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SOME CONSIDERATIONS OF MATRIX EQUATIONS USING THE CONCEPT OF REPRODUCTIVITY

BRANKO MALEŠEVIĆ [1] AND BILJANA RADIČIĆ [2]

Abstract. In this paper we analyse Cline's matrix equation, generalized Penrose's matrix system and a matrix system for k-commutative $\{1\}$-inverses. We determine reproductive and non-reproductive general solutions of analysed matrix equation and analysed matrix systems.

1. Introduction

In this paper we determine general and reproductive general solutions of analysed matrix equation and analysed matrix systems. We are going to use the concept of reproductivity in order to prove that certain formulas represent the general solutions of analysed matrix equation and analysed matrix systems. The concept of reproductive equations was introduced by S. B. Prešić [2] in 1968.
Let S be a given non-empty set and J be a given unary relation of S. Then an equation $J(x)$ is *consistent* if there is at least one element $x_0 \in S$, so-called *the solution*, such that $J(x_0)$ is true. A formula $x = \phi(t)$, where $\phi : S \to S$ is a given function, represents *the general solution* [20] of the equation $J(x)$ if and only if

$$(\forall t)J(\phi(t)) \wedge (\forall x)(J(x) \Longrightarrow (\exists t)x = \phi(t)).$$

Let us cite the definition of reproductive equations according by S. B. Prešić [2].

Definition 1.1. *The reproductive equations* are the equations of the following form:

$$x = \varphi(x),$$

where x is a unknown, S is a given set and $\varphi : S \longrightarrow S$ is a given function which satisfies the following condition:

$$\varphi \circ \varphi = \varphi. \tag{1.1}$$

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The condition (1.1) is called *the condition of reproductivity* [2]. The fundamental properties of reproductive equations are given by the following two statements (S. B. Prešić [2]) (see also [5], [6] and [17]).

Theorem 1.1. *For any consistent equation $J(x)$ there is an equation of the form $x = \varphi(x)$, which is equivalent to $J(x)$ being in the same time reproductive as well.*

Theorem 1.2. *If a certain equation $J(x)$ is equivalent to the reproductive one $x = \varphi(x)$, the general solution is given by the formula $x = \varphi(y)$, for any value $y \in S$.*

Let us remark that a formula $x = \phi(t)$, where $\phi : S \to S$ is a given function, represents *the reproductive general solution* [20] of the equation $J(x)$ if and only if

$$(\forall t) J(\phi(t)) \wedge (\forall t)(J(t) \Longrightarrow t = \phi(t)).$$

Reproductivity of some equations of mathematical analysis was studied by J. D. Kečkić in [9], [10]. In [15] J. D. Kečkić and S. B. Prešić considered the general applications of the concept of reproductivity. The general applications of the concept of reproductivity in various mathematical structures can also be found in [7], [8], [13], [14], [16] and [17].

2. Main results

Let $m, n \in \mathbb{N}$ and $\mathbb{C}$ is the field of complex numbers. The set of all $m \times n$ matrices over $\mathbb{C}$ is denoted by $\mathbb{C}^{m\times n}$. By $\mathbb{C}_a^{m\times n}$ we denote the set of all matrices from $\mathbb{C}^{m\times n}$ with a rank a. For $A \in \mathbb{C}^{m\times n}$ the rank of A is denoted by $rank(A)$. The unit matrix of order m is denoted by I_m (if the dimension of unit matrix is known from the context, we omit the index which indicates the dimension and we use designation I). Let $A \in \mathbb{C}^{m\times n}$, then a solution of the matrix equation

$$AXA = A$$

is called $\{1\}$-*inverse* of A and it is denoted by $A^{(1)}$. In the general case $\{1\}$-inverse of A is not uniquely determined. The set of all $\{1\}$-inverses of A is denoted by $A\{1\}$. It can be shown that $A\{1\}$ is not empty. $\{1\}$-inverse of A is uniquely determined if A is regular. In that case $\{1\}$-inverse $A^{(1)}$ corresponds to A^{-1} i.e. $A\{1\} = \{A^{-1}\}$. There are also other types of inverses. More informations about $\{1\}$-inverse and other types of inverses can be found in [18] and [19]. For $A \in \mathbb{C}^{m\times m}$ the smallest non-negative integer k such that $rank(A^k) = rank(A^{k+1})$ is called the index of A and it is denoted by $Ind(A)$.

