each positive set, according as $\lambda_4 > \lambda_{12}, \lambda_4 < \lambda_{12}$ or $\lambda_4 = \lambda_{12}$. The eight possible positive sets and the corresponding boundary sets are given below.

If $P_f = V - \{v_3\}$, then $B_f = B_1$ or B_2 or B_3.
If $P_f = V - \{v_1\}$, then $B_f = B_4$ or B_5 or B_6.
If $P_f = V - \{v_7\}$, then $B_f = B_4$ or B_5 or B_6.
If $P_f = V - \{v_9\}$, then $B_f = B_1 \cup \{v_{10}\}$ or $B_2 \cup \{v_{10}\}$ or $B_3 \cup \{v_{10}\}$.
If $P_f = V - \{v_1, v_7\}$, then $B_f = B_4$ or B_5 or B_6.
If $P_f = V - \{v_3, v_9\}$, then $B_f = B_1 \cup \{v_{10}\}$ or $B_2 \cup \{v_{10}\}$ or $B_3 \cup \{v_{10}\}$.
If $P_f = V - \{v_3, v_7\}$, then $B_f = B_7$ or B_8 or B_9.
If $P_f = V - \{v_1, v_9, v_5\}$, then $B_f = B_7 \cup \{v_{10}\}$ or $B_8 \cup \{v_{10}\}$ or $B_9 \cup \{v_{10}\}$.
Thus we get 24 equivalence classes of $MTDFs$.

Similarly if $0 < \lambda_1, \lambda_9 < 1$ and at least one of the values λ_4, λ_{12} is equal to zero or one, then for the corresponding $MTDFs$ we have eight positive sets and three possible boundary sets for each positive set according as $\lambda_1 > \lambda_9, \lambda_1 < \lambda_9$ or $\lambda_1 = \lambda_9$.

For the remaining sixty four positive sets, the positive set uniquely determines the boundary set. Hence there exist exactly 121 equivalence classes of $MTDFs$ for P_{12}.

Let

$$
\begin{aligned}
V_1 &= \{f \in V(\mathcal{C}_T(P_{12})) : B_f \supseteq B_9\},\\
V_2 &= \{f \in V(\mathcal{C}_T(P_{12})) : B_f \supseteq B_1\},\\
V_3 &= \{f \in V(\mathcal{C}_T(P_{12})) : B_f \supseteq B_2\},\\
V_4 &= \{f \in V(\mathcal{C}_T(P_{12})) : B_f \supseteq B_3\},\\
V_5 &= \{f \in V(\mathcal{C}_T(P_{12})) : B_f \supseteq B_4\},\\
V_6 &= \{f \in V(\mathcal{C}_T(P_{12})) : B_f \supseteq B_5\},\\
V_7 &= \{f \in V(\mathcal{C}_T(P_{12})) : B_f \supseteq B_6\},\\
V_8 &= \{f \in V(\mathcal{C}_T(P_{12})) : B_f \supseteq B_7\} \quad \text{and}\\
V_9 &= \{f \in V(\mathcal{C}_T(P_{12})) : B_f \supseteq B_8\}.
\end{aligned}
$$

Now $B_f \supseteq B_9$ only when $t_1 = \lambda_1 = \lambda_9$ and $t_2 = \lambda_4 = \lambda_{12}$. Total number of positive sets when $\lambda_1 = \lambda_9$ and $\lambda_4 = \lambda_{12}$ is 9. Therefore $|V_1| = 9$. Also $B_f \supseteq B_1$ only when $t_1 = \lambda_1$ and $t_2 = \lambda_4$. Hence $\lambda_1 > \lambda_9$ and $\lambda_4 > \lambda_{12}$. Since $\lambda_1 > \lambda_9$, we have four possible choices for the pair (λ_1, λ_9), namely, $(1, 0)$ $(1, \lambda_9), (\lambda_1, 0)$ and (λ_1, λ_9). Similarly we have four possible choices for the pair $(\lambda_4, \lambda_{12})$. Hence the total number of positive sets when $\lambda_1 > \lambda_9$ and $\lambda_4 > \lambda_{12}$ is 16. Therefore $|V_2| = 16$. Similarly $|V_3| = |V_5| = |V_6| = 16$. Now, $B_f \supseteq B_3$ only when $t_1 = \lambda_1$ and $t_2 = \lambda_4 = \lambda_{12}$. Hence $\lambda_1 > \lambda_4$ and $\lambda_4 = \lambda_{12}$. Since $\lambda_4 = \lambda_{12}$, we have three possible choices for he pair $(\lambda_4, \lambda_{12})$, namely, $(1, 1), (0, 0)$ and $(\lambda_4, \lambda_{12})$. Hence the total number of positive sets when $\lambda_1 > \lambda_9$ and $\lambda_4 = \lambda_{12}$ is 12. Therefore $|V_4| = 12$. Similarly $|V_7| = |V_8| = |V_9| = 12$.

