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SPECTRUM OF TWO NEW JOINS OF GRAPHS AND INFINITE FAMILIES OF INTEGRAL GRAPHS

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Abstract. Let G_1 and G_2 be two graphs with $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$, $E(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$. Let $S(G)$ denote the subdivision graph of G [3]. The S_{vertex} join of G_1 with G_2, denoted by $G_1 \dot{\vee} G_2$ is obtained from $S(G_1)$ and G_2 by joining all vertices of G_1 with all vertices of G_2. The S_{edge} join of G_1 with G_2, denoted by $G_1 \underline{\vee} G_2$ is obtained from $S(G_1)$ and G_2 by joining all vertices of $S(G_1)$ corresponding to the edges of G_1 with all vertices of G_2. In this paper we obtain the spectrum of these two new joins of graphs. As an application some infinite family of new classes of integral graphs are constructed.

1. Introduction

Let G be a simple graph on n vertices. The spectrum of an adjacency matrix $A(G)$ of G is the multiset of its eigenvalues and forms the spectrum of G denoted by $Spec(G)$. As $A(G)$ is real and symmetric, its eigenvalues can be ordered as $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n$. Results related to $A(G)$ and its spectrum can be found in [3, 2, 10, 8]. Some recent results related to matrices of graphs can be found in [7].

A graph G is A-integral if the spectrum of $A(G)$ consists only of integers. In [1] constructions and properties of integral graphs are discussed in detail. The graphs K_p, C_4, C_6 and $K_{n,n}$ are examples of integral graphs. Some work on these lines pertaining to the class of trees is in [9]. Moreover, several graph operations such as cartesian product, strong sum and product on integral graphs can be used for constructing infinite families of integral graphs [1]. For some other works see [6, 5, 4] and also the references cited therein.

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In this paper we define two new joins of graphs, obtain their spectrum and use them to construct some infinite families of new integral graphs. The discussion in subsequent sections are based on the following definitions and lemmas.

Definition 1.1 (Subdivision graph)**.** [3] The subdivision graph $S(G)$ of G is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a path of length 2. Thus $V(S(G)) = V(G) \bigcup E(G)$ and consists of edges obtained by joining vertices corresponding to edges to the vertices with which they are incident in G.

Lemma 1.1. [3] *Let G be a r-regular (n,n) graph with an adjacency matrix A and an incidence matrix R. Let $L(G)$ be its line graph. Then $RR^T = A + rI$ and $R^TR = A(L(G)) + 2I$. Moreover if G has an eigenvalue equals to $-r$ with an eigenvector X, then G is bipartite and $R^TX = 0$.*

Lemma 1.2. [3] *Let G be an r-regular (n,m) graph with spec$(G) = \{r, \lambda_2, \dots, \lambda_n\}$. Then spec$(L(G)) = \begin{pmatrix} 2r-2 & \lambda_2 + r - 2 & \dots & \lambda_n + r - 2 & -2 \\ 1 & 1 & \dots & 1 & m-n \end{pmatrix}$. Also Z is an eigenvector belonging to the eigenvalue -2 if and only if $RZ = 0$.*

1.1. Two new joins of graphs and their spectra

In this section we define two new joins namely the S_{vertex} join and S_{edge} join of two graphs G_1 and G_2 and obtain their spectra. Throughout the paper we set $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ and $V_1 = \{u_1, u_2, \dots, u_{n_1}\}$, $V_2 = \{v_1, v_2, \dots, v_{n_2}\}, E_1 = \{e_1, e_2, \dots, e_{m_1}\}$. Henceforth $\mathbf{1}_k$ and 0_k are the vectors of order k with all elements equal to 1 and 0 respectively. Moreover $0_{j\times k}$ denotes the $j \times k$ all zeros matrix and $J_{j\times k}$, the $j \times k$ all one matrix.

Definition 1.2. The S_{vertex} join of two graphs G_1 and G_2 denoted by $G_1 \veebar G_2$ is obtained from $S(G_1)$ and G_2 by joining each vertex of $V(G_1)$ with every vertex of $V(G_2)$.

Definition 1.3. The S_{edge} join of two graphs G_1 and G_2 denoted by $G_1 \underline{\vee} G_2$ is obtained from $S(G_1)$ and G_2 by joining each vertex of $E(G_1)$ with every vertex of $V(G_2)$.

