KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 36 NUMBER 1 (2012), PAGES 141–150.

ODD MEAN LABELING OF THE GRAPHS $P_{a,b}, P_a^b$ AND $P_{\langle 2a\rangle}^b$

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ABSTRACT. Let $G(V,E)$ be a graph with p vertices and q edges. A graph G is said to be odd mean if there exists a function $f : V(G) \to \{0,1,2,3,\dots,2q-1\}$ satisfying f is $1-1$ and the induced map $f^* : E(G) \to \{1,3,5,\dots,2q-1\}$ defined by

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$$

is a bijection. If a graph G admits an odd mean labeling then G is called an odd mean graph. In this paper we study the odd meanness of the class of graphs $P_{a,b}, P_a^b$ and $P_{\langle 2a\rangle}^b$ and we prove that the graphs $P_{2r,m}, P_{2r+1,2m+1}, P_{2r}^m, P_{2r+1}^{2m+1}$ and $P_{\langle 2r,m\rangle}$ for all values of r and m are odd mean graphs.

1. INTRODUCTION

Throughout this paper, by a graph we mean a finite, undirected, simple graph. Let $G(V,E)$ be a graph with p vertices and q edges. For notations and terminology we follow [1].

The concept of mean labeling was first introduced by S. Somasundaram and R. Ponraj [7]. The concept of odd mean labeling was introduced by K. Manickam and M. Marudai [5]. They have studied in [5] the odd meanness of many standard graphs.

A graph G with p vertices and q edges is said to be odd mean if there exists a function $f : V(G) \to \{0,1,2,3,\dots,2q-1\}$ satisfying f is $1-1$ and the induced map $f^* : E(G) \to \{1,3,5,\dots,2q-1\}$ defined by

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd,} \end{cases}$$

Key words and phrases. Labeling, Odd mean labeling, Odd mean graphs.
2010 *Mathematics Subject Classification.* Primary: 05C78.
Received: May 23, 2011.

is a bijection. If a graph G has an odd mean labeling, then we say that G is an odd mean graph.

An odd mean labeling of the cube is given in Figure 1.

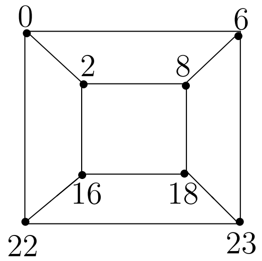


FIGURE 1.

In [2], Kathiresan established that the graph $P_{r,2m+1}$ is graceful for all values of r and m and conjectured that $P_{a,b}$ is graceful except when $a = 2r + 1$ and $b = 4s + 2$. In [6] Sekar proved the conjecture except in one case where $a = 4r + 1 (r > 1)$ with the corresponding $b = 4m (m > r)$. In [3, 4], Ganesan discussed the magic labeling of the type $(1, 1, 1)$ and consecutive labeling of the type $(1, 1, 1)$ of the plane graphs $P_{a,b}$ and d-anti magic labeling of the plane graphs P_a^b. Meanness of the graphs $P_{a,b}$ and P_a^b are discussed in [8]. Motivated by these works, in this paper, we study the odd meanness of the class of graphs $P_{a,b}, P_a^b$ and $P^b_{\langle 2a\rangle}$ and we prove that the graphs $P_{2r,m}, P_{2r+1,2m+1}, P^m_{2r}, P^{2m+1}_{2r+1}$ and $P^m_{\langle 2r\rangle}$ for all values of r and m are odd mean graphs.

2. ODD MEANNESS OF THE GRAPHS $P_{a,b}$

Let u and v be two fixed vertices. We connect u and v by means of $b \geq 2$ internally disjoint paths of length $a \geq 2$ each. The resulting graph embedded in a plane is denoted by $P_{a,b}$.

Let $v_0^i, v_1^i, v_2^i, \ldots, v_a^i$ be the vertices of the i^{th} copy of the path of length a where $i = 1, 2, \ldots, b$, $v_0^i = u$ and $v_a^i = v$ for all i. We observe that the graph $P_{a,b}$ has $(a-1)b + 2$ vertices and ab edges.