This section of paper is divided into *three* parts. The *first* part is devoted to the matrix equation

$$A^m X B^n = C, \tag{2.1}$$

where $A \in \mathbb{C}^{p\times p}$, $B \in \mathbb{C}^{q\times q}$, $C \in \mathbb{C}^{p\times q}$, $m \geq k = ind(A)$ and $n \geq l = ind(B)$. In the *second* part we consider the matrix system

$$(2.2.1.)\quad A^m X = B \qquad \wedge \qquad (2.2.2.)\quad X D^n = E, \tag{2.2}$$

where $A \in \mathbb{C}^{p\times p}$, $B \in \mathbb{C}^{p\times q}$, $D \in \mathbb{C}^{q\times q}$, $E \in \mathbb{C}^{p\times q}$, $m \geq Ind(A)$ and $n \geq Ind(D)$. A solution of the matrix system

$$AXA = A \quad \wedge \quad A^k X = XA^k, \tag{2.3}$$

where $A \in \mathbb{C}^{p\times p}$ is a singular matrix and $k \in \mathbb{N}$, is analysed in the *third* part of this section.

2.1. In this part we analyse the matrix equation (2.1). In the paper [3] R. E. Cline was the first one who considered the matrix equation (2.1). Using Penrose's condition for the consistence of the matrix equation $AXB = C$, R. E. Cline concluded that the matrix equation (2.1) is consistent if and only if

$$A^m(A^m)^{(1)}C(B^n)^{(1)}B^n = C. \tag{2.4}$$

In the paper [3] it was shown that the matrix equation (2.1) is consistent for any $m > k$ and any $n > l$ if and only if the matrix equation $A^kXB^l = C$ is consistent. Based on the results in the paper [22] the condition of consistence (2.4) for the matrix equation (2.1) can be also considered in a new form (see Theorem 2.1).

Lemma 2.1. *If the matrix equation* (2.1) *is consistent, the equivalence*

$$A^mXB^n = C \iff X = f(X) = X - (A^m)^{(1)}(A^mXB^n - C)(B^n)^{(1)}$$

is true.

Proof. $\Longrightarrow)$: Suppose that $A^mXB^n = C$. Then, the equality

$$(A^m)^{(1)}A^mXB^n(B^n)^{(1)} = (A^m)^{(1)}C(B^n)^{(1)}$$

is also true and

$$\begin{aligned} X &= X - (A^m)^{(1)}A^mXB^n(B^n)^{(1)} + (A^m)^{(1)}C(B^n)^{(1)} \\ &= X - (A^m)^{(1)}(A^mXB^n - C)(B^n)^{(1)} \\ &= f(X) \end{aligned}$$

$\Longleftarrow)$: Suppose that $X = f(X) = X - (A^m)^{(1)}(A^mXB^n - C)(B^n)^{(1)}$. Then,

$$\begin{aligned} A^mXB^n &= A^mf(X)B^n \\ &= A^m\Big(X - (A^m)^{(1)}(A^mXB^n - C)(B^n)^{(1)}\Big)B^n \\ &= A^mXB^n - \underbrace{A^m(A^m)^{(1)}A^m}_{(=A^m)}\,X\,\underbrace{B^n(B^n)^{(1)}B^n}_{(=B^n)} + \underbrace{A^m(A^m)^{(1)}C(B^n)^{(1)}B^n}_{(\underset{(2.4)}{=}C)} \\ &= A^mXB^n - A^mXB^n + C \\ &= C. \end{aligned}$$

$\square$

Remark 2.1. It is easy to show that $f^2(Y) = f(Y)$ i.e. the function f satisfies the condition of reproductivity. Therefore, if the matrix equation (2.1) is consistent, it is equivalent to the reproductive matrix equation $X = f(X)$.

Based on the previous remark and Theorem 1.2 we conclude that the following theorem is true.

