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K-DOMINATION ON HEXAGONAL CACTUS CHAINS

SNJEŽANA MAJSTOROVIĆ [1], TOMISLAV DOŠLIĆ [2], AND ANTOANETA KLOBUČAR [3]

Abstract. In this paper we use the concept of k-domination, where $k \geq 1$. We determine minimum k-dominating sets and k-domination numbers of three special types of hexagonal cactus chains. Those are para-, meta- and ortho-chains.

For an arbitrary hexagonal chain G_h of length $h \geq 1$ we establish the lower and the upper bound for k-domination number γ_k. As a consequence, we find the extremal chains due to γ_k.

1. Introduction and terminology

We will first give some mathematical definitions. For any graph G we denote the vertex-set and edge-set of G by $V(G)$ and $E(G)$ respectively. A subset D of $V(G)$ is called a k-dominating set, if for every vertex y not in D, there exists at least one vertex x in D, such that distance between them is $\leq k$. For convenience we also say that D k-dominates G. The k-domination number is the cardinality of the smallest k-dominating set. 1-domination number is also called domination number and 1-domination set is called dominating set.

A dominating set D of a graph G is perfect if each vertex of G is dominated with exactly one vertex in D. A perfect dominating set of G is necessarily a minimum dominating set of it as well.

Chemical structures are conveniently represented by graphs, where atoms correspond to vertices and chemical bonds correspond to edges [1], [2]. However, this representation does not just provide the visual insight of the molecular structures, but inherits many useful information about chemical properties of molecules. It has been shown in QSAR and QSPR studies that many physical and chemical properties of molecules are well correlated with graph theoretical invariants that are termed

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topological indices or molecular descriptors [3]. One of such graph theoretical invariants is domination number. It has been shown that this number discriminates well even the slightest changes in trees and hence it is very suitable for the analysis of the RNA structures [4]. From previous it follows that domination number is just the simplest variant of k-domination numbers well known in mathematics [5].

A cactus graph is connected graph in which no edge lies in more than one cycle. The study of these objects started in 1950's under the name of Husimi trees. In papers by Husimi [6] and Riddell [7] those graphs were used in studies of cluster integrals in the theory of condensation in statistical mechanics [8]. Later they found applications in the theory of electrical and communication networks [9] and in chemistry [10].

In this paper, we analyze k-domination of hexagonal cactus chains which are generalizations of chain benzenoids. Usage of topological indices for the analysis of graphite samples has already shown to be useful and there is quite a substantial amount of the literature covering connection between benzenoids and topological indices. In some papers was investigated k-domination on the cartesian product of two paths which is equivalent to rectangular square grid [11]. The matching-related properties of hexagonal cacti were investigated in a series of papers by Farrell [12], [13], [14] and their matching and independence polynomials were studied in a recent paper by one of the present authors [15].

A **hexagonal cactus** G is a cactus graph consisting only of cycles with 6 vertices, i.e. hexagons. A vertex shared by two or more hexagons is called a cut-vertex. If every hexagon in G has at most two cut-vertices, and every cut-vertex is shared by exactly two hexagons, we call G a **hexagonal cactus chain**. The number of hexagons in G is called the **length** of a chain. With G_h we denote a hexagonal cactus chain of length h and write $G_h = C^1C^2 \dots C^h$, where C^i are consecutive hexagons, $i = 1, \dots, h$. Let $c_i = min\{d(y, w) : y \in C^i, w \in C^{i+2}\}$, $i = 1, 2, \dots, h-2$. We say that c_i is the distance between hexagons C^i and C^{i+2}.

An example of a hexagonal cactus chain is given in Figure 1.

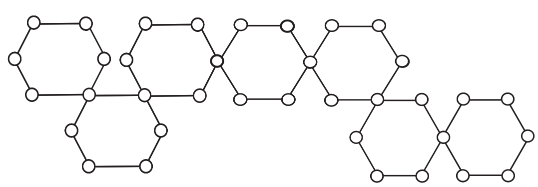

Figure 1. Hexagonal cactus chain of length 7.