For example $P_{4,5}$ is shown in Figure 2.

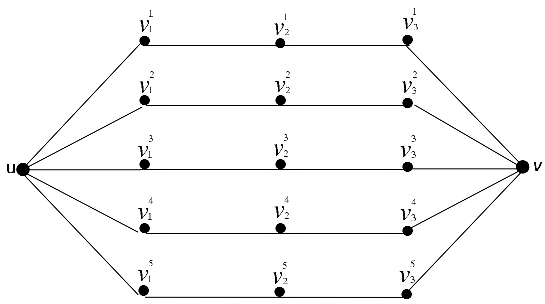


FIGURE 2.

Theorem 2.1. *$P_{2r,m}$ is an odd mean graph for all values of r and m.*

Proof. Let $v_0^i, v_1^i, v_2^i, \ldots, v_{2r}^i$ be the vertices of the i^{th} copy of the path of length $2r$ where $i = 1, 2, \ldots, m$, $v_0^i = u$ and $v_{2r}^i = v$ for all i. We observe that the number of vertices of the graph $P_{2r,m}$ has $(2r-1)m+2$ vertices and the number of edges of the graph is $2rm$.

Case(i) When m is odd.

Let $m = 2k+1$ for some $k \in Z^+$.

Define f on $V(P_{2r,2k+1})$ as follows:

$$\begin{aligned} f(u) &= 0, \\ f(v) &= 4rm - 1, \\ f(v_{2j+1}^i) &= 4mj + 4i - 2, \ i = 1, 2, \ldots, m, \ j = 0, 1, 2, \ldots, r-1 \end{aligned}$$

$$\text{and } f(v_{2j}^i) = \begin{cases} (4m+3) + 4m(j-1) + 4(i-1), & 1 \le i \le k \\ (2m+1) + 4m(j-1) + 4(i-(k+1)), & k+1 \le i \le 2k+1, \\ & j = 1, 2, \ldots, r-1. \end{cases}$$

It can be verified that the label of the edges of the graph are $1, 3, 5, \ldots, 2q-1$. Hence, $P_{2r,2k+1}$ is an odd mean graph for all values of r and k.

Case(ii) When m is even.

Let $m = 2k$ for some $k \in Z^+$.

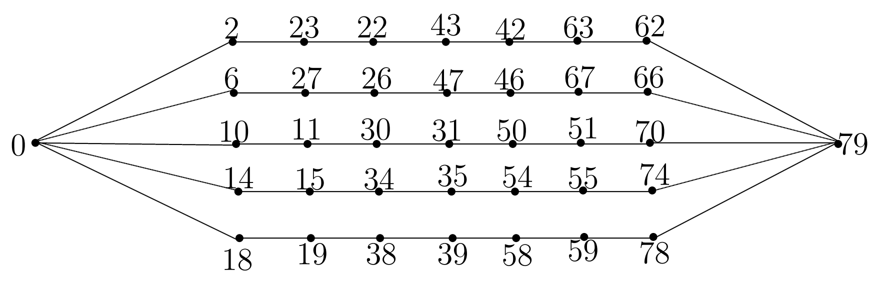
Define f on $V(P_{2r,2k})$ as follows:

$$\begin{aligned} f(u) &= 0, \\ f(v) &= 4rm - 1, \\ f(v_{2j+1}^i) &= 4mj + 4i - 2, \ i = 1, 2, \ldots, m, \ j = 0, 1, 2, \ldots, r-1 \end{aligned}$$