Theorem 2.1. *If the matrix equation* (2.1) *is consistent, the general solution of the matrix equation* (2.1) *is given by the formula*

$$X = f(Y) = (A^m)^{(1)}C(B^n)^{(1)} + Y - (A^m)^{(1)}A^mYB^n(B^n)^{(1)},$$

where Y is an arbitrary matrix corresponding dimensions.

The following theorem is an extension of the previous theorem.

Theorem 2.2. *If X_0 is a particular solution of the matrix equation* (2.1)*, the general solution of the matrix equation* (2.1) *is given by the formula*

$$X = g(Y) = X_0 + Y - (A^m)^{(1)}A^mYB^n(B^n)^{(1)}, \tag{2.5}$$

where Y is an arbitrary matrix corresponding dimensions.

Proof. It is easy to see that the solution of the matrix equation (2.1) is given by (2.5). On the contrary, let X be any solution of the matrix equation (2.1), then

$$\begin{aligned} X &= X - (A^m)^{(1)}C(B^n)^{(1)} + (A^m)^{(1)}C(B^n)^{(1)} \\ &= X - (A^m)^{(1)}A^mXB^n(B^n)^{(1)} + (A^m)^{(1)}A^mX_0B^n(B^n)^{(1)} \\ &= X - (A^m)^{(1)}A^m(X - X_0)B^n(B^n)^{(1)} \\ &= X_0 + (X - X_0) - (A^m)^{(1)}A^m(X - X_0)B^n(B^n)^{(1)} \\ &= X_0 + Y - (A^m)^{(1)}A^mYB^n(B^n)^{(1)} \\ &= g(Y), \end{aligned}$$

where $Y = X - X_0$. From this we see that every solution X of the matrix equation (2.1) can be represented in the form (2.5). $\square$

Remark 2.2. From $g^2(Y) = g(Y) + \Big(X_0 - (A^m)^{(1)}C(B^n)^{(1)}\Big)$ we conclude that the function g is reproductive if and only if $X_0 = (A^m)^{(1)}C(B^n)^{(1)}$.

Remark 2.3. Theorem 2.2 is an extension, as we mentioned, of Theorem 2.1 because there is a matrix equation (2.1) and a particular solution X_0 such that $X_0 \neq (A^m)^{(1)}C(B^n)^{(1)}$ for any choice of $\{1\}$-inverses $(A^m)^{(1)}$ and $(B^n)^{(1)}$ similar to the corresponding example from [21].

2.2. In this part we analyse the matrix system (2.2), as a special extension of Penrose's matrix system [1]:

$$AX = B \qquad \wedge \qquad XD = E,$$

using the concept of reproductivity.

Based on the result from [1] we conclude that one common solution of the matrix system (2.2) is given by

$$X_1 = (A^m)^{(1)}B + E(D^n)^{(1)} - (A^m)^{(1)}(A^m)AE(D^n)^{(1)}.$$

The results which follow are extensions of the results from [18] (pp. 54–55) and [22].

Lemma 2.2. *The matrix equations* (2.2.1.) *and* (2.2.2.) *have a common solution if and only if each equation separately has a solution and*

$$A^m E = BD^n.$$

Proof. The proof is similar to the proof in [18]. □

Lemma 2.3. *If the matrix system* (2.2) *is consistent, the equivalence*

$$(A^m X = B \ \wedge \ XD^n = E) \iff$$
$$X = f(X) = X_1 + (I - (A^m)^{(1)}A^m)X(I - D^n(D^n)^{(1)})$$

is true.

Proof. The proof is similar to the proof in [22]. □

Remark 2.4. It is easy to show that $f^2(Y) = f(Y)$ i.e. the function f satisfies the condition of reproductivity. Therefore, if the matrix system (2.2) is consistent, it is equivalent to the reproductive matrix equation $X = f(X)$.

Based on the previous remark and Theorem 1.2 we conclude that the following theorem is true.

Theorem 2.3. *If the matrix system* (2.2) *is consistent, the general solution of the matrix system* (2.2) *is given by the formula*

$$X = f(Y) = X_1 + (I - (A^m)^{(1)}A^m)Y(I - D^n(D^n)^{(1)}),$$

where Y is an arbitrary matrix corresponding dimensions.

