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DERIVED GRAPHS OF SOME GRAPHS

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Abstract. The derived graph of a simple graph G, denoted by $G^\dagger$, is the graph having the same vertex set as G, in which two vertices are adjacent if and only if their distance in G is two. Continuing the studies communicated in Kragujevac J. Math. **34** (2010), 139–146, we examined derived graphs of some graphs and determine their spectra.

1. Introduction

In two recent papers [1, 2], the so-called derived graphs were considered, with emphasis on their spectral properties. In the present paper we obtain a few more results along the same lines.

In this paper, we consider simple graphs, that is, graphs without directed, multiple, or weighted edges, and without self loops. Let G be such a graph and let its vertex set be $V(G) = \{v_1, v_2, \dots, v_n\}$. The distance between the vertices v_i and v_j is equal to length of a shortest path between v_i and v_j.

Definition 1.1. Let G be a simple graph with vertex set $V(G)$. The *derived graph* of G, denoted by $G^\dagger$ is the graph with vertex set $V(G)$, in which two vertices are adjacent if and only if their distance in G is two.

Definition 1.2. The spectrum of the derived graph of the graph G (that is, the multiset of the eigenvalues of the adjacency matrix of G) is said to be the *second-stage spectrum* of G.

It is needless to say that the second–stage spectrum of the graph G is just the ordinary spectrum of its derived graph $G^\dagger$.

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Definition 1.3. The *energy* of a graph G is the sum of the absolute values of the eigenvalues of G. The energy of the derived graph of a graph G is referred to as the *second–stage energy* of G.

In [2] graphs whose derived graphs are connected are characterized and upper bounds for the eigenvalues of $G^\dagger$ are established. In [1], results for spectra and energy of derived graphs, in particular for graphs of diameter 2, are communicated.

In order to state our main results, we need some preparations.

2. Auxiliary results

Let G be a simple graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = \{e_1, e_2, \dots, e_m\}$. Then the vertex-edge incidence matrix of G is the $n \times m$ matrix $\mathcal{J} = \mathcal{J}(G)$ whose (i, j) entry is equal to unity if the vertex v_i is incident to the edge e_j, and is zero otherwise.

Let $R(G)$ be the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the endpoints of the edge corresponding to it. It will be called *semi total point graph*. The construction of $R(G)$ is illustrated by the following example:

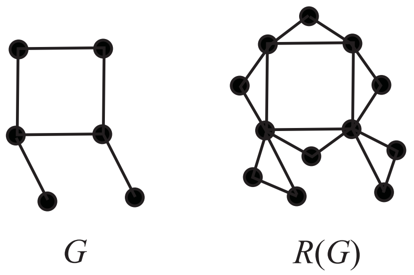

Figure 1. A graph and its semi total point graph.

The adjacency matrix of $R(G)$ has the form

$$R(G) = \begin{bmatrix} \mathbf{0}_m & \mathcal{J}^t \\ \mathcal{J} & \mathbf{A} \end{bmatrix}$$

where $\mathbf{A}$ and $\mathcal{J}$ are, respectively, the adjacency and incidence matrices of G.

Theorem 2.1. [4] *If G is a regular graph of degree r with n vertices and $m = nr/2$ edges, then the characteristic polynomial of $R(G)$ is given by*

$$\phi(R(G), \lambda) = \lambda^{m-n} (\lambda + 1)^n \, \phi\Big(G, \frac{\lambda^2 - r}{\lambda + 1}\Big).$$

Lemma 2.1. [5] *If M is a nonsingular square matrix then,*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M| \, \Big|Q - P\, M^{-1}\, N\Big|.$$

3. Main results

We now generalize the concept of semi total point graph as follows.

Definition 3.1. Let G be a simple graph of order n possessing m edges. The k-*th semi total point graph* of G, denoted by $R^k(G)$, is the graph obtained by adding k vertices to each edge of G and joining them to the endpoints of the respective edge. Obviously, this is equivalent to adding k triangles to each edge of G.