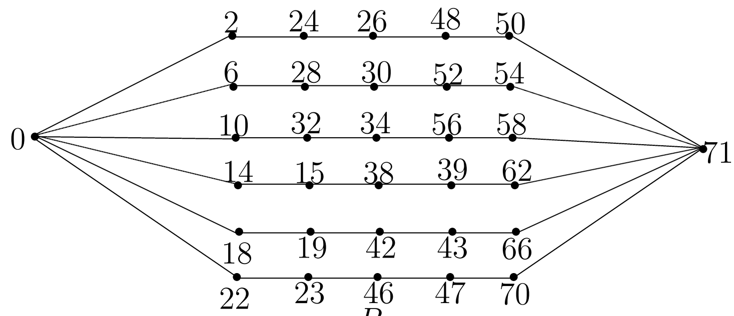
$$\text{and } f(v_{2j}^i) = \begin{cases} 4m + 4m(j-1) + 4(i-1), & 1 \le i \le k \\ 2m+3 + 4m(j-1) + 4(i-(k+1)), & k+1 \le i \le 2k, \\ & j = 1, 2, \ldots, r-1. \end{cases}$$

It is easy to check that the label of the edges of the graph are $1, 3, 5, \ldots, 2q-1$. Hence, $P_{2r,2k}$ is an odd mean graph. Thus $P_{2r,m}$ is an odd mean graph for all r and m.

For example, odd mean labelings of the graphs $P_{8,5}$ and $P_{6,6}$ are shown in Figure 3. □



$P_{8,5}$



$P_{6,6}$

FIGURE 3.

Theorem 2.2. *$P_{2r+1,2m+1}$ is an odd mean graph for all values of r and m.*

Proof. Let $v_0^i, v_1^i, v_2^i, \dots, v_{2r+1}^i$ be the vertices of the i^{th} copy of the path of length $2r+1$ where $i = 1, 2, \dots, 2m+1$, $v_0^i = u$ and $v_{2r+1}^i = v$ for all i. We observe that the number of vertices of the graph $P_{2r+1,2m+1}$ is $2r(2m+1)+2$ and the number of edges of the graph is $(2r+1)(2m+1)$.

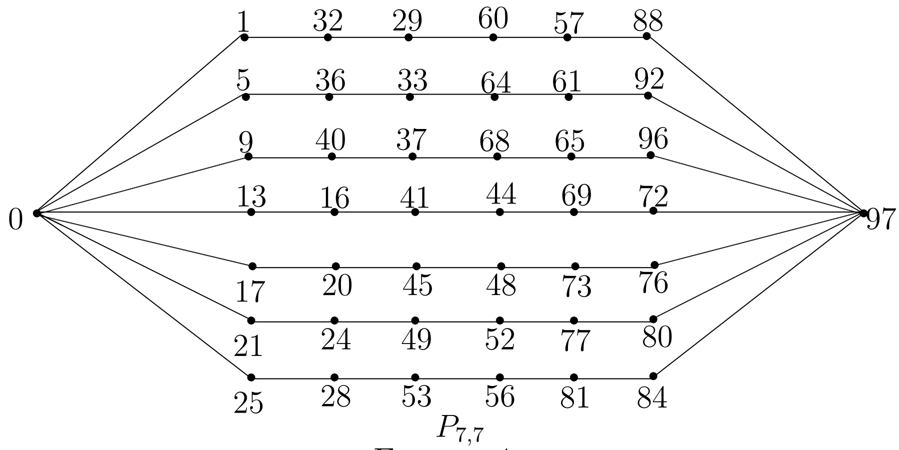
Define f on $V(P_{2r+1,2m+1})$ as follows:

$$\begin{aligned} f(u) &= 0, \\ f(v) &= 2(2r+1)(2m+1) - 1, \\ f(v_{2j+1}^i) &= (4(2m+1)+3)j + 4i - 3, \ i = 1, 2, \dots, 2m+1, \ j = 0, 1, 2, \dots, (r-1) \end{aligned}$$

$$\text{and } f(v_{2j}^i) = \begin{cases} (4(2m+1)+4) + (4(2m+1)+3)(j-1) & \\ +4(i-1), & 1 \le i \le m \\ (2m+2) + (4(2m+1)+3)(j-1) & \\ +4(i-(m+1)), & m+1 \le i \le 2m+1, \\ & j = 1, 2, \dots, r. \end{cases}$$

It can be verified that the label of the edges of the graph are $1, 3, 5, \dots, 2q-1$. Hence, $P_{2r+1,2m+1}$ is an odd mean graph.