The following theorem is an extension of the previous theorem.

Theorem 2.4. *If X_0 is a particular solution of the matrix system* (2.2)*, the general solution of the matrix system* (2.2)*) is given by the formula*

$$X = g(Y) = X_0 + (I - (A^m)^{(1)}A^m)Y(I - D^n(D^n)^{(1)}),$$

where Y is an arbitrary matrix corresponding dimensions.

Proof. The proof is similar to the proof in [22]. □

Remark 2.5. From $g^2(Y) = g(Y) + (X_0 - X_1)$ we conclude that the function g is reproductive if and only if $X_0 = X_1$.

2.3. In this part we analyse the matrix system (2.3). The second equation of the matrix system (2.3) determines $\{5^k\}$-*inverse* of A. A solution of the matrix system (2.3) is $\{1, 5^k\}$-inverse which is called k-*commutative* $\{1\}$-*inverse* and is denoted by $\bar{A}$. k-commutative $\{1\}$-inverses were considered in [11], [12] and [16]. It is easy to check that one solution of the matrix system (2.3) is given by $\hat{X} = \bar{A}A\bar{A}$. In [12] J. D. Kečkić gave the condition for the consistency of the matrix system (2.3). We are going to represent the formula of the general reproductive solution for the consistent matrix system (2.3) using the concept of reproductive equations. We need the following three lemmas.

Lemma 2.4. $A^k\bar{A}^k = \bar{A}^kA^k$.

Proof. $A^k\bar{A}^k = \underbrace{A^k\bar{A}}_{(=\bar{A}A^k)}\bar{A}^{k-1} = \bar{A}A^k\bar{A}^{k-1} = \bar{A}\underbrace{A^k\bar{A}}_{(=\bar{A}A^k)}\bar{A}^{k-2} = \bar{A}^2A^k\bar{A}^{k-2} = \ldots = \bar{A}^kA^k.$ □

Lemma 2.5. *For any particular solution X_0 of the matrix system* (2.3) *the equalities*

$$X_0A^k\bar{A}^k = A^k\bar{A}^{k+1} \quad \textit{and} \quad \bar{A}^kA^kX_0 = A^k\bar{A}^{k+1}$$

are true.

Proof. We are going to prove the first equality.

$$\begin{aligned}
X_0\underbrace{A^k}_{(=A^k\bar{A}^kA^k)}\bar{A}^k &= \underbrace{X_0A^k}_{(=A^kX_0)}\underbrace{\bar{A}^kA^k}_{(\underset{L.2.4.}{=}A^k\bar{A}^k)}\bar{A}^k = A^kX_0A^k\bar{A}^{2k} \\
&= A^{k-1}\underbrace{AX_0A}_{(=A)}A^{k-1}\bar{A}^{2k} = A^{k-1}AA^{k-1}\bar{A}^{2k} \\
&= A^{2k-1}\bar{A}^{2k}, \\
\underbrace{A^k}_{(=A^k\bar{A}^kA^k)}\bar{A}^{k+1} &= A^k\underbrace{\bar{A}^kA^k}_{(\underset{L.2.4.}{=}A^k\bar{A}^k)}\bar{A}^{k+1} = A^{2k}\bar{A}^{2k+1} \\
&= A^k\underbrace{A^k\bar{A}}_{(=\bar{A}A^k)}\bar{A}^{2k} = A^k\bar{A}A^k\bar{A}^{2k} \\
&= A^{k-1}\underbrace{A\bar{A}A}_{(=A)}A^{k-1}\bar{A}^{2k} = A^{k-1}AA^{k-1}\bar{A}^{2k} \\
&= A^{2k-1}\bar{A}^{2k}.
\end{aligned}$$

Therefore, $X_0A^k\bar{A}^k = A^k\bar{A}^{k+1}$. The second equality is proved similarly. □

Lemma 2.6. *Let* $\hat{X} = \bar{A}A\bar{A}$. *If the matrix system* (2.3) *is consistent, the equivalence*

$$AXA = A \;\wedge\; A^kX = XA^k \quad \Longleftrightarrow$$
$$X = f(X) = \hat{X} + X - (I - \bar{A}A)XA^k\bar{A}^k - \bar{A}^kA^kX(I - A\bar{A}) - \bar{A}AXA\bar{A}$$

is true.

