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CHROMATIC NUMBER AND SOME MULTIPLICATIVE VERTEX-DEGREE-BASED INDICES OF GRAPHS

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Abstract. For a (molecular) graph, the first and second Zagreb indices (M_1 and M_2) are two well-known topological indices in chemical graph theory introduced in 1972 by Gutman and Trinajstić. Multiplicative versions of Zagreb indices, such as Narumi-Katayama index, multiplicative Zagreb index and multiplicative sum Zagreb index, have been much studied in the past. Let $\mathbf{G}(n,k)$ be the set of connected graphs of order n and with chromatic number k. In this paper we show that, in $\mathbf{G}(n,k)$, Turán graph $T_n(k)$ has the maximal Narumi-Katayama index, the maximal multiplicative Zagreb index and the maximal multiplicative sum Zagreb index. And the extremal graphs from $\mathbf{G}(n,k)$ with $k=2$ or 3 are determined with minimal values of these above indices.

1. Introduction

We only consider finite, undirected and simple graphs throughout this paper. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *degree* of $v \in V(G)$, denoted by $d_G(v)$, is the number of vertices in G adjacent to v. For a subset W of $V(G)$, let $G-W$ be the subgraph of G obtained by deleting the vertices of W and the edges incident with them. Similarly, for a subset E' of $E(G)$, we denote by $G-E'$ the subgraph of G obtained by deleting the edges of E'. If $W=\{v\}$ and $E'=\{xy\}$, the subgraphs $G-W$ and $G-E'$ will be written as $G-v$ and $G-xy$ for short, respectively. For any two nonadjacent vertices x and y of graph G, we let $G+xy$ be the graph obtained from G by adding an edge xy. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colors such that G can be colored

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with these colors in order that no two adjacent vertices have the same color. Other undefined notations and terminology on the graph theory can be found in [2].

A graphical invariant is a number related to a graph which is a structural invariant, in other words, it is a fixed number under graph automorphisms. In chemical graph theory, these invariants are also known as the topological indices. Two of the oldest graph invariants are the well-known Zagreb indices first introduced in [11] where Gutman and Trinajstić examined the dependence of total π-electron energy on molecular structure and elaborated in [12]. For a (molecular) graph G, the first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ are, respectively, defined as follows:

$$M_1 = M_1(G) = \sum_{v \in V(G)} d_G(v)^2, \qquad M_2 = M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

These two classical topological indices reflect the extent of branching of the molecular carbon-atom skeleton [1, 20]. The first Zagreb index M_1 was also termed as "Gutman index" by some scholars ([20]). The main properties of M_1 and M_2 were summarized in [4, 5, 8, 15, 17]. In particular, Deng [5] gave a unified approach to determine extremal values of Zagreb indices for trees, unicyclic, and bicyclic graphs, respectively. Other recent results on Zagreb indices can be found in [24] and the references cited therein.

Recently, Todeschini et al. [19, 21] have proposed the multiplicative variants of ordinary Zagreb indices, which are defined as follows:

$$\prod\nolimits_1 = \prod\nolimits_1(G) = \prod_{v \in V(G)} d_G(v)^2, \qquad \prod\nolimits_2 = \prod\nolimits_2(G) = \prod_{uv \in E(G)} d_G(u) d_G(v).$$

These two graph invariants are called "first and second multiplicative Zagreb indices" by Gutman [7]. In the same paper, Gutman determined that among all trees of order $n \geq 4$, the extremal trees with respect to these multiplicative Zagreb indices are path P_n (with maximal $\prod_1$ and with minimal $\prod_2$) and star S_n (with maximal $\prod_2$ and with minimal $\prod_1$). A molecular graph which models the skeleton of a molecule ([23]) is a connected graph of maximum degree at most 4. The bounds of a molecular topological descriptor are important information of a (molecular) graph in the sense that they establish the approximate range of the descriptor in terms of molecular structural parameters.