Every G_h, $h \geq 2$, has exactly two hexagons with only one cut-vertex. Such hexagons are called **terminal hexagons**. All other hexagons in a chain are called internal hexagons. An internal hexagon in G_h is called an **ortho-hexagon** if its cut-vertices are adjacent, a **meta-hexagon** if the distance between its cut-vertices is 2, and a **para-hexagon** if the distance between its cut-vertices is 3. A hexagonal cactus chain is said to be **uniform** if all its internal hexagons are of the same type. So a chain G_h is called an **ortho-chain** if all its internal hexagons are ortho-hexagons. In the same way we define meta- and para-chain. An ortho-chain of length h is denoted by O_h, a meta-chain by M_h, and a para-chain by L_h. Notice that for O_h we have $c_i = 1$, for M_h $c_i = 2$ and for L_h $c_i = 3$, $i = 1, \ldots, h-2$. See Figure 2.

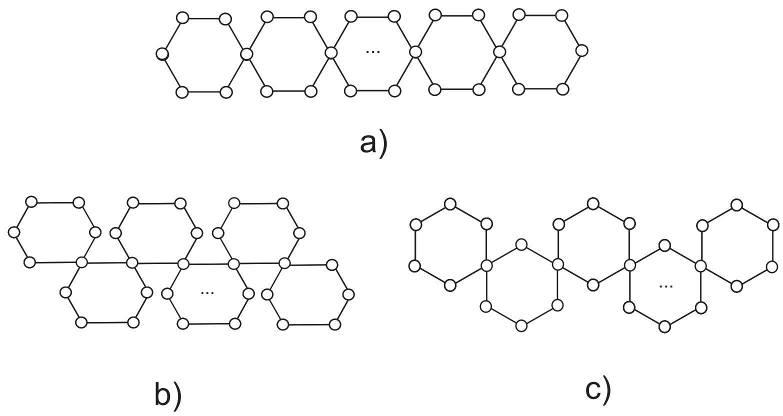

Figure 2: a) para-chain, b) ortho-chain, c) meta-chain.

The **open k-neighborhood** $N_k(v)$ of $v \in V(G)$ is the set of vertices in $V(G) \setminus \{v\}$ at distance at most k from v.

In this paper we first deal with k-domination on L_h, M_h and O_h, where $h \geq 1$. Then we extend these investigations to general hexagonal cactus chains and find the extremal ones.

2. Domination number of uniform hexagonal cactus chains

In this section we consider the 1-domination or just domination for three uniform hexagonal cactus chains L_h, M_h and O_h. We start by labeling their vertices in the way shown in Figure 3.

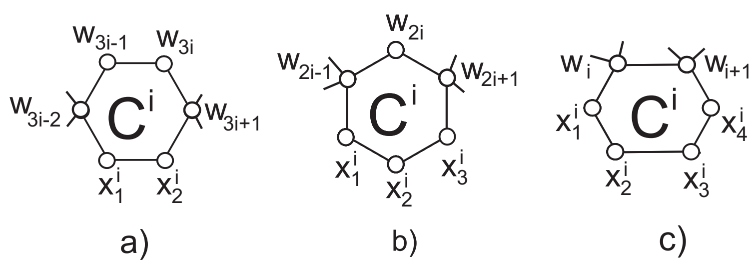

Figure 3. Labeling of vertices in uniform chains

Before we present main results, we will need the following proposition:

Proposition 2.1. [20] *Let P_n be a path and C_n be a cycle with n vertices. Then*

$$\gamma_k(P_n) = \gamma_k(C_n) = \left\lceil \frac{n}{2k+1} \right\rceil.$$

The case of L_h is the simplest and we treat it first.

Theorem 2.1. $\gamma(L_h) = h + 1$.