The graph $R^k(G)$ is of order $n+mk$ and has $(1+2k)m$ edges. Of course, the semi total point graph discussed in the preceding section is just the special case of $R^k(G)$ for $k=1$. The construction of $R^k(G)$ is illustrated by the following example:

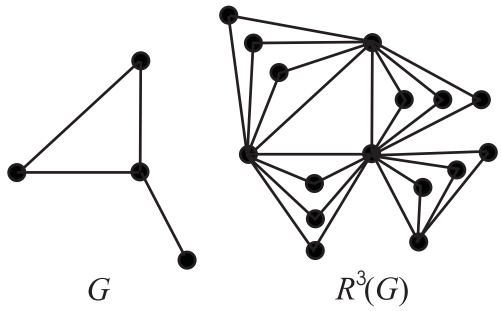

Figure 2. A graph and its k-th semi total point graph for $k=3$.

Claim 3.1. *Let G be a simple graph with m edges, Δ triangles, and degree sequence* $\Big[d_1, d_2, \ldots, d_n\Big]$.

1. *The number of triangles of $R^k(G)$ is equal to $\Delta + mk$.*
2. *The degree sequences of $R^k(G)$ is*

$$\Big[(k+1)d_1, (k+1)d_2, \ldots, (k+1)d_n, 2, 2, \ldots, 2 \ (mk \ times)\Big].$$

We now generalize Theorem 2.1:

Theorem 3.1. *If G is a regular graph of order n and degree r, then for any $k \geq 1$, the characteristic polynomial of its k-th semi total point graph $R^k(G)$ is given by*

$$\phi(R^k(G), \lambda) = \lambda^{mk-n} \, (\lambda+k)^n \, \phi\Big(G, \frac{\lambda^2 - kr}{\lambda+k}\Big) \tag{3.1}$$

where $m = nr/2$ is the number of edges of G.

Proof. By a pertinent labeling of the vertices of $R^k(G)$, its characteristic polynomial assumes the form

$$\phi(R^k(G), \lambda) = \begin{vmatrix} \lambda \, \mathbf{I}_{mk} & -\mathbf{\Gamma}^t \\ -\mathbf{\Gamma} & \lambda \, \mathbf{I}_n - \mathbf{A}(G) \end{vmatrix}$$

where $\mathbf{I}_p$ stands for the unit matrix of order p, $\mathbf{A}(G)$ is the adjacency matrix of G, and $\mathbf{\Gamma} = (\mathcal{J}(G), \mathcal{J}(G), \ldots, \mathcal{J}(G))$. Then by applying Lemma 2.1,

$$\phi(R^k(G), \lambda) = \lambda^{mk} \left| \lambda\, \mathbf{I}_n - \mathbf{A}(G) - \frac{\mathbf{\Gamma}\, \mathbf{\Gamma}^t}{\lambda\, \mathbf{I}_{mk}} \right| .$$

Since G is regular,

$$\mathbf{\Gamma}\, \mathbf{\Gamma}^t = k\, \mathbf{A}(G) + kr\, \mathbf{I}_n$$

from which

$$\begin{aligned} \phi(R^k(G), \lambda) &= \lambda^{mk} \left| \frac{(\lambda^2 - kr)\mathbf{I}_n - (\lambda + k)\mathbf{A}(G)}{\lambda} \right| \\ &= \lambda^{mk-n}\, (\lambda + k)^n \left| \frac{\lambda^2 - kr}{\lambda + k}\, \mathbf{I}_n - \mathbf{A}(G) \right| \end{aligned}$$

and equation (3.1) follows straightforwardly. □

In what follows we consider a class of graphs constructed by attaching k new pendent vertices to each vertex of the underlying graph. These graphs are often referred to as *thorny graphs or thorn graphs* and have been much studied in the mathematical literature (see, for instance [3, 7–9]). The thorny graph pertaining to the graph G will be denoted by G^{+k}. The spectrum of G^{+k} was determined in [6]

We now establish a few elementary properties of the derived graphs of thorny graphs.