In 1984, Narumi and Katayama [16] first introduced the following product index which is named as Narumi-Katayama index ([13, 14, 22])

$$NK = NK(G) = \prod_{v \in V(G)} d_G(v).$$

Note that, for any graph G, we have $\prod_1(G) = NK(G)^2$. Thus the first multiplicative Zagreb index ($\prod_1$) can not be viewed as new topological index of graph, and in this paper we only need to deal with the case of $NK(G)$ rather than $\prod_1(G)$. Very recently, Eliasi, Iranmanesh and Gutman [6] first introduced another multiplicative version of first Zagreb index, which is called as *multiplicative sum Zagreb index* [25] to distinguish from first multiplicative Zagreb index ($\prod_1$), as follows:

$$\prod\nolimits_1^*(G) = \prod_{uv\in E(G)} (d_G(u) + d_G(v)).$$

Some new results can be found in [9, 25, 26] on Narumi-Katayama index, second multiplicative Zagreb index and multiplicative sum Zagreb index.

Let $\mathbf{G}(n,k)$ be the set of connected graphs of order n and with chromatic number k. In this paper we show that, in $\mathbf{G}(n,k)$, Turán graph $T_n(k)$ has the maximal Narumi-Katayama index (NK), the maximal second multiplicative Zagreb index ($\prod_2$) and the maximal multiplicative sum Zagreb index ($\prod_1^*$). Moreover the extremal graphs from $\mathbf{G}(n,k)$ with minimal values of these above three indices are determined for $k = 2, 3$.

2. Some lemmas

In this section we will list or prove some lemmas as preliminaries, which will play an important role in the next proofs.

By the definitions of Narumi-Katayama index, second multiplicative Zagreb index and multiplicative sum Zagreb index, these two lemmas below can be easily obtained.

Lemma 2.1. *Let G be a graph with two nonadjacent vertices $u, v \in V(G)$. Then we have*
(1) $NK(G + uv) > NK(G)$;
(2) $\prod_2(G + uv) > \prod_2(G)$;
(3) $\prod_1^*(G + uv) > \prod_1^*(G)$.

Lemma 2.2. *Let G be a graph with $e \in E(G)$. Then we have*
(1) $NK(G - e) < NK(G)$;
(2) $\prod_2(G - e) < \prod_2(G)$;
(3) $\prod_1^*(G - e) < \prod_1^*(G)$.

Lemma 2.3. [7] *For any graph G, we have* $\prod_2(G) = \prod_{x\in V(G)} d_G(x)^{d_G(x)}$.

Recalling that $\prod_1(G) = NK(G)^2$ for any graph G, the following remark is obvious.

Remark 2.1. Let G_1 and G_2 be two connected graphs. Then we have $\prod_1(G_1) > \prod_1(G_2)$ if and only if $NK(G_1) > NK(G_2)$.

Remark 2.2. [25, 26] Any tree T of size t attached to a graph G can be changed into a path P_{t+1}. During this process, the first multiplicative Zagreb index $\prod_1$ increases, while the second multiplicative Zagreb index $\prod_2$ and multiplicative sum Zagreb index $\prod_1^*$ all decrease.

Combining Remarks 2.1 and 2.2, we can easily obtain the following remark.

Remark 2.3. Any tree T of size t attached to a graph G can be changed into a path P_{t+1}. During this process, Narumi-Katayama index NK increases, while the second multiplicative Zagreb index $\prod_2$ and multiplicative sum Zagreb index $\prod_1^*$ all decrease.

Now we consider a graph transformation, which will make different effects on Narumi-Katayama index, second multiplicative Zagreb index and multiplicative sum Zagreb index.

Transformation A. Assume that a pendent path $P = v_1v_2 \cdots v_{t-1}v_t$ is attached at v_1 in graph G and there are two neighbors x and y of v_1 different from v_2. Let $G' = G - xv_1 + xv_t$, see Figure 1.

Lemma 2.4. [25, 26] *Let G and G' be two graphs as shown in Figure 1. Then we have*
(1) $\prod_1(G) < \prod_1(G')$;
(2) $\prod_2(G) > \prod_2(G')$;
(3) $\prod_1^*(G) > \prod_1^*(G')$.

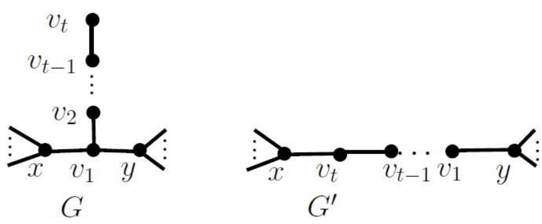

FIGURE 1. Transformation A

Based on Remark 2.1 and Lemma 2.4, the following lemma follows immediately.