Proof. $\Longrightarrow)$: Suppose that $AXA = A \;\wedge\; A^kX = XA^k$. Then,

$$\bar{A}AXA^k\bar{A}^k = \bar{A}\underbrace{AXA}_{(=A)}A^{k-1}\bar{A}^k = \underbrace{\bar{A}A^k}_{(=A^k\bar{A})}\bar{A}^k = A^k\bar{A}\bar{A}^k = A^k\bar{A}^{k+1}.$$

Bearing in mind that $XA^k\bar{A}^k = A^k\bar{A}^{k+1}$ (Lemma 2.5.), we conclude that

$$XA^k\bar{A}^k = \bar{A}AXA^k\bar{A}^k.$$

In a similar way we get that $\bar{A}^kA^kX = \bar{A}^kA^kXA\bar{A}$.

Therefore,

$$\begin{aligned} X &= \bar{A}A\bar{A} + X - XA^k\bar{A}^k + \bar{A}AXA^k\bar{A}^k - \bar{A}^kA^kX + \bar{A}^kA^kXA\bar{A} - \bar{A}\underbrace{A}_{(=AXA)}\bar{A} \\ &= \bar{A}A\bar{A} + X - XA^k\bar{A}^k + \bar{A}AXA^k\bar{A}^k - \bar{A}^kA^kX + \bar{A}^kA^kXA\bar{A} - \bar{A}AXA\bar{A} \\ &= \bar{A}A\bar{A} + X - (I - \bar{A}A)XA^k\bar{A}^k - \bar{A}^kA^kX(I - A\bar{A}) - \bar{A}AXA\bar{A} \\ &= f(X). \end{aligned}$$

$\Longleftarrow)$: Suppose that $X = f(X) = \bar{A}A\bar{A} + X - (I - \bar{A}A)XA^k\bar{A}^k - \bar{A}^kA^kX(I - A\bar{A})$ $-\bar{A}AXA\bar{A}$. Then,

$$\begin{aligned} AXA &= Af(X)A \\ &= A\Big(\bar{A}A\bar{A} + X - (I - \bar{A}A)XA^k\bar{A}^k - \bar{A}^kA^kX(I - A\bar{A}) - \bar{A}AXA\bar{A}\Big)A \\ &= \underbrace{A\bar{A}A}_{(=A)}\bar{A}A + AXA - \underbrace{A(I - \bar{A}A)}_{(=0)}XA^k\bar{A}^kA - A\bar{A}^kA^kX\underbrace{(I - A\bar{A})A}_{(=0)} - \underbrace{A\bar{A}A}_{(=A)}X\underbrace{A\bar{A}A}_{(=A)} \\ &= \underbrace{A\bar{A}A}_{(=A)} + AXA - AXA \\ &= A, \end{aligned}$$

$$
\begin{aligned}
A^kX &= A^kf(X)\\
&= A^k\Big(\bar{A}A\bar{A} + X - (I-\bar{A}A)XA^k\bar{A}^k - \bar{A}^kA^kX(I-A\bar{A}) - \bar{A}AXA\bar{A}\Big)\\
&= A^k\bar{A}A\bar{A} + A^kX - A^k(I-\bar{A}A)XA^k\bar{A}^k - \underbrace{A^k\bar{A}^kA^k}_{(=A^k)}X(I-A\bar{A}) - A^k\bar{A}\underbrace{AXA}_{(=A)}\bar{A}\\
&= A^k\bar{A}A\bar{A} + A^kX - A^kXA^k\bar{A}^k + A^k\bar{A}AXA^k\bar{A}^k - A^kX + A^kXA\bar{A} - A^k\bar{A}A\bar{A}\\
&= -A^k\underbrace{XA^k\bar{A}^k}_{(\underset{L.2.5.}{=}A^k\bar{A}^{k+1})} + A^k\bar{A}\underbrace{AXA}_{(=A)}A^{k-1}\bar{A}^k + A^{k-1}\underbrace{AXA}_{(=A)}\bar{A}\\
&= -A^kA^k\bar{A}^{k+1} + A^k\underbrace{\bar{A}A^k}_{(=A^k\bar{A})}\bar{A}^k + A^k\bar{A}\\
&= -A^{2k}\bar{A}^{k+1} + A^kA^k\bar{A}\bar{A}^k + A^k\bar{A}\\
&= -A^{2k}\bar{A}^{k+1} + A^{2k}\bar{A}^{k+1} + A^k\bar{A}\\
&= A^k\bar{A},
\end{aligned}
$$