Now the boundary sets of V_1, V_2, V_4 and V_8 contain the total dominating set B_1 and hence $\langle V_1 \cup V_2 \cup V_4 \cup V_8 \rangle = K_{49}$. Also the boundary sets of V_1, V_3, V_4 and V_9 contain the total dominating set B_2 and hence $\langle V_1 \cup V_3 \cup V_4 \cup V_9 \rangle = K_{49}$. The boundary sets of V_1, V_5, V_7 and V_8 contain the total dominating set B_4 and hence $\langle V_1 \cup V_5 \cup V_7 \cup V_8 \rangle = K_{49}$. The boundary sets of V_1, V_6, V_7 and V_9 contain the total dominating set B_5 and hence $\langle V_1 \cup V_6 \cup V_7 \cup V_9 \rangle = K_{49}$. □

Theorem 2.7. *The total convexity graph of K_3 is the graph with exactly one cut vertex and having three blocks each isomorphic to K_3.*

Proof. Let $V(K_3) = \{v_1, v_2, v_3\}$ and let f be any $MTDF$ of K_3. Then $f(v_1)+f(v_2) \geq 1$, $f(v_2) + f(v_3) \geq 1$ and $f(v_3) + f(v_1) \geq 1$. Adding these inequalities we get $|f| \geq \frac{3}{2}$.

We claim that $|B_f| \geq 2$. Suppose $|B_f| = 1$ and let $B_f = \{v_1\}$. Then $f(v_2)+f(v_3) = 1$. Since $|f| \geq \frac{3}{2}$, it follows that $v_1 \in P_f$ and hence B_f does not totally dominate P_f, which is a contradiction. Thus $|B_f| \geq 2$.

We now claim that if $A \subseteq V(K_3)$ and $|A| = 2$, then there exist exactly two equivalence classes of $MTDFs$ with $B_f = A$. Let $A = \{v_1, v_2\}$. Let f be any $MTDF$ of K_3 with $B_f = A$. Then $f(v_2)+f(v_3) = 1$ and $f(v_3)+f(v_1) = 1$. Hence $f(v_2) = f(v_1)$ and since $|f| \geq \frac{3}{2}$, it follows that $\{v_1, v_2\} \subseteq P_f$. Hence $P_f = \{v_1, v_2\}$ or $\{v_1, v_2, v_3\}$.

Now, $f_1 = (1, 1, 0)$ is an $MTDF$ of K_3 with $P_{f_1} = \{v_1, v_2\}$ and $B_{f_1} = \{v_1, v_2\}$. Also $f_2 = (\frac{3}{4}, \frac{3}{4}, \frac{1}{4})$ is an $MTDF$ of K_3 with $P_{f_2} = V$ and $B_{f_2} = \{v_1, v_2\}$. Thus we have exactly two equivalence classes of $MTDFs$ with $B_f = A$.

Also when $B_f = V$, we have $f(v_1)+f(v_2) = f(v_2)+f(v_3) = f(v_3)+f(v_1) = 1$. Hence $f(v_1) = f(v_2) = f(v_3)$, so that $P_f = V$. Hence there exists exactly one equivalence class of $MTDF$ with $P_f = V$. Thus there exist exactly seven equivalence classes of $MTDFs$, namely $f_1, f_2, f_3, f_4, f_5, f_6$ and f_7 whose positive sets and boundary sets are given below.

$$
\begin{array}{ll}
P_{f_1} = \{v_1, v_2\}, & B_{f_1} = \{v_1, v_2\}; \\
P_{f_2} = V, & B_{f_2} = \{v_1, v_2\}; \\
P_{f_3} = \{v_2, v_3\}, & B_{f_3} = \{v_2, v_3\}; \\
P_{f_4} = V, & B_{f_4} = \{v_2, v_3\}; \\
P_{f_5} = \{v_3, v_1\}, & B_{f_5} = \{v_3, v_1\};
\end{array}
$$

$$P_{f_6} = V, \qquad B_{f_6} = \{v_3, v_1\};$$
$$P_{f_7} = V, \qquad B_{f_7} = V \quad \text{respectively.}$$

Let $V_1 = \{f_1, f_2\}, V_2 = \{f_3, f_4\}, V_3 = \{f_5, f_6\}$ and $V_4 = \{f_7\}$.