Lemma 2.5. *Let G and G' be two graphs as shown in Figure 1. Then we have*
(1) $NK(G) < NK(G')$;
(2) $\prod_2(G) > \prod_2(G')$;
(3) $\prod_1^*(G) > \prod_1^*(G')$.

Lemma 2.6. [7, 9, 25] *Let T be a tree of order $n \geq 5$ different from S_n and P_n. Then*
(1) $NK(S_n) < NK(T)$;
(2) $\prod_2(P_n) < \prod_2(T)$;
(3) $\prod_1^*(P_n) < \prod_1^*(T)$.

Let C_n^k be a graph obtained by attaching $n - k$ pendent edges to a vertex of C_k. The extremal unicyclic graphs with minimal values of Narumi-Katayama index, second multiplicative Zagreb index and multiplicative sum Zagreb index have been completely determined in the following lemma.

Lemma 2.7. [7, 25, 26] *Let G be a connected unicyclic graph of order $n \geq 4$ different from C_n^3 and C_n. Then*
(1) $NK(C_n^3) < NK(G)$;
(2) $\prod_2(C_n) < \prod_2(G)$;
(3) $\prod_1^*(C_n) < \prod_1^*(G)$.

Hereafter we always assume that $n_1 \leq n_2 \leq \cdots \leq n_k$ are positive integers with $\sum_{i=1}^{k} = n$. Denote by $K_{n_1,n_2,\cdots,n_k}$ a complete k-partite graph of order n whose partition sets are of size $n_1, n_2, \cdots, n_k$, respectively. The lemma below presents the values of Narumi-Katayama index, second multiplicative Zagreb index and multiplicative sum Zagreb index of $K_{n_1,n_2,\cdots,n_k}$, respectively.

Lemma 2.8. *Assume that $K_{n_1,n_2,\cdots,n_k}$ is the graph defined as above. Then we have*
(1) $NK(K_{n_1,n_2,\cdots,n_k}) = \prod_{i=1}^{k}(n-n_i)^{n_i}$;
(2) $\prod_2(K_{n_1,n_2,\cdots,n_k}) = \prod_{i=1}^{k}(n-n_i)^{(n-n_i)n_i}$;
(3) $\prod_1^*(K_{n_1,n_2,\cdots,n_k}) = \prod_{1\leq i<j\leq k}(2n-n_i-n_j)^{n_in_j}$.

Proof. For $j \in \{1, 2, \cdots, k\}$, in partition set of size n_j in $K_{n_1,n_2,\cdots,n_k}$, each vertex is of degree $n - n_j$. By the definition of Narumi-Katayama index and Lemma 2.3, it is easy to see that

$$NK(K_{n_1,n_2,\cdots,n_k}) = \prod_{i=1}^{k}(n-n_i)^{n_i} \quad \text{and} \quad \prod\nolimits_2(K_{n_1,n_2,\cdots,n_k}) = \prod_{i=1}^{k}(n-n_i)^{(n-n_i)n_i}.$$

Between two partition sets of sizes n_i, n_j with $1 \leq i < j \leq k$, respectively, in $K_{n_1,n_2,\cdots,n_k}$, there exist n_in_j edges linking these two sets. Moreover, the two vertices incident with each of these edges are of degrees $n - n_i$ and $n - n_j$, respectively. From the definition of multiplicative sum Zagreb index,

$$\prod\nolimits_1^*(K_{n_1,n_2,\cdots,n_k}) = \prod_{1\leq i<j\leq k}(2n-n_i-n_j)^{n_in_j},$$

ending the proof of this lemma. □

3. Main results

In this section we will determine the extremal graphs from $\mathbf{G}(n, k)$ with respect to Narumi-Katayama index, second multiplicative Zagreb index and multiplicative sum Zagreb index, respectively. To do it, we first prove a related lemma below.

Lemma 3.1. *Let $K_{n_1,n_2,\cdots,n_k}$ be a graph defined as above with $n_j - n_i \geq 2$ for $i < j$. Then*
(1) $NK(K_{n_1,n_2,\cdots,n_i,\cdots,n_j,\cdots,n_k}) < NK(K_{n_1,n_2,\cdots,n_i+1,\cdots,n_j-1,\cdots,n_k})$;
(2) $\prod_2(K_{n_1,n_2,\cdots,n_i,\cdots,n_j,\cdots,n_k}) < \prod_2(K_{n_1,n_2,\cdots,n_i+1,\cdots,n_j-1,\cdots,n_k})$;

(3) $\prod_1^*(K_{n_1,n_2,\cdots,n_i,\cdots,n_j,\cdots,n_k}) < \prod_1^*(K_{n_1,n_2,\cdots,n_i+1,\cdots,n_j-1,\cdots,n_k})$.