$$
\begin{aligned}
XA^k &= f(X)A^k\\
&= \Big(\bar{A}A\bar{A} + X - (I-\bar{A}A)XA^k\bar{A}^k - \bar{A}^kA^kX(I-A\bar{A}) - \bar{A}AXA\bar{A}\Big)A^k\\
&= \bar{A}A\bar{A}A^k + XA^k - (I-\bar{A}A)X\underbrace{A^k\bar{A}^kA^k}_{(=A^k)} - \bar{A}^kA^kX(I-A\bar{A})A^k - \bar{A}\underbrace{AXA}_{(=A)}\bar{A}A^k\\
&= \bar{A}A\bar{A}A^k + XA^k - XA^k + \bar{A}AXA^k - \bar{A}^kA^kXA^k + \bar{A}^kA^kXA\bar{A}A^k - \bar{A}A\bar{A}A^k\\
&= \bar{A}\underbrace{AXA}_{(=A)}A^{k-1} - \underbrace{\bar{A}^kA^kX}_{(\underset{L.2.5.}{=}A^k\bar{A}^{k+1})}A^k + \bar{A}^kA^kXA\bar{A}A^k\\
&= \bar{A}A^k - A^k\bar{A}^{k+1}A^k + \bar{A}^kA^kXA\bar{A}A^k\\
&= \bar{A}A^k - A^k\bar{A}^{k+1}A^k + \bar{A}^kA^{k-1}\underbrace{AXA}_{(=A)}\bar{A}A^k\\
&= \bar{A}A^k - A^k\bar{A}^{k+1}A^k + \underbrace{\bar{A}^kA^k}_{(\underset{L.2.4.}{=}A^k\bar{A}^k)}\bar{A}A^k\\
&= \bar{A}A^k - A^k\bar{A}^{k+1}A^k + A^k\bar{A}^{k+1}A^k\\
&= \bar{A}A^k.
\end{aligned}
$$

From $A^k\bar{A} = \bar{A}A^k$ we see that $A^kX = XA^k$. □

Remark 2.6. It is easy to show that $f^2(Y) = f(Y)$ i.e. the function f satisfies the condition of reproductivity. Therefore, if the matrix system (2.3) is consistent, it is equivalent to the reproductive matrix equation $X = f(X)$.

Based on the previous remark and Theorem 1.2 we conclude that the following theorem is true.

Theorem 2.5. *If the matrix system* (2.3) *is consistent, the general solution of the matrix system* (2.3) *is given by the formula*

$$X = f(Y) = \bar{A}A\bar{A} + Y - (I - \bar{A}A)YA^k\bar{A}^k - \bar{A}^kA^kY(I - A\bar{A}) - \bar{A}AYA\bar{A},$$

where Y is an arbitrary matrix corresponding dimensions.

In [12] J. D. Kečkić also proved this theorem, but his proof is different from the previously exposed proof.
The following theorem is an extension of the previous theorem.

Theorem 2.6. *If X_0 is a particular solution of the matrix equation* (2.3), *the general solution of the matrix equation* (2.3) *is given by the formula*

$$X = g(Y) = X_0 + Y - (I - \bar{A}A)YA^k\bar{A}^k - \bar{A}^kA^kY(I - A\bar{A}) - \bar{A}AYA\bar{A}. \tag{2.6}$$

where Y is an arbitrary matrix corresponding dimensions.

Proof. It is easy to see that the solution of the matrix system (2.3) is given by (2.6). On the contrary, let X be any solution of the matrix equation (2.3), then

$$\begin{aligned} X = {} & X - \underbrace{A^k\bar{A}^{k+1}}_{(\underset{L.2.5.}{=} XA^k\bar{A}^k)} + \underbrace{A^k\bar{A}^{k+1}}_{(\underset{L.2.5.}{=} X_0A^k\bar{A}^k)} + \bar{A}A^{k+1}\bar{A}^{k+1} - \bar{A}A^{k+1}\bar{A}^{k+1} - \underbrace{A^k\bar{A}^{k+1}}_{(\underset{L.2.5.}{=} \bar{A}^kA^kX)} \\ & + \underbrace{A^k\bar{A}^{k+1}}_{(\underset{L.2.5.}{=} \bar{A}^kA^kX)}A\bar{A} + \underbrace{A^k\bar{A}^{k+1}}_{(\underset{L.2.5.}{=} \bar{A}^kA^kX_0)} - \underbrace{A^k\bar{A}^{k+1}}_{(\underset{L.2.5.}{=} \bar{A}^kA^kX_0)}A\bar{A} - \bar{A}\underbrace{A}_{(=AXA)}\bar{A} + \bar{A}\underbrace{A}_{(=AX_0A)}\bar{A} \\ = {} & X - XA^k\bar{A}^k + X_0A^k\bar{A}^k + \bar{A}A\underbrace{A^k\bar{A}^{k+1}}_{(\underset{L.2.5.}{=} XA^k\bar{A}^k)} - \bar{A}A\underbrace{A^k\bar{A}^{k+1}}_{(\underset{L.2.5.}{=} X_0A^k\bar{A}^k)} - \bar{A}^kA^kX \\ & + \bar{A}^kA^kXA\bar{A} + \bar{A}^kA^kX_0 - \bar{A}^kA^kX_0A\bar{A} - \bar{A}AXA\bar{A} + \bar{A}AX_0A\bar{A} \end{aligned}$$

$$
\begin{aligned}
&= X_0+(X-X_0)-(X-X_0)A^k\bar{A}^k+\bar{A}AXA^k\bar{A}^k-\bar{A}AX_0A^k\bar{A}^k-\bar{A}^kA^k(X-X_0)\\
&\quad +\bar{A}^kA^k(X-X_0)A\bar{A}-\bar{A}A(X-X_0)A\bar{A}\\
&= X_0+(X-X_0)-(X-X_0)A^k\bar{A}^k+\bar{A}A(X-X_0)A^k\bar{A}^k-\bar{A}^kA^k(X-X_0)\\
&\quad +\bar{A}^kA^k(X-X_0)A\bar{A}-\bar{A}A(X-X_0)A\bar{A}\\
&= X_0+(X-X_0)-(I-\bar{A}A)(X-X_0)A^k\bar{A}^k-\bar{A}^kA^k(X-X_0)(I-A\bar{A})\\
&\quad -\bar{A}A(X-X_0)A\bar{A}\\
&= X_0+Y-(I-\bar{A}A)YA^k\bar{A}^k-\bar{A}^kA^kY(I-A\bar{A})-\bar{A}AYA\bar{A}\\
&= g(Y),
\end{aligned}
$$

where $Y = X - X_0$. From this we see that every solution X of the matrix system (2.2) can be represented in the form (2.6). □

Remark 2.7. From $g^2(Y) = g(Y) + (X_0 - \hat{X})$ we conclude that the function g is reproductive if and only if $X_0 = \hat{X}$.

Remark 2.8. The preceding result is an extension of the consideration which is given in [22] (see Application 2.2)

3. Conclusion

We want to emphasize that there are also other matrix equations and matrix systems whose solutions can be analysed in the same way as we done with (2.1), (2.2) and (2.3).

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[1]Department of Applied Mathematics
Faculty of Electrical Engineering, University of Belgrade
Serbia
E-mail address: `malesevic@etf.rs`

[2] part-time job
Department of Mathematics, Physics and Descriptive Geometry
Faculty of Civil Engineering, University of Belgrade
Serbia
E-mail address: `radicic.biljana@yahoo.com`