Figure 1 illustrates the above definitions.

We now determine the spectrum of $G_1 \veebar G_2$ and $G_1 \underline{\vee} G_2$ where G_i is r_i regular, for $i = 1, 2$.

1.2. Spectrum of $G_1 \veebar G_2$

Theorem 1.1. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices, an incidence matrix R and eigenvalues of the adjacency matrix A_{G_i}, $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,2} \geq \cdots \geq$*

(a) $K_4 \veebar K_2$ (b) $K_4 \underline{\vee} K_2$

FIGURE 1. S_{vertex} and S_{edge} joins

λ_{i,n_i}. *Then the spectrum of* $G_1 \veebar G_2$ *consists of* $\pm\sqrt{\lambda_{1,j} + r_1}, j = 2, 3, \ldots, n_1$; $\lambda_{2,j}, j = 2, 3, \ldots, n_2$; 0 *of multiplicity* $m_1 - n_1$ *together with three more which are roots of the cubic* $x^3 - r_2x^2 - (n_1n_2 + 2r_1)x + 2r_1r_2 = 0$.

Proof. For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and an adjacency matrix A_{G_i}. Let R be the incidence matrix of G_1. Then by a proper labeling of vertices, the adjacency matrix $\mathcal{A}$ of $G_1 \veebar G_2$ can be written as

$$\mathcal{A} = \begin{bmatrix} 0_{n_1} & R & J_{n_1\times n_2} \\ R^T & 0_{m_1} & 0_{m_1\times n_2} \\ J_{n_2\times n_1} & 0_{n_2\times m_1} & A(G_2) \end{bmatrix}.$$

As a regular graph, G_1 has all-one vector $\mathbf{1}_{n_1}$ as an eigenvector corresponding to eigenvalue r_1, while all other eigenvectors are orthogonal to $\mathbf{1}_{n_1}$.

Let X be an eigenvector of G_1 corresponding to an eigenvalue $\lambda \neq r_1$. Then $\Phi = \begin{bmatrix} \sqrt{\lambda + r_1}X \\ R^TX \\ 0 \end{bmatrix}$ is an eigenvector of $\mathcal{A}$ corresponding to $\sqrt{\lambda + r_1}$. This is because

$$\begin{aligned} \mathcal{A} \cdot \Phi &= \begin{bmatrix} 0_{n_1\times n_1} & R & J_{n_1\times n_2} \\ R^T & 0_{m_1\times m_1} & 0_{m_1\times n_2} \\ J_{n_2\times n_1} & 0_{n_2\times m_1} & A(G_2) \end{bmatrix} \cdot \begin{bmatrix} \sqrt{\lambda + r_1}X \\ R^TX \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda + r_1X \\ \sqrt{\lambda + r_1}R^TX \\ 0 \end{bmatrix} = \sqrt{\lambda + r_1} \begin{bmatrix} \sqrt{\lambda + r_1}X \\ R^TX \\ 0 \end{bmatrix} \\ &= \sqrt{\lambda + r_1}\Phi. \end{aligned}$$

Similarly $\hat{\Phi} = \begin{bmatrix} -\sqrt{\lambda + r_1}X \\ R^TX \\ 0 \end{bmatrix}$ is an eigenvector of $\mathcal{A}$ corresponding to $-\sqrt{\lambda + r_1}$.

Now consider the $m_1 - n_1$ linearly independent eigenvectors $Y_l, l = 1, 2, \ldots, m_1 - n_1$ of $L(G_1)$ corresponding to the eigenvalue -2. Then by Lemma 1.2, $RY_l = 0$. Consequently $\Psi_l = \begin{bmatrix} 0 \\ Y_l \\ 0 \end{bmatrix}$ is an eigenvector of $\mathcal{A}$ with an eigenvalue 0. This is because