For example, an odd mean labeling of the graph $P_{7,7}$ is shown in Figure 4. $\square$



$P_{7,7}$

FIGURE 4.

3. Odd meanness of the Graphs P_a^b

Let a and b be integers such that $a \geq 2$ and $b \geq 2$. Let $y_1, y_2, \ldots, y_a$ be the fixed vertices. We connect the vertices y_i and y_{i+1} by means of b internally disjoint paths P_i^j of length $i+1$ each, $1 \leq i \leq a-1, 1 \leq j \leq b$. Let y_i, $x_{i,j,1}, x_{i,j,2}, \ldots, x_{i,j,i}, y_{i+1}$ be the vertices of the path P_i^j, where $1 \leq i \leq a-1$ and $1 \leq j \leq b$. The resulting graph embedded in a plane is denoted by P_a^b, where

$$V(P_a^b) = \{y_i : 1 \leq i \leq a\} \cup \bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{x_{i,j,k} : 1 \leq k \leq i\}$$

and

$$E(P_a^b) = \bigcup_{i=1}^{a-1} \{y_i, x_{i,j,1} : 1 \leq j \leq b\} \cup \bigcup_{i=2}^{a-1} \bigcup_{j=1}^{b} \{x_{i,j,k} x_{i,j,k+1} : 1 \leq k \leq i-1\}$$
$$\bigcup_{i=1}^{a-1} \{x_{i,j,i} y_{i+1} : 1 \leq j \leq b\}$$

we observe that the number of vertices of the graph P_a^b is $\frac{ab(a-1)}{2} + a$ and the number of edges is $\frac{b(a-1)(a+2)}{2}$.

For example, the plane graph P_4^3 is shown in Figure 5.

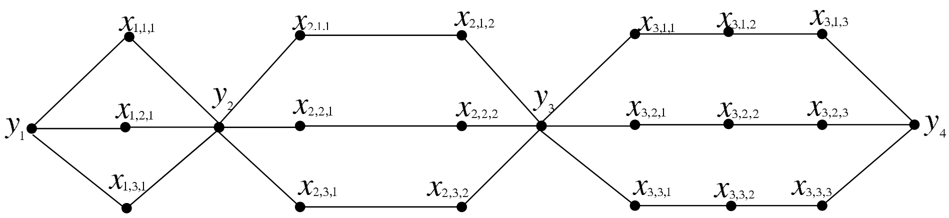


Figure 5.

Theorem 3.1. *P_r^{2m+1} is an odd mean graph for all values of r and m.*

Proof. Let $y_1, y_2, \ldots, y_r$ be the fixed vertices. We connect the vertices y_i and y_{i+1} by means of $2m+1$ internally disjoint paths p_i^j of length $i+1$ each, $1 \leq i \leq r-1, 1 \leq j \leq 2m+1$. Let $y_i, x_{i,j,1}, x_{i,j,2}, \ldots, x_{i,j,i}, y_{i+1}$ be the vertices of the path P_i^j, where $1 \leq i \leq r-1$ and $1 \leq j \leq 2m+1$. We observe that the number of vertices of the graph P_r^{2m+1} is $\frac{(2m+1)r(r-1)}{2} + r$ and the number of edges is $\frac{(2m+1)(r-1)(r+2)}{2}$

Define f on $V(P_r^{2m+1})$ as follows:

$$\begin{aligned} f(y_1) &= 0, \\ f(y_i) &= f(y_{i-1}) + (2m+1)2i, 2 \leq i \leq r-1, \\ f(y_r) &= (2m+1)(r-1)(r+2) - 1, \\ f(x_{1,j,1}) &= 4j - 3, 1 \leq j \leq 2m+1, \end{aligned}$$

$$f(x_{i,j,1}) = \begin{cases} f(y_i) + f(x_{1,j,1}) + 1 & \text{if } i \text{ is odd} \\ f(y_i) + f(x_{1,j,1}) & \text{if } i \text{ is even, } 2 \le i \le r-1, \end{cases}$$