Proof. Let $D_{L_h} = \left\{w_{3i-2} : i = 1, \ldots, \left\lfloor \frac{h}{3} \right\rfloor\right\}$. The set D_{L_h} is a dominating set of L_h and therefore $\gamma(L_h) \leq |D_{L_h}| = h + 1$.

To prove that $\gamma(L_h) \geq h + 1$, we need the following lemma:

Lemma 2.1. *Let $h \geq 2$. If D is a minimum dominating set of L_h, then*

$$\left\{w_{3i-2} : i = 1, \ldots, \left\lfloor \frac{h}{3} \right\rfloor\right\} \subseteq D.$$

Proof. Let D be a minimum dominating set of L_h such that $w_{3t-2} \notin D$ for some fixed $t \in \left\{2, \ldots, \left\lfloor \frac{h}{3} \right\rfloor - 1\right\}$. Then at least one of the vertices from the set

$$\{w_{3t-3}, w_{3t-1}, x_2^t, x_1^{t+1}\}$$

is in D. Let $x_2^t \in D$ and $S = D \cap C^t C^{t+1}$. Then C^t contains one more dominating vertex and C^{t+1} contains two dominating vertices. We have $|S| = 4$. Let $S' = \{w_{3t-5}, w_{3t-2}, w_{3t+1}\}$. We define $D' = (D \setminus S) \cup S'$. Set D' also dominates L_h and $|D'| < |D|$. This is a contradiction to the assumption that D is a minimum dominating set of L_h. Therefore, we have $w_{3t-2} \in D$, for any minimum dominating set D of L_h, $t = 2, \ldots, \left\lfloor \frac{h}{3} \right\rfloor - 1$.

The similar approach is used for the case $t \in \{1, \left\lfloor \frac{h}{3} \right\rfloor\}$. □

From Lemma 2.1 we conclude that $\gamma(L_h) \geq h + 1$ and D_{L_h} is unique (see Figure 4). This completes the proof of Theorem 2.1. □

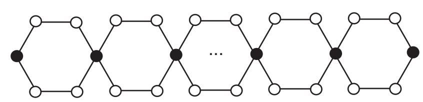

Figure 4. Minimum dominating set of L_h.

Corollary 2.1. $D_{L_h} \subset D_{L_{h+1}}$ *and* $\gamma(L_{h+1}) = \gamma(L_h) + 1, \quad \forall h \geq 1$. □

The remaining two cases are similar and we treat them together.

Theorem 2.2. $\gamma(M_h) = \gamma(O_h) = \left\lceil \frac{3h}{2} \right\rceil$.

Proof. Let us consider the set

$$D_{M_h} = \left\{w_{4i-1}, x_1^{2i-1} : i = 1, \ldots, \left\lceil \frac{h}{2} \right\rceil\right\} \cup \left\{x_3^{2i} : i = 1, \ldots, \left\lfloor \frac{h}{2} \right\rfloor\right\}.$$

The set D_{M_h} is a dominating set of M_h and hence $\gamma(M_h) \leq |D_{M_h}| = \left\lceil \frac{3h}{2} \right\rceil$.

Similarly, the set

$$D_{O_h} = \left\{w_{2i}, x_2^{2i-1} : i = 1, \ldots, \left\lceil \frac{h}{2} \right\rceil\right\} \cup \left\{x_3^{2i} : i = 1, \ldots, \left\lfloor \frac{h}{2} \right\rfloor\right\}$$

is a dominating set of O_h and hence $\gamma(O_h) \le |D_{O_h}| = \left\lceil \frac{3h}{2} \right\rceil$.

The sets D_{M_h} and D_{O_h} are presented in Figure 5.