Proof. Set

$$\Delta_1 = \frac{NK(K_{n_1,n_2,\cdots,n_i+1,\cdots,n_j-1,\cdots,n_k})}{NK(K_{n_1,n_2,\cdots,n_i,\cdots,n_j,\cdots,n_k})}, \qquad \Delta_2 = \frac{\prod_2(K_{n_1,n_2,\cdots,n_i+1,\cdots,n_j-1,\cdots,n_k})}{\prod_2(K_{n_1,n_2,\cdots,n_i,\cdots,n_j,\cdots,n_k})}$$

and

$$\Delta_3 = \frac{\prod_1^*(K_{n_1,n_2,\cdots,n_i+1,\cdots,n_j-1,\cdots,n_k})}{\prod_1^*(K_{n_1,n_2,\cdots,n_i,\cdots,n_j,\cdots,n_k})}.$$

Since the values of Narumi-Katayama index, second multiplicative Zagreb index and multiplicative sum Zagreb index of $K_{n_1,n_2,\cdots,n_i,\cdots,n_j,\cdots,n_k}$ are all positive, it suffices to prove that $\Delta_i > 1$ for $i = 1, 2, 3$.

From Lemma 2.8 (1), we obtain

$$\begin{aligned}
\Delta_1 &= \frac{(n-n_i-1)^{n_i+1}(n-n_j+1)^{n_j-1}}{(n-n_i)^{n_i}(n-n_j)^{n_j}} \\
&= \left(1-\frac{1}{n-n_i}\right)^{n_i} \times \frac{n-n_i-1}{n-n_j} \times \left(1+\frac{1}{n-n_j}\right)^{n_j-1} \\
&> \left[\left(1-\frac{1}{n-n_i}\right)\left(1+\frac{1}{n-n_j}\right)\right]^{n_i} > 1.
\end{aligned}$$

Note that the last inequality holds since $(1-\frac{1}{n-n_i})(1+\frac{1}{n-n_j}) > 1$ when $n_j - n_i \geq 2$. So we complete the proof of result in (1).

In view of Lemma 2.8 (2), we get

$$\begin{aligned}
\Delta_2 &= \frac{(n-n_i-1)^{(n-n_i-1)(n_i+1)}(n-n_j+1)^{(n-n_j+1)(n_j-1)}}{(n-n_i)^{(n-n_i)n_i}(n-n_j)^{(n-n_j)n_j}} \\
&= \frac{(n-n_i-1)^{(n-n_i)n_i}(n-n_i-1)^{n-2n_i+1}}{(n-n_i)^{(n-n_i)n_i}} \cdot \frac{(n-n_j+1)^{(n-n_j)n_j}(n-n_j+1)^{2n_j-n-1}}{(n-n_j)^{(n-n_j)n_j}} \\
&= \left[\left(1-\frac{1}{n-n_i}\right)\left(1+\frac{1}{n-n_j}\right)\right]^{(n-n_i)n_i} \frac{(n-n_i-1)^{n-2n_i-1}}{(n-n_j+1)^{n-2n_j+1}} \\
&\qquad \text{as } (n-n_j)n_j > (n-n_i)n_i \text{ when } n_j - n_i \geq 2 \\
&> \left(\frac{n-n_i-1}{n-n_j+1}\right)^{n-2n_j+1} \qquad \text{as } \left(1-\frac{1}{n-n_i}\right)\left(1+\frac{1}{n-n_j}\right) > 1 \\
&\qquad \text{and } n-2n_i-1 > n-2n_j+1 \text{ when } n_j - n_i \geq 2 \\
&\geq 1.
\end{aligned}$$

And the last inequality holds since $\frac{n-n_i-1}{n-n_j+1} \geq 1$ when $n_j - n_i \geq 2$. Thus the proof of the result in (2) is over.