$$\mathcal{A} \cdot \Psi_l = \begin{bmatrix} 0_{n_1 \times n_1} & R & J_{n_1 \times n_2} \\ R^T & 0_{m_1 \times m_1} & 0_{m_1 \times n_2} \\ J_{n_2 \times n_1} & 0_{n_2 \times m_1} & A(G_2) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ Y_l \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now as a regular graph, G_2 has all-one vector $\mathbf{1}_{n_2}$ as an eigenvector corresponding to eigenvalue r_2, while all other eigenvectors are orthogonal to $\mathbf{1}_{n_2}$. Let Z be an eigenvector of G_2 corresponding to an eigenvalue $\mu \neq r_2$. Then using similar arguments as above we have $\Omega = \begin{bmatrix} 0 \\ 0 \\ Z \end{bmatrix}$ as an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue μ. Thus we have altogether constructed $m_1 + n_1 + n_2 - 3$ eigenvectors and corresponding eigenvalues. By the very construction all are linearly independent. Now there remains three. The already constructed eigenvectors are orthogonal to $\begin{bmatrix} J_{n_1 \times 1} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ J_{m_1 \times 1} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ J_{n_2 \times 1} \end{bmatrix}$ and hence the remaining three are spanned by these three vectors. Thus they are of the form $\Theta = \begin{bmatrix} \alpha J_{n_1 \times 1} \\ \beta J_{m_1 \times 1} \\ \gamma J_{n_2 \times 1} \end{bmatrix}$ for some $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. But then from $\mathcal{A} \cdot \Theta = x \cdot \Theta$ we have that Θ is an eigenvector of $\mathcal{A}$ with eigenvalue x if and only if $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ is an eigenvector of the matrix $M = \begin{bmatrix} 0 & r_1 & n_2 \\ 2 & 0 & 0 \\ n_1 & 0 & r_2 \end{bmatrix}$ with eigenvalue x. Since the characteristic equation of M is $x^3 - r_2 x^2 - (n_1 n_2 + 2r_1)x + 2r_1 r_2 = 0$, the theorem is proved. □

Corollary 1.1. *If G_2 is a totally disconnected graph, then $r_2 = 0$ and hence the spectrum of $G_1 \veebar G_2$ consists of $\pm\sqrt{\lambda_{1,j} + r_1}, j = 2, 3, \ldots, n_1$, 0 of multiplicity $m_1 + n_2 - n_1$ and $\pm\sqrt{n_1 n_2 + 2r_1}$.*

1.3. Spectrum of $G_1 \underline{\vee} G_2$

In this section we have the following theorem, proof of which are on similar lines as that of Theorem 1.1.

Theorem 1.2. *For $i = 1,2$, let G_i be an r_i-regular graph with n_i vertices, an incidence matrix R and eigenvalues of the adjacency matrix A_{G_i}, $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,n_i}$. Then the spectrum of $G_1 \veebar G_2$ consists of $\pm\sqrt{\lambda_{1,j} + r_1}$, $j = 2,3,\ldots,n_1$; $\lambda_{2,j}, j = 2,3,\ldots,n_2$; 0 of multiplicity $m_1 - n_1$ together with three more which are roots of the cubic $x^3 - r_2x^2 - (m_1n_2 + 2r_1)x + 2r_1r_2 = 0$.*

Corollary 1.2. *If G_2 is a totally disconnected graph, then $r_2 = 0$ and hence the spectrum of $G_1 \veebar G_2$ consists of $\pm\sqrt{\lambda_{1,j} + r_1}, j = 2,3,\ldots,n_1, 0$ of multiplicity $m_1+n_2-n_1$ and $\pm\sqrt{m_1n_2 + 2r_1}$.*

2. New infinite families of integral graphs

The following propositions give a necessary and sufficient condition for the S_{vertex} and S_{edge} joins of G_1 and G_2 to be integral whose proof follows from Theorems 1.1 and 1.2.

Proposition 2.1. *For $i = 1,2$, let G_i be an r_i-regular graph with n_i vertices and eigenvalues of the adjacency matrix A_{G_i}, $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,n_i}$. Then $G_1 \dot{\vee} G_2$ is integral if and only if $\pm\sqrt{\lambda_{1,j} + r_1}, j = 2,3,\ldots,n_1$, the three roots of $x^3 - r_2x^2 - (n_1n_2 + 2r_1)x + 2r_1r_2 = 0$ are integers and G_2 is integral.*

Proposition 2.2. *For $i = 1,2$, let G_i be an r_i-regular graph with n_i vertices and eigenvalues of the adjacency matrix A_{G_i}, $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,n_i}$. Then $G_1 \veebar G_2$ is integral if and only if $\pm\sqrt{\lambda_{1,j} + r_1}, j = 2,3,\ldots,n_1$, the three roots of $x^3 - r_2x^2 - (m_1n_2 + 2r_1)x + 2r_1r_2 = 0$ are integers and G_2 is integral.*

In particular if $G_2 = \overline{K_{n_2}}$, a totally disconnected graph, then $r_2 = 0$ and hence in view of Corollaries 1.1 and 1.2 we have the following necessary and sufficient conditions for the graphs $G_1 \dot{\vee} \overline{K_{n_2}}$ and $G_1 \veebar \overline{K_{n_2}}$ to be integral.