Clearly each of the induced subgraphs $\langle V_1 \rangle, \langle V_2 \rangle$ and $\langle V_3 \rangle$ is K_2 and f_7 is adjacent to all the remaining vertices. Further any element of V_i is not adjacent to any element of V_j for any i, j with $i \neq j$, $1 \leq i, j \leq 3$. □

Theorem 2.8. *Let G be the total convexity graph of K_4. Then G has the following properties.*

(i) $|V(G)| = 33$.
(ii) *The graph G has a unique vertex v with deg $v = 32$.*
(iii) *The set $V(G) - \{v\}$ can be partitioned into seven subsets $A, V_{12}, V_{13}, V_{14}, V_{23}, V_{24}$ and V_{34} such that $\langle A \rangle = K_8$ and $\langle V_{ij} \rangle = K_4$.*
(iv) *The set A can be partitioned into four subsets A_1, A_2, A_3 and A_4 such that $|A_i| = 2$ for each i and $\langle V_{ij} \cup A_k \cup A_l \rangle = K_8$, where $1 \leq i < j \leq 4, 1 \leq k, l \leq 4$ and i, j, k, l are all distinct.*

Proof. Let $V(K_4) = \{v_1, v_2, v_3, v_4\}$ and let f be any $MTDF$ of K_4. Then for any v_i, $1 \leq i \leq 4$, we have $\sum\limits_{u \in N(v_i)} f(u) \geq 1$. Adding these four inequalities we get $3|f| \geq 4$ and hence $|f| \geq \frac{4}{3}$.

We claim that $|B_f| \geq 2$. Suppose $|B_f| = 1$ and let $B_f = \{v_1\}$. Then $f(v_2) + f(v_3) + f(v_4) = 1$. Since $|f| \geq \frac{4}{3}$, it follows that $v_1 \in P_f$ and hence B_f does not totally dominate P_f, which is a contradiction. Thus $|B_f| \geq 2$.

We now claim that if $X \subseteq V(K_4)$ and $|X| = 2$, then there exist exactly four equivalence classes of $MTDFs$ with $B_f = X$. Let $X = \{v_1, v_2\}$. Let f be any $MTDF$ with $B_f = X$. Then $f(v_2) + f(v_3) + f(v_4) = 1$ and $f(v_1) + f(v_3) + f(v_4) = 1$. Hence $f(v_2) = f(v_1)$ and since $|f| \geq \frac{4}{3}$, it follows that $f(v_1) > 0$. Thus $\{v_1, v_2\} \subseteq P_f$. Hence $P_f = \{v_1, v_2\}$ or $\{v_1, v_2, v_3\}$ or $\{v_1, v_2, v_4\}$ or V. Now,

$f_1 = (1, 1, 0, 0)$ is an $MTDF$ with $P_{f_1} = \{v_1, v_2\}$, $B_{f_1} = \{v_1, v_2\}$; $f_2 = (\frac{3}{4}, \frac{3}{4}, \frac{1}{4}, 0)$ is an $MTDF$ with $P_{f_2} = \{v_1, v_2, v_3\}$, $B_{f_2} = \{v_1, v_2\}$; $f_3 = (\frac{3}{4}, \frac{3}{4}, 0, \frac{1}{4})$ is an $MTDF$ with $P_{f_3} = \{v_1, v_2, v_4\}$, $B_{f_3} = \{v_1, v_2\}$ and $f_4 = (\frac{3}{4}, \frac{3}{4}, \frac{1}{8}, \frac{1}{8})$ is an $MTDF$ with $P_{f_4} = \{v_1, v_2, v_3, v_4\}$, $B_{f_4} = \{v_1, v_2\}$.