By Lemma 2.8 (3), we have

$$\begin{aligned}
\Delta_3 &= \frac{(2n-n_i-n_j)^{(n_i+1)(n_j-1)}}{(2n-n_i-n_j)^{n_in_j}} \prod_{1\le p\le k, p\ne i,j} \frac{(2n-n_p-n_i-1)^{n_p(n_i+1)}}{(2n-n_p-n_i)^{n_pn_i}} \\
&\quad \times \prod_{1\le p\le k, p\ne i,j} \frac{(2n-n_p-n_j+1)^{n_p(n_j-1)}}{(2n-n_p-n_j)^{n_pn_j}} \\
&> \prod_{1\le p\le k, p\ne i,j} \frac{(2n-n_p-n_i-1)^{n_p(n_i+1)}}{(2n-n_p-n_i)^{n_pn_i}} \prod_{1\le p\le k, p\ne i,j} \frac{(2n-n_p-n_j+1)^{n_p(n_j-1)}}{(2n-n_p-n_j)^{n_pn_j}} \\
&= \prod_{1\le p\le k, p\ne i,j} \left(1-\frac{1}{2n-n_p-n_i}\right)^{n_pn_i} (2n-n_p-n_i)^{n_p} \\
&\quad \times \prod_{1\le p\le k, p\ne i,j} \left(1+\frac{1}{2n-n_p-n_j}\right)^{n_pn_j} (2n-n_p-n_j)^{-n_p} \\
&= \prod_{1\le p\le k, p\ne i,j} \left[\left(1-\frac{1}{2n-n_p-n_i}\right)^{n_i}\left(1+\frac{1}{2n-n_p-n_j}\right)^{n_j}\right]^{n_p} \\
&\quad \times \prod_{1\le p\le k, p\ne i,j} \left(\frac{2n-n_i-n_p}{2n-n_j-n_p}\right)^{n_p} \\
&> \prod_{1\le p\le k, p\ne i,j} \left[\left(1-\frac{1}{2n-n_p-n_i}\right)^{n_i}\left(1+\frac{1}{2n-n_p-n_j}\right)^{n_j}\right]^{n_p} \\
&\qquad\qquad \text{as } \frac{2n-n_i-n_p}{2n-n_j-n_p} > 1 \text{ when } n_j-n_i\ge 2 \\
&> \prod_{1\le p\le k, p\ne i,j} \left[\left(1-\frac{1}{2n-n_p-n_i}\right)\left(1+\frac{1}{2n-n_p-n_j}\right)\right]^{n_in_p} > 1.
\end{aligned}$$

Moreover, the last inequality holds since

$$\left(1-\frac{1}{2n-n_p-n_i}\right)\left(1+\frac{1}{2n-n_p-n_j}\right) > 1 \text{ when } n_j-n_i\ge 2.$$

This completes the proof of the lemma. □

For $k=1$, the set $\mathbf{G}(n,k)$ contains a single connected graph K_1. When $k=n$, the only graph in $\mathbf{G}(n,k)$ is K_n. So, in the following, we always assume that $1<k<n$ and $n=kq+r$ where $0\le r<k$, i.e., $q=\lfloor \frac{n}{k}\rfloor$. The following theorem presents the extremal graph from $\mathbf{G}(n,k)$ having maximal Narumi-Katayama index, maximal second multiplicative Zagreb indices and maximal multiplicative sum Zagreb index, respectively.

Theorem 3.1. *For any graph $G\in\mathbf{G}(n,k)$, we have*

(1) $NK(G)\le NK(T_n(k)) = (n-\lfloor\frac{n}{k}\rfloor)^{\lfloor\frac{n}{k}\rfloor(k-r)}(n-\lceil\frac{n}{k}\rceil)^{\lceil\frac{n}{k}\rceil r}$ *with equality holding if and only if $G\cong T_n(k)$;*

(2) $\prod_2(G) \le \prod_2(T_n(k)) = \left(n - \lfloor \frac{n}{k} \rfloor\right)^{(n-\lfloor \frac{n}{k}\rfloor)\lfloor \frac{n}{k}\rfloor(k-r)} \left(n - \lceil \frac{n}{k} \rceil\right)^{(n-\lceil \frac{n}{k}\rceil)\lceil \frac{n}{k}\rceil r}$ *with equality holding if and only if* $G \cong T_n(k)$*;*