Proposition 2.3. *$G_1 \dot{\vee} \overline{K_{n_2}}$ is integral if and only if $\pm\sqrt{\lambda_{1,j} + r_1}, j = 2,3,\ldots,n_1$ and $\pm\sqrt{n_1n_2 + 2r_1}$ are integers.*

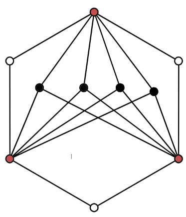
Proposition 2.4. *$G_1 \veebar \overline{K_{n_2}}$ is integral if and only if $\pm\sqrt{\lambda_{1,j} + r_1}, j = 2,3,\ldots,n_1$ and $\pm\sqrt{m_1n_2 + 2r_1}$ are integers.*

2.1. Infinite families of integral graphs of the form $G_1 \dot{\vee} G_2$

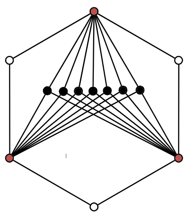
We have that the complete graph K_n is $n-1$ regular with $Spec(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$ and the complete bipartite graph $K_{n,n}$ is $n-$ regular with $Spec(K_{n,n}) = \begin{pmatrix} n & 0 & -n \\ 1 & n-2 & 1 \end{pmatrix}$. Then

Proposition 2.5. *The infinite family of graphs $K_{n_1} \dot{\vee} \overline{K_{n_2}}$ is integral for $n_1 = t^2 + 2$ and $n_2 = h^2(t^2 + 2) \pm 2ht - 1, h = 1,2,\ldots$ and $t = 1,2,\ldots$.*

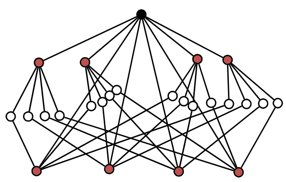
Proposition 2.6. *The infinite family of graphs* $K_{n_1,n_1} \vee \overline{K_{n_2}}$ *is integral for* $n_1 = t^2$ *and* $n_2 = 2h^2 - 1, h = 1, 2, \ldots$ *and* $t = 1, 2, \ldots$.



(a) $K_3 \vee \overline{K_4}$ with spectrum $\{4^1, 1^2, 0^4, -1^2, -4^1\}$.



(b) $K_3 \vee \overline{K_7}$ with spectrum $\{5^1, 1^2, 0^7, -1^2, -5^1\}$.



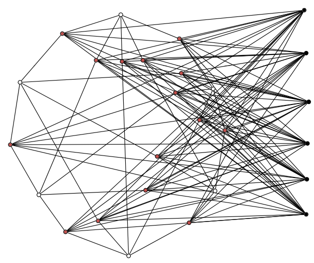
(c) $K_{4,4} \vee \overline{K_1}$ with spectrum $\{4^1, 2^6, 0^{11}, -2^6, -4^1\}$.

FIGURE 2. Examples of integral graphs of the form $G_1 \vee G_2$

2.2. Infinite families of integral graphs of the form $G_1 \underline{\vee} G_2$

Proposition 2.7. *The infinite family of graphs* $K_{n_1} \underline{\vee} \overline{K_{n_2}}$ *is integral for* $n_1 = t^2 + 2$ *and* $n_2 = 2(t^2 - 1), t = 1, 2, \ldots$.

Note: When G_1 is a complete bipartite graph, integral graphs of the form $G_1 \veebar G_2$ seem to be very rare and to name a few are $K_{49,49} \veebar \overline{K_2}$, $K_{49,49} \veebar \overline{K_{31}}$, $K_{49,49} \veebar \overline{K_{71}}$, $K_{49,49} \veebar \overline{K_{158}}$ and $K_{289,289} \veebar \overline{K_7}$.



$K_6 \veebar \overline{K_6}$ with spectrum $\{10^1, 2^5, 0^{15}, -2^5, -10^1\}$.

Figure 3. Example of integral graph of the form $G_1 \veebar G_2$

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