Thus for each subset X of $V(K_4)$ with $|X| = 2$, there are exactly four equivalence classes of $MTDFs$ with boundary set X and hence there exist twenty four such equivalence classes.

We now claim that if $X \subseteq V(K_4)$ and $|X| = 3$, then there exist exactly two equivalence classes of $MTDFs$ with $B_f = X$. Let $X = \{v_1, v_2, v_3\}$. Let f be any $MTDF$ with $B_f = X$. Then

$f(v_2) + f(v_3) + f(v_4) = 1$,
$f(v_1) + f(v_3) + f(v_4) = 1$ and
$f(v_1) + f(v_2) + f(v_4) = 1$.

Hence $f(v_2) = f(v_1) = f(v_3)$ and P_f contains $\{v_1, v_2, v_3\}$. Hence $P_f = \{v_1, v_2, v_3\}$ or V.

Now, $f_5 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ is an $MTDF$ with $B_{f_5} = P_{f_5} = \{v_1, v_2, v_3\}$. Also $f_6 = (\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5})$ is an $MTDF$ with $B_{f_6} = \{v_1, v_2, v_3\}$ and $P_{f_6} = V$.

Thus for each subset X of $V(K_4)$ with $|X| = 3$, there are exactly two equivalence classes of $MTDFs$ with boundary set X and hence there exist eight such equivalence classes.

Now we claim that when $B_f = V$, there exists exactly one $MTDF$ f with $P_f = V$. Since $B_f = V$, we have $f(N(v_1)) = f(N(v_2)) = f(N(v_3)) = f(N(v_4)) = 1$. Hence $f(v_1) = f(v_2) = f(v_3) = f(v_4) = \frac{1}{3}$.

Thus we have 24 equivalence classes with $|B_f| = 2$, 8 equivalence classes with $|B_f| = 3$ and 1 equivalence class with $|B_f| = 4$. Hence we have 33 equivalence classes of $MTDFs$ for the complete graph K_4 and $|V(G)| = 33$.

Now for any subset $\{v_i, v_j\} \subseteq \{v_1, v_2, v_3, v_4\}$ with $i < j$, let V_{ij} denote the set of all equivalence classes of $MTDFs$ of K_4 with boundary set $\{v_i, v_j\}$. Then $|V_{ij}| = 4$ and since boundary set B_f of any $MTDF$ $f \in V_{ij}$ contains the total dominating set $\{v_i, v_j\}$, it follows from Observation 2.3 that $\langle V_{ij} \rangle = K_4$.

Now, let A_1, A_2, A_3 and A_4 denote the set of all equivalence classes of $MTDFs$ of G with boundary set $V - \{v_1\}, V - \{v_2\}$, $V - \{v_3\}$ and $V - \{v_4\}$ respectively. Then $|A_i| = 2$. Let $A = A_1 \cup A_2 \cup A_3 \cup A_4$. If $f, g \in A$ then $|B_f \cap B_g| = 2$ and $B_f \cap B_g$ is a total dominating set of K_4. Hence $\langle A \rangle = K_8$. Also $\langle V_{12} \cup A_3 \cup A_4 \rangle$ is complete since B_f contains the dominating set $\{v_1, v_2\}$ for all $f \in V_{12} \cup A_3 \cup A_4$. Similarly $\langle V_{ij} \cup A_k \cup A_l \rangle$ where $i < j$ and i, j, k, l are all distinct is a complete graph. Now if f and g are in two different sets of the form V_{ij}, then $B_f \cap B_g$ is either singleton or empty and hence f and g are not adjacent. Now, the unique $MTDF$ f with $B_f = V$ is adjacent to any other $MTDF$ g since $|B_f \cap B_g| \geq 2$ and $B_f \cap B_g$ is a total dominating set of K_4. This completes the proof of the theorem. □

Theorem 2.9. $\mathcal{C}_T(C_4) = K_9$.