(3) $\prod_1^*(G) \le \prod_1^*(T_n(k)) = \left(2n - 2\lfloor \frac{n}{k} \rfloor\right)^{\lfloor \frac{n}{k}\rfloor^2 \binom{k-r}{2}} \times \left(2n - 2\lceil \frac{n}{k} \rceil\right)^{\lceil \frac{n}{k}\rceil^2 \binom{r}{2}} \times \left(2n - \lfloor \frac{n}{k} \rfloor - \lceil \frac{n}{k} \rceil\right)^{\lfloor \frac{n}{k}\rfloor \lceil \frac{n}{k}\rceil r(k-r)}$ *with equality holding if and only if* $G \cong T_n(k)$*.*

Proof. From the definition of chromatic number, any graph G from $\mathbf{G}(n,k)$ has k color classes each of which is an independent set. Suppose that the k classes have order $n_1, n_2, \cdots, n_k$, respectively. By Lemma 2.1, we find that extremal graph from $\mathbf{G}(n,k)$ with maximal Narumi-Katayama index, maximal second multiplicative Zagreb index and maximal multiplicative sum Zagreb index must be a complete k-partite graph $K_{n_1,n_2,\cdots,n_k}$.

By Lemma 3.1, we claim that the maximal values of $NK(G)$, $\prod_2(G)$ and $\prod_1^*(G)$, respectively, are attained for $G \cong T_n(k)$.

Conversely, one can see easily that the first equality holds in (1) or (2), or (3) when $G \cong T_n(k)$. Recalling that $n = k\lfloor \frac{n}{k} \rfloor + r = (k-r)\lfloor \frac{n}{k} \rfloor + r\lceil \frac{n}{k} \rceil$, the values of $NK(T_n(k))$, $\prod_2(T_n(k))$ and $\prod_1^*(T_n(k))$ can be easily obtained by Lemma 2.8.
This finishes the proof of this lemma. □

Next we turn to determine the minimal values of Narumi-Katayama index, second multiplicative Zagreb index and multiplicative sum Zagreb index of graphs from $\mathbf{G}(n,k)$.

Theorem 3.2. *Let* G *be a graph in* $\mathbf{G}(n,2)$ *with* $n \ge 5$*. Then we have*
(1) $NK(G) \ge NK(S_n)$ *with equality holding if and only if* $G \cong S_n$*;*
(2) $\prod_2(G) \ge \prod_2(P_n)$ *with equality holding if and only if* $G \cong P_n$*;*
(3) $\prod_1^*(G) \ge \prod_1^*(P_n)$ *with equality holding if and only if* $G \cong P_n$*.*

Proof. From Lemma 2.2, we find that the graph from $\mathbf{G}(n,2)$ with minimal Narumi-Katayama index, minimal second multiplicative Zagreb index and minimal multiplicative sum Zagreb index, respectively, must be a tree. Therefore the three above results in this theorem follow immediately from Lemma 2.6. □

Before characterizing the extremal graphs from $\mathbf{G}(n,3)$ with minimal Narumi-Katayama index, minimal second multiplicative Zagreb index and minimal multiplicative sum Zagreb index, we first prove the following lemma.

Lemma 3.2. *Let* $n \ge 4$ *and* G *be a graph from* $\mathbf{G}(n,3)$ *with minimal Narumi-Katayama index, or minimal second multiplicative Zagreb index or minimal multiplicative sum Zagreb index. Then* G *must be a unicyclic graph with its girth being odd.*

Proof. From the definition of set $\mathbf{G}(n,3)$, we have $\chi(G) = 3$ for any graph $G \in \mathbf{G}(n,3)$. So there is at least one cycle in any $G \in \mathbf{G}(n,3)$. From Lemma 2.2, we conclude that extremal graph $G \in \mathbf{G}(n,3)$ with minimal Narumi-Katayama index, or minimal second multiplicative Zagreb index or minimal multiplicative sum Zagreb index must be a unicyclic graph. Thus this lemma follows immediately by the fact that a uncyclic graph with its girth being even is a bipartite graph. □

Theorem 3.3. *Let G be a graph in $\mathbf{G}(n,3)$ with $n \geq 5$. Then we have $NK(C_n^3) \leq NK(G)$ with equality holding if and only if $G \cong C_n^3$.*

Proof. From Lemma 3.2, we claim that the graph from $\mathbf{G}(n,3)$ with minimal Narumi-Katayama index must be a unicyclic graph with an odd girth. It follows from Lemma 2.7 that the connected uncyclic graph of order n with minimal Narumi-Katayama index is C_n^3 with girth 3, which finishes the proof of this theorem. □