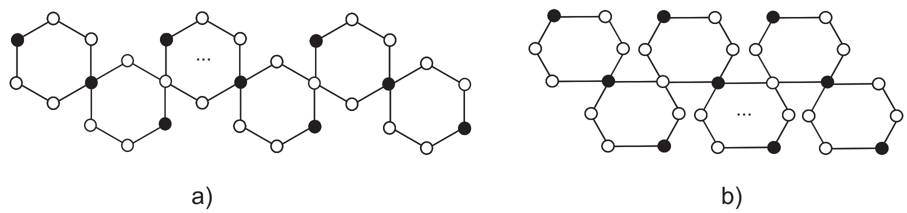

Figure 5. Minimum dominating sets of (a) M_h and (b) O_h.

Let us prove that D_{M_h} is a dominating set of minimum cardinality. Let $\gamma(M_h) < \left\lceil \frac{3h}{2} \right\rceil$. We split M_h into subchains $C^{2i-1}C^{2i}$, $i = 0, 1, \ldots, \left\lfloor \frac{h}{2} \right\rfloor$ and the last chain C^h, if h odd. If h is even, then by the Pigeonhole Principle there exists i such that the subchain $C^{2i-1}C^{2i}$ contains fewer than three vertices from D_{M_h}. This is impossible because $C^{2i-1}C^{2i}$ is isomorphic to L_2 and from Theorem 2.1 we have $\gamma(L_2) = 3$. If h is odd, then either one of considered subchains contains fewer than three vertices from D_{M_h} or C^h contains at most one such vertex. Both cases are impossible since two hexagons cannot be dominated with fewer that three vertices, and one hexagon cannot be dominated with fewer than two vertices. We conclude $\gamma(M_h) \ge \left\lceil \frac{3h}{2} \right\rceil$.

The proof of $\gamma(O_h) \ge \left\lceil \frac{3h}{2} \right\rceil$ is similar and we omit the details. □

Remark 2.1. For h even, dominating sets D_{M_h} and D_{O_h} are unique.

Corollary 2.2. $D_{M_h} \subset D_{M_{h+1}}$ *and* $D_{O_h} \subset D_{O_{h+1}}$, $\forall h \ge 1$. □

Corollary 2.3. *Let* $G_h \in \{M_h, O_h\}$. *Then* $\gamma(G_1) = 2$, $\gamma(G_2) = 3$ *and for* $h \ge 3$

$$\gamma(G_h) = \begin{cases} \gamma(G_{h-1}) + 1, & \text{for} \quad h \quad \text{even}, \\ \gamma(G_{h-1}) + 2, & \text{for} \quad h \quad \text{odd}. \end{cases}$$

Proof. Let us prove the recurrence for M_h. We proved $\gamma(M_2) = 3$. By adding one new hexagon, there are 5 vertices to consider. Since D_{M_2} is unique, we cannot rearrange dominating vertices in M_2 so that w_5 is dominating vertex. Therefore, we need two more dominating vertices in the last hexagon and we have $\gamma(M_3) = \gamma(M_2) + 2$. For M_4, the dominating vertices in C^3 can be arranged so that w_7 is dominating vertex. Since w_7 is a cut-vertex, it dominates 3 vertices in C^4 and we need only one more dominating vertex. We have $\gamma(M_4) = \gamma(M_3) + 1$. The same procedure is followed for M_5 and M_6. Inductively, we conclude that for h even $\gamma(M_h) = \gamma(M_{h-1}) + 1$, and for h odd $\gamma(M_h) = \gamma(M_{h-1}) + 2$. The proof for O_h is essentially the same. □

3. Domination number of arbitrary hexagonal chains

In the following we deal with an arbitrary hexagonal cactus chain G_h and present some results about its domination number. With D_h we denote the minimum dominating set of G_h.

Theorem 3.1. *Let G_h be a hexagonal cactus chain of length h. Then:*
1) *either* $\gamma(G_h) = \gamma(G_{h-1}) + 1$ *or* $\gamma(G_h) = \gamma(G_{h-1}) + 2$, $h \geq 2$;
2) *If* $\gamma(G_{h-1}) = \gamma(G_{h-2}) + 2$, *then* $\gamma(G_h) = \gamma(G_{h-1}) + 1$, $h \geq 3$.