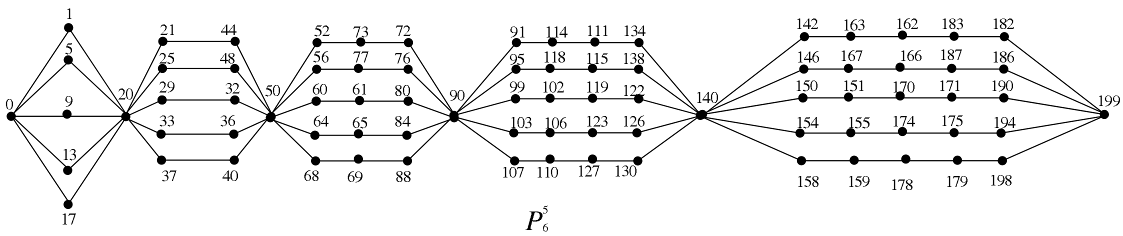
$$f(x_{i,j,2}) = \begin{cases} f(y_i) + (4(2m+1)+4) + 4(i-1), & 1 \le i \le m, \\ f(y_i) + (2(2m+1)+2) & \\ +4(i-(m+1)), & m+1 \le i \le 2m+1, \\ & \text{if } i \text{ is even, } 2 \le i \le r-1 \\ f(y_i) + (4(2m+1)+3) + 4(i-1), & 1 \le i \le m \\ f(y_i) + (2(2m+1)+1) & \\ +4(i-(m+1)), & m+1 \le i \le 2m+1, \\ & \text{if } i \text{ is odd, } 3 \le i \le r-1 \end{cases}$$

and

$$f(x_{i,j,k}) = \begin{cases} f(x_{i,j,1}) + 2(k-1)(2m+1), & \text{if } k \text{ is odd, } 3 \le k < r, k \le i \le r-1 \\ f(x_{i,j,2}) + 2(k-2)(2m+1), & \text{if } k \text{ is even, } 4 \le k < r, k \le i \le r-1. \end{cases}$$

It can be verified that the label of the edges of the graph are $1, 3, 5, \ldots, 2(2m+1)(r-1)(r+2)-1$. Then the resultant graph is an odd mean graph.

For example, an odd mean labeling of P_6^5 is shown in Figure 6. □



P_6^5

FIGURE 6.

4. Odd meanness of the graph $P^b_{\langle 2a \rangle}$

Let a and b be integers such that $a \ge 1$ and $b \ge 2$. Let $y_1, y_2, \ldots, y_{a+1}$ be the fixed vertices. We connect the vertices y_i and y_{i+1} by means of b internally disjoint path P_i^j of length $2i$ each, $1 \le i \le a, 1 \le j \le b$. Let $y_i, x_{i,j,1}, x_{i,j,2}, \ldots, x_{i,j,2i-1}, y_{i+1}$ be the vertices of the path P_i^j, where $1 \le i \le a$ and $1 \le j \le b$. The resulting graph embedded in a plane is denoted by $P^b_{\langle 2a \rangle}$ where

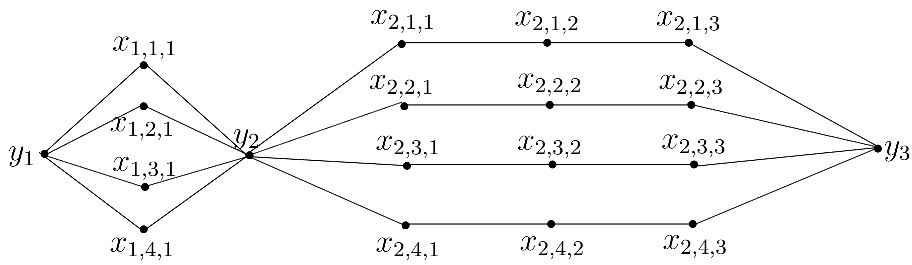
$$V(P^b_{\langle 2a \rangle}) = \{y_i : 1 \le i \le a+1\} \cup \bigcup_{i=1}^{a} \bigcup_{j=1}^{b} \{x_{i,j,k} : 1 \le k \le 2i-1\}$$