Proof. Let $C_4 = (v_1, v_2, v_3, v_4, v_1)$ and let $f = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be any $MTDF$ of C_4. Since f is minimal, $\lambda_3 = 1 - \lambda_1$ and $\lambda_2 = 1 - \lambda_4$. Hence $f = (\lambda_1, 1 - \lambda_4, 1 - \lambda_1, \lambda_4)$ where $0 \leq \lambda_1, \lambda_4 \leq 1$. Further any function f of the above form is a TDF with $B_f = V$. Thus any $MTDF$ of C_4 is of the form $(\lambda_1, 1 - \lambda_4, 1 - \lambda_1, \lambda_4)$ where $0 \leq \lambda_1, \lambda_4 \leq 1$. Hence there exist exactly nine $MTDFs$ with distinct positive sets. Since $B_f = V$ for all $MTDFs$ f, it follows that $\mathcal{C}_T(C_4) = K_9$. □

Theorem 2.10. *Let H be the 2-corona of a connected graph G of order n, where the 2-corona of G is obtained by attaching a path of length 2 at each vertex of G. Then $\mathcal{C}_T(H) = K_{3^n}$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, $S_i = \{u_1, u_2, \ldots, u_n\}$ be the set of all supports of H and $L_i = \{w_1, w_2, \ldots, w_n\}$ be the set of all leaves of H.
Let f be any $MTDF$ of H. Clearly

$f(v_i) = \lambda_i$,
$f(u_i) = 1$ and
$f(w_i) = 1 - \lambda_i$, where $0 \leq \lambda_i \leq 1$ and $1 \leq i \leq n$.
If $\lambda_i = 1$, then $v_i \in P_f$ and $w_i \notin P_f$.
If $\lambda_i = 0$, then $v_i \notin P_f$ and $w_i \in P_f$.
If $0 < \lambda_i < 1$, then $\{v_i, w_i\} \subseteq P_f$.
Hence if $|P_f \cap V(G)| = r$, then $P_f = (P_f \cap V(G)) \cup \{w_i : \lambda_i < 1\} \cup S_i$. Hence there exist 3^n $MTDFs$ f having different positive sets.

We now claim that if f and g are two $MTDFs$ of H with $P_f = P_g$, then $B_f = B_g$. If $P_f \cap V(G) = P_g \cap V(G) = \emptyset$, then $B_f = B_g = V(H)$. If $P_f \cap V(G) = P_g \cap V(G) = V(G)$, then $B_f = B_g = S_i \cup L_i$. Now, suppose $P_f \cap V(G)$ is a proper nonempty subset of $V(G)$. Let $v \in B_f$. If $v \in L_i \cup S_i$, then $v \in B_g$. If $v \in V(G)$, then $N(v) \cap V(G) \cap P_f = \emptyset$. Hence $N(v) \cap V(G) \cap P_g = \emptyset$, so that $v \in B_g$. Thus $B_f \subseteq B_g$ and by a similar argument we get $B_g \subseteq B_f$. Hence $B_f = B_g$.

Thus the number of equivalence classes of $MTDFs$ of H is 3^n. Since B_f contains the total dominating set $S_i \cup L_i$, it follows from Observation 2.3 that $\mathcal{C}_T(H) = K_{3^n}$. □

Theorem 2.11. $\mathcal{C}_T(K_{r,s}) = K_{(2^r-1)\times(2^s-1)}$.

Proof. Let $V_1 = \{v_1, v_2, \ldots, v_r\}$ and $V_2 = \{u_1, u_2, \ldots, u_s\}$ be the bipartition of $K_{r,s}$. Let f be any $MTDF$ of $K_{r,s}$. Then
$f(N(v_i)) = f(u_1) + f(u_2) + f(u_3) + \cdots + f(u_s) = 1$ and
$f(N(u_i)) = f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_r) = 1$, where $0 \leq f(u_i), f(v_i) \leq 1$.
Hence $P_f = X \cup Y$, where X is any nonempty subset of V_1 and Y is any nonempty subset of V_2. Thus the total number of positive sets is $(2^s - 1) \times (2^r - 1)$ and since $B_f = V$, for any $MTDF$ f, it follows that $\mathcal{C}_T(K_{r,s})$ is complete. Thus $\mathcal{C}_T(K_{r,s}) = K_{(2^r-1)\times(2^s-1)}$. □

3. Conclusion and scope

In this paper we have determined the total convexity graphs of some standard graphs. Even finding the number of elements in the total convexity graph of P_n for arbitrary n seems to be a difficult problem. One can also attempt to find the total convexity graphs of other families of graphs such as cycles C_n where $n \geq 5$, Petersen graph, hypercubes etc. Further one can investigate necessary/sufficient conditions for a given graph G to be the total convexity graph of some graph H.

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