Proof. 1) Let G_{h-1} be an arbitrary hexagonal cactus chain of length $h-1$ with a minimum dominating set D_{h-1}, and let u be a cut-vertex as in Figure 6. Adding one new hexagon to G_{h-1} results in 5 new vertices. We consider the following cases:

1° $u \in D_{h-1}$. Vertex u dominates three vertices in C^h. For the remaining three vertices in C_h we need at most one dominating vertex. See Figure 6a.

We have $|D_h| \leq |D_{h-1}| + 1$ and $u \in D_h$.

For any dominating set D_h of G_h we have $|D_h \cap C^j| = 2$, $j = 1, h$. To prove this, notice that the vertices from $C^2 \setminus \{u_1\}$, where $\{u_1\} \in C^1 \cap C^2$, can dominate at most one vertex from C^1. For the remaining 5 vertices in C^1 we need at least 2 dominating vertices. Vertices from C^2 can dominate at most three vertices in C^1. But then $u_1 \in D_h$ and we need at least one more vertex to dominate C^1. Since $u_1 \in C^1$, it follows that $|D_h \cap C^1| \geq 2$. Since the domination number of a hexagon is 2, we have $|D_h \cap C^1| \leq 2$, that is, $|D_h \cap C^1| = 2$. For C^h the proof is essentially the same.

From $u \in D_{h-1} \Rightarrow u \in D_h$ and $|D_h \cap C^h| = 2$, we conclude that $|D_{h-1}| \leq |D_h| - 1$, that is, $|D_h| \geq |D_{h-1}| + 1$. We obtain $|D_h| = |D_{h-1}| + 1$ and $\gamma(G_h) = \gamma(G_{h-1}) + 1$.

2° $u \notin D_{h-1}$. If there exists another minimum dominating set D'_{h-1} such that $u \in D'_{h-1}$, then we consider D'_{h-1} instead of D_{h-1}, and continue as in previous case. Otherwise, u is dominated with at least one vertex from C^{h-1}. Then $|D_h| \leq |D_{h-1}|+2$, since we have 5 undominated vertices in C^h. From $|D_h \cap C^h| = 2$ it follows that $|D_{h-1}| \leq |D_h| - 2$. We conclude $|D_h| = |D_{h-1}| + 2$ and $\gamma(G_h) = \gamma(G_{h-1}) + 2$. The case is shown in Figure 6b.

2) If $\gamma(G_{h-1}) = \gamma(G_{h-2}) + 2$ for some $h \geq 3$, then at least 4 vertices in C^{h-1} are not dominated with D_{h-2}. That means that $u \notin D_{h-2}$, where $\{u\} = C^{h-2} \cap C^{h-1}$. From part 1) of the theorem we conclude $|D_{h-1}| = |D_{h-2}| + 2$, and dominating vertices in C_{h-1} can be chosen arbitrarily, as long as not both of them are adjacent to u. If we attach one more hexagon to C^{h-1}, we can set $v \in D_{h-1}$, where $\{v\} \in C^{h-1} \cap C^h$. Now we have case 1° in 1). It follows that $\gamma(G_h) = \gamma(G_{h-1}) + 1$. See Figure 6c. □

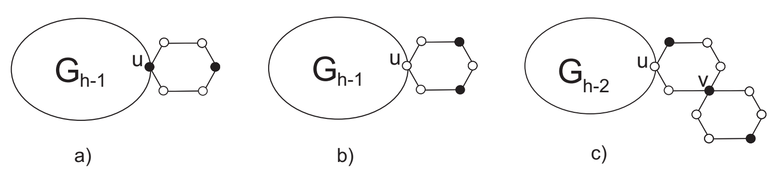

Figure 6.

We close this section by showing that the uniform chains are extremal among all hexagonal cactus chains with respect to the dominating number γ.

Theorem 3.2. *Let G_h be a hexagonal cactus chain of length $h \geq 1$. Then*

$$\gamma(L_h) \leq \gamma(G_h) \leq \gamma(M_h) = \gamma(O_h).$$

Proof. The left inequality follows from Theorem 3.1(1) and Corollary 2.1, while the right inequality follows from Theorem 3.1 and Corollary 2.3. □

4. k-DOMINATION NUMBERS OF UNIFORM HEXAGONAL CHAINS, $k \geq 2$

Theorem 4.1. *Let $G_h \in \{L_h, M_h, O_h\}$ and let c be the distance between the nearest two cut-vertices in G_h. Then*

$$\gamma_k(G_h) = \begin{cases} h+1, & \text{for} \quad k=2 \\ \left\lceil \dfrac{ch+1}{2(k+c)-5} \right\rceil, & \text{for} \quad k \geq 3 \end{cases},$$

with G_h being O_h, M_h and L_h when c is equal to $1, 2$ and 3, respectively.

Proof. **Case** $k = 2$.

For L_h set $DL = \{w_{3i-2} : i = 1, \ldots, h+1\}$ is 2-dominating set of L_h and $\gamma_2(L_h) \leq |DL| = h+1$.

For M_h we have dominating set

$$DM = \left\{x_1^{2i-1} : i = 1, \ldots, \left\lceil \frac{h}{2} \right\rceil\right\} \cup \left\{x_2^{2i} : i = 1, \ldots, \left\lfloor \frac{h}{2} \right\rfloor\right\} \cup \{w_{2h+1}\},$$

and $\gamma_2(M_h) \leq |DM| = h+1$.

For O_h dominating set is $DO = \{x_2^i : i = 0, 1, \ldots, h\} \cup \{w_{h+1}\}$ and $\gamma_2(O_h) \leq |DO| = h+1$.

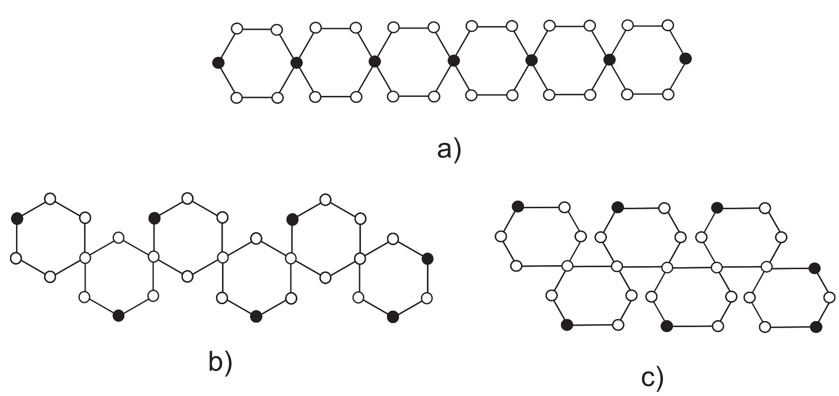

Figure 7: Minimum 2-dominating sets of a) L_6, b) M_6 and c) O_6.

In the following we prove that sets DL, DM and DO have minimum cardinality among all 2-dominating sets of L_h, M_h and O_h, respectively.

Let D_h be the minimum 2-dominating set of L_h. Then from Proposition 2.1 we have $|D_1| = 2$. If we consider G_{h-1}, $h \geq 2$, then by adding one new hexagon, we have 5 new vertices. Vertices from D_{h-1} 2-dominate at most 5 vertices in C^h. Still, there is one not dominated vertex for which we need at least one 2-dominating vertex. Therefore, $|D_h| \geq |D_{h-1}| + 1$, $\forall h \geq 2$. We obtain $|D_h| \geq |D_1| + h - 1$, that is, $|D_h| \geq h + 1$. We proved $\gamma_2(L_h) = h + 1$. The same conclusions are obtained for M_h and L_h.