and

$$E(P^b_{\langle 2a\rangle}) = \bigcup_{i=1}^{a}\{y_i x_{i,j,1} : 1 \le j \le b\} \cup \bigcup_{i=2}^{a}\bigcup_{j=1}^{b}\{x_{i,j,k}x_{i,j,k+1} : 1 \le k \le 2i-2\}$$
$$\bigcup_{i=1}^{a}\{x_{i,j,2i-1}y_{i+1} : 1 \le j \le b\}.$$

We observe that the number of vertices of the graph $P^b_{\langle 2a\rangle}$ is $a^2b + a + 1$ and the number of edges is $a(a+1)b$.

For example, the plane graph $P^4_{\langle 4\rangle}$ is shown in Figure 7.



$P^4_{\langle 4\rangle}$

FIGURE 7.

Theorem 4.1. *$P^m_{\langle 2r\rangle}$ is an odd mean graph for all values of r and m.*

Proof. Let $y_1, y_2, \ldots, y_{r+1}$ be the fixed vertices. We connect the vertices y_i and y_{i+1} by means of m internally disjoint paths P_i^j of length $2i$ each, $1 \le i \le r$, $1 \le j \le m$. Let $y_i, x_{i,j,1}, x_{i,j,2}, \ldots, x_{i,j,2i-1}, y_{i+1}$ be the vertices of the path P_i^j, where $1 \le i \le r$ and $1 \le j \le m$. We observe that the number of vertices of the graph $P^m_{<2r>}$ is $r^2m + r + 1$ and the number of edges is $r(r+1)m$.

Case(i) When m is odd.

Let $m = 2t + 1, t \in Z^+$.

Define f on $V(P^m_{<2r>})$ as follows:

$$f(y_1) = 0,$$
$$f(y_i) = f(y_{i-1}) + 2(2i-2)m, 2 \le i \le r,$$
$$f(y_{r+1}) = 2r(r+1)m - 1,$$
$$f(x_{1,j,1}) = 4j - 2, 1 \le j \le m,$$
$$f(x_{i,j,1}) = f(y_i) + f(x_{1,j,1}), 2 \le i \le r,$$
$$f(x_{i,j,2}) = \begin{cases} f(y_i) + (4m+3) + 4(i-1), & 1 \le i \le t \\ f(y_i) + (2m+1) + 4(i-(k+1)), & t+1 \le i \le 2t+1, 2 \le i \le r \end{cases}$$

and

$$f(x_{i,j,k}) = \begin{cases} f(x_{i,j,1}) + 2(k-1)m & \text{if } k \text{ is odd, } 3 \leq k < 2r, \frac{k+1}{2} \leq i \leq r \\ f(x_{i,j,2}) + 2(k-2)m & \text{if } k \text{ is even, } 4 \leq k < 2r, \frac{k+2}{2} \leq i \leq r. \end{cases}$$

It can be verified that, the label of the edges of the graph are $1, 3, 5, \ldots, 2r(r+1)m-1$. Hence $P^{2t+1}_{\langle 2r\rangle}$ is an odd mean graph for all values of r and t.

Case(ii) when m is even.

Let $m = 2t, t \in Z^+$.

Define f on $V(P^{2t}_{2r})$ as follows:

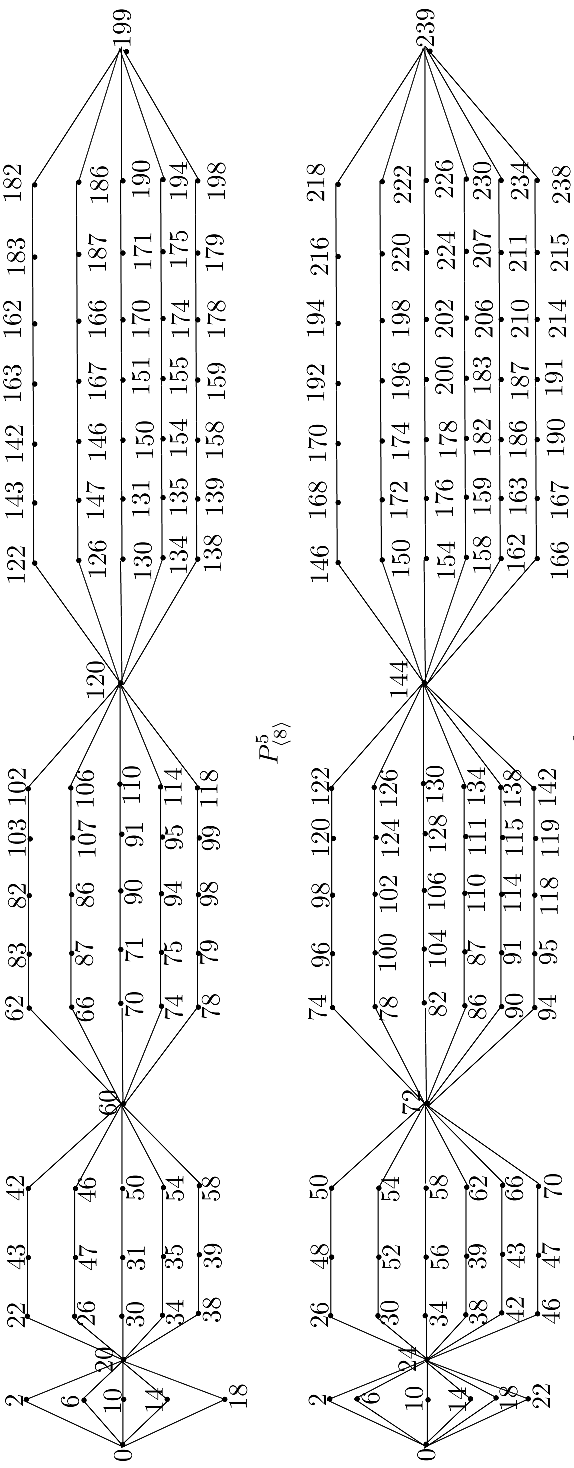
$$f(y_1) = 0,$$
$$f(y_i) = f(y_{i-1}) + 2(2i-2)m,$$
$$f(y_{r+1}) = 2r(r+1)m - 1,$$
$$f(x_{1,j,1}) = 4j - 2,\ 1 \leq j \leq m,$$
$$f(x_{i,j,1}) = f(y_i) + f(x_{1,j,1}),\ 2 \leq i \leq r,\ 1 \leq j \leq m,$$
$$f(x_{i,j,2}) = \begin{cases} f(y_i) + 4m + 4(i-1), & 1 \leq i \leq t \\ f(y_i) + (2m+3) + 4(i-(t+1)), & t+1 \leq i \leq 2t \end{cases}$$

and

$$f(x_{i,j,k}) = \begin{cases} f(x_{i,j,1}) + 2(k-1)m & \text{if } k \text{ is odd, } 3 \leq k < 2r, \frac{k+1}{2} \leq i \leq r \\ f(x_{i,j,2}) + 2(k-2)m & \text{if } k \text{ is even, } 4 \leq k < 2r, \frac{k+2}{2} \leq i \leq r. \end{cases}$$

It is easy to check that, the label of the edges of the graph are $1, 3, 5, \ldots, 2r(r+1)m-1$. Hence $P^{2t}_{\langle 2r\rangle}$ is an odd mean graph for all values of r and t. Thus $P^{m}_{\langle 2r\rangle}$ is an odd mean graph for all values of r and m.

For example, odd mean labelings of the graphs $P^{6}_{\langle 8\rangle}$ and $P^{5}_{\langle 8\rangle}$ are shown in Figure 8. □



$P_{\langle 8 \rangle}^5$

$P_{\langle 8 \rangle}^6$

FIGURE 8.

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