To characterize the graph from $\mathbf{G}(n,3)$ with minimal second multiplicative Zagreb index or minimal multiplicative sum Zagreb index, we need to introduce some definitions. We denote by $C_k((n-k)^1)$ the graph obtained by attaching to one vertex of C_k a pendent path of length $n-k$. Let $\mathbf{G}^0(n,3) = \{C_k((n-k)^1) : k \text{ is odd}\}$. Obviously, $\mathbf{G}^0(n,3)$ is a subset of $\mathbf{G}(n,3)$. A unicyclic graph G is said to be a *sun graph* ([18]) if cycle vertices have degrees at most three and remaining vertices have degrees at most two.

Theorem 3.4. *Let G be a graph in $\mathbf{G}(n,3)$ with $n \geq 5$ being odd. Then we have*
(1) $\prod_2(G) \geq \prod_2(C_n)$ *with equality holding if and only if* $G \cong C_n$;
(2) $\prod_1^*(G) \geq \prod_1^*(C_n)$ *with equality holding if and only if* $G \cong C_n$.

Proof. By Lemma 3.2, we find that the graph from $\mathbf{G}(n,3)$ with $n \geq 5$ being odd must be an uncyclic graph. Considering Lemma 2.7 (2) and (3), the uncyclic graph of order n ($n \geq 5$ is odd) with minimal second multiplicative Zagreb index or minimal multiplicative sum Zagreb index is C_n, and $\chi(C_n) = 3$ when n is odd. Thus this theorem follows immediately. □

Theorem 3.5. *Let G be a graph in $\mathbf{G}(n,3)$ with $n \geq 4$ being even. Then we have*
(1) $\prod_2(G) \geq 9 \times 4^{n-2}$ *with equality holding if and only if G is isomorphic to any graph from* $\mathbf{G}^0(n,3)$;
(2) $\prod_1^*(G) \geq 3 \times 5^3 \times 4^{n-4}$ *with equality holding if and only if G is isomorphic to any graph from* $\mathbf{G}^0(n,3) \setminus \{C_{n-1}(1^1)\}$.

Proof. (1) Assume that G_0 is the graph from $\mathbf{G}(n,3)$, with $n \geq 4$ being even, having the minimal second multiplicative Zagreb index. By Lemmas 3.2 and 2.7, we claim that G_0 must be a unicyclic graph different from C_n.

In view of Remark 2.2, any unicyclic graph can be changed into a sun graph with a smaller second multiplicative Zagreb index. So we deduce that G_0 must be a sun graph. By running Transformation A, considering Lemma 2.6, any sun graph can be changed into a graph in $\mathbf{G}^0(n,3)$ with a smaller second multiplicative Zagreb index. Thus we claim that G_0 must be in the set $\mathbf{G}^0(n,3)$. From the definition of second multiplicative Zagreb index, we get $\prod_2(H) = 9 \times 4^{n-2}$ for any graph $H \in \mathbf{G}^0(n, 3)$. Therefore the result in (1) holds.

(2) Assume that G_0' is the graph from $\mathbf{G}(n, 3)$, with $n \geq 4$ being even, having the minimal multiplicative sum Zagreb index. By a very similar reasoning as above, we can find that G_0' belongs to the set $\mathbf{G}^0(n, 3)$. From the definition of multiplicative sum Zagreb index, we have

$\prod_1^*(C_{n-1}(1^1)) = 5^2 4^{n-2}$,
$\prod_1^*(H) = 3 \times 5^3 4^{n-4}$ for any graph H from $\mathbf{G}^0(n, 3) \setminus \{C_{n-1}(1^1)\}$, and
$\prod_1^*(C_{n-1}(1^1)) - \prod_1^*(H) = 5^2 4^{n-2} - 3 \times 5^3 4^{n-4} = 25 \times 4^{n-4} > 0$.
Then the result in (2) follows immediately. □

Unfortunately, by now we do not know the extremal graph from $\mathbf{G}(n, k)$ with $3 < k < n$ having minimal Narumi-Katayama index, minimal second multiplicative Zagreb index and minimal multiplicative sum Zagreb index, respectively. Maybe it will be an interesting topic for the further research in the future.

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