Case: $k \geq 3$.

We will first consider L_h. Let $t = \left\lceil \dfrac{3h+1}{2k+1} \right\rceil$. We consider the set

$$S = \{w_{(2k+1)i+k+1} : i = 0, 1, \ldots, t-1\}.$$

If $(2k+1)(t-1) + k + 1 \leq 3h + 1$, then S is a k-dominating set for L_h. Otherwise,

$$S' = (S \setminus \{w_{(2k+1)(t-1)+k+1}\}) \cup \{w_{3h+1}\}$$

is k-dominating set of L_h. We have $\gamma_k(L_h) \leq |S| = |S'| = t$.

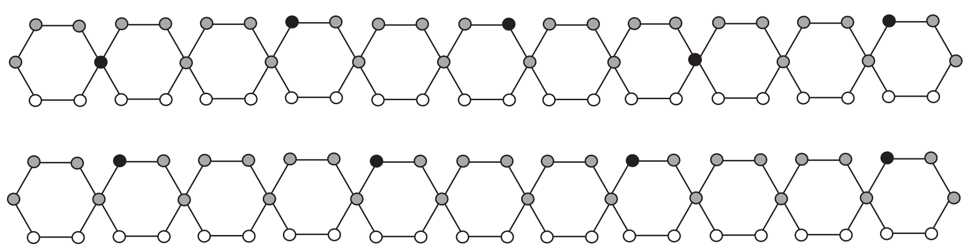

Figure 8. Minimum k-dominating sets of L_{11} with $k = 3$ and $k = 4$.

Let us prove that $|D| \geq t$ for any k-dominating set D of L_h.

Lemma 4.1. *Let P_{3h+1} be an induced subgraph of L_h with vertex-set*

$$V(P_{3h+1}) = \{w_i : i = 0, 1, \ldots, 3h+1\}.$$

There exists a minimum k-dominating set D of L_h such that $D \subset V(P_{3h+1})$.

Proof. Let D be the minimum k-dominating set of L_h such that $x_1^j \in D$, $j \in \{1, 2, \ldots, h\}$. Then $N_k[x_1^j] = N_k[w_{3j-1}]$ since both vertices w_{3j-1} and x_1^j k-dominate C^j, they are at the same distance from cut-vertices w_{3j-2} and w_{3j+1}, which means that they k-dominate same set of vertices in other hexagons. Let $D' = D \setminus \{x_1^j\} \cup \{w_{3j-1}\}$. Set D' also k-dominates L_h, and since $|D'| = |D|$, we concluded that D' is a minimum k-dominating set of L_h. In a similar way we conclude that if $x_2^j \in D$, then $N_k[x_2^j] = N_k[w_{3j}]$, $j \in \{1, \ldots, h\}$ and $D' = D \setminus \{x_2^j\} \cup \{w_{3j}\}$ is a minimum k-dominating set of L_h. We proved that for any minimum k-dominating set containing vertices x_1^j or x_2^j, there exists another minimum k-dominating set D of L_h such that $D \subset V(P_{3h+1})$. □

Let D be the set that satisfy Lemma 4.1. Then D k-dominates P_{3h+1}. Therefore,

$$|D| \geq \gamma_k(P_{3h+1}) = t.$$

For M_h let $t = \left\lceil \dfrac{2h+1}{2k-1} \right\rceil$ and $S = \{w_{(2k-1)i+k} : i = 0, 1, \ldots, t-1\}$.

If $(2k-1)(t-1)+k \leq 2h+1$, set S is a k-dominating set of M_h. Otherwise,

$$S' = (S \setminus \{w_{(2k-1)(t-1)+k}\}) \cup \{w_{2h+1}\}$$

is k-dominating set of M_h. Hence, $\gamma_k(M_h) \leq |S| = |S'| = t$.

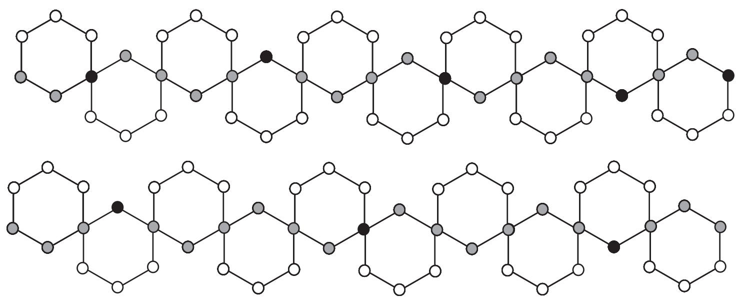

Figure 9. Minimum k-dominating sets of M_{10} with $k = 3$ and $k = 4$.

To prove that $\gamma_k(M_h) \geq t$, we consider the following lemma:

Lemma 4.2. *Let P_{2h+1} be an induced subgraph of M_h with vertex-set*

$$V(P_{2h+1}) = \{w_i : i = 1, \ldots, 2h+1\}.$$

There exists a minimum k-dominating set D of M_h such that $D \subset V(P_{2h+1})$. With the notation $D = \{w_{i_1}, w_{i_2}, \ldots, w_{i_p}\}$, $1 \leq i_1 < i_2 < \ldots < i_p \leq 2h+1$, D has the following properties:

(i) $d(w_{i_j}, w_{i_{j+1}}) \leq 2k-1$, $\forall j = 1, \ldots, p-1$.
(ii) $d(w_1, w_{i_1}) \leq k-1$ *and* $d(w_{i_p}, w_{2h+1}) \leq k-1$.

Proof. Let D be the minimum k-dominating set of M_h such that $x_1^j \in D$, $j \in \{1, \ldots, h\}$. Then $N_k[x_1^j] \subseteq N_k[w_{2j-1}]$ since x_1^j k-dominates C^j, $d(x_1^j, w_{2j-1}) = 1$, $d(x_1^j, w_{2j+1}) = 3$ and w_{2j-1} k-dominates at least $N_k[x_1^j]$. Set $D' = D \setminus \{x_1^j\} \cup \{w_{2j-1}\}$ also k-dominates M_h. Since $|D| = |D'|$, D' is minimum k-dominating set. With the similar approach we conclude that if $x_2^j \in D$, then $N_k[x_2^j] \subseteq N_k[w_{2j-1}]$ and $D' = D \setminus \{x_2^j\} \cup \{w_{2j-1}\}$ is also a minimum k-dominating set of M_h. (For this case, the choice of D' can also be $D' = D \setminus \{x_2^j\} \cup \{w_{2j+1}\}$.) For $x_3^j \in D$, we choose $D' = D \setminus \{x_2^j\} \cup \{w_{2j+1}\}$. Since j is an arbitrary element of the set $\{1, \ldots, h\}$, the first part of lemma is proved.

(i) Let us assume that $D = \{w_{i_1}, w_{i_2}, \ldots, w_{i_p}\}$, $1 \leq i_1 < i_2 < \ldots < i_p \leq 2h+1$, is minimum k-dominating set of M_h, and let $d(w_{i_r}, w_{i_{r+1}}) \geq 2k$ for some fixed $r \in \{1, \ldots, p\}$. If $i_r \equiv 1(mod 2)$, w_{i_r} is a cut-vertex and there are at least k hexagons between w_{i_r} and $w_{i_{r+1}}$. We denote the corresponding subchain with $C^{j_1}C^{j_2}\cdots C^{j_l}$, $l \geq k$, $1 \leq j_l \leq h$.

For k odd, we have $x_2^{j_{\lceil \frac{k}{2} \rceil}} \notin N_k[w_{i_r}] \cap N_k[w_{i_{r+1}}]$. Then there exists a vertex $