Lemma 3.1. *Let C_n be the cycle on n vertices. Then*

$$\begin{aligned} (C_3^{+1})^\dagger &\cong C_6, \\ (C_3^{+2})^\dagger &\;\; \textit{is biregular of degrees } 4 \textit{ and } 3, \\ (C_3^{+3})^\dagger &\;\; \textit{is biregular of degrees } 6 \textit{ and } 4, \\ (C_{2p+1}^{+k})^\dagger &\cong R^k(C_{2p+1}),\;\; p \geq 2, \\ (C_{2p}^{+k})^\dagger &\cong R^k(C_p) \cup R^k(C_p),\;\; p \geq 2. \end{aligned}$$

Proof. Follows by construction. □

The below results can be obtained by simple, yet lengthy calculation, which we skip.

Claim 3.2. *Let K_n be the complete graph on n vertices. Then for $k \geq 1$, the second–stage spectrum of K_n^{+k} consists of:*

-1	$nk - 1$ *times,*
k	$n - 1$ *times,*
$\frac{1}{2}\left[k - 1 + \sqrt{(k-1)^2 + 4k(n-1)^2}\right]$	*once,*
$\frac{1}{2}\left[k - 1 - \sqrt{(k-1)^2 + 4k(n-1)^2}\right]$	*once.*

Consequently, the second–stage energy of K_n^{+k}, *that is the energy of* $(K_n^{+k})^\dagger$, *is equal to* $2nk - k - 1 + \sqrt{(k-1)^2 + 4k(n-1)^2}$.

In the special case $k = n$ the second–stage energy of K_n^{+k} is equal to $2n^2 - n - 1 + (n-1)\sqrt{1+4n}$. Therefore, for $n = 2, 6, 12, 20, \ldots$ i.e., for $n = p(p+1)$, the second–stage energy of K_n^{+k} is integer.

Claim 3.3. *Let* $K_{a,b}$ *be the complete bipartite graph on* $a+b$ *vertices. Then for* $k \geq 1$, *the second–stage spectrum of* $K_{a,b}^{+k}$ *consists of:*

$$
\begin{array}{ll}
-1 & (a+b)k - 2 \text{ times}, \\
k-1 & a+b-2 \text{ times}, \\
\frac{1}{2}\left[k + a - 2 + \sqrt{(k-a)^2 + 4kab}\right] & \text{once}, \\
\frac{1}{2}\left[k + a - 2 - \sqrt{(k-a)^2 + 4kab}\right] & \text{once}, \\
\frac{1}{2}\left[k + b - 2 + \sqrt{(k-b)^2 + 4kab}\right] & \text{once}, \\
\frac{1}{2}\left[k + b - 2 - \sqrt{(k-b)^2 + 4kab}\right] & \text{once}.
\end{array}
$$

Consequently, the second–stage energy of $K_{a,b}^{+k}$, *that is, the energy of* $(K_{a,b}^{+k})^\dagger$, *is equal to* $(a+b)(2k-1) - 2k + \sqrt{(k-a)^2 + 4kab} + \sqrt{(k-b)^2 + 4kab}$.

For the special case $a = b$ we have:

Claim 3.4. *Let* $K_{a,a}$ *be the complete bipartite graph on* $2a$ *vertices. Then for* $k \geq 1$, *the second–stage spectrum of* $K_{a,a}^{+k}$ *consists of:*

$$
\begin{array}{ll}
-1 & 2(ak-1) \text{ times} \\
k-1 & 2(a-1) \text{ times} \\
\frac{1}{2}\left[k + a - 2 + \sqrt{(k-a)^2 + 4ka^2}\right] & 2 \text{ times} \\
\frac{1}{2}\left[k + a - 2 - \sqrt{(k-a)^2 + 4ka^2}\right] & 2 \text{ times}
\end{array}
$$

Consequently, the second–stage energy of $K_{a,a}^{+k}$, *that is, the energy of* $(K_{a,a}^{+k})^\dagger$, *is equal to* $2a(2k-1) - 2k + 2\sqrt{(k-a)^2 + 4ka^2}$.

In the special case $a = b = k$ the second–stage energy of $K_{a,b}^{+k}$ is equal to $4a^2 - 4a + 4a\sqrt{a}$. Therefore, for $a = p^2$, the second–stage energy of $K_{a,b}^{+k}$ is an integer divisible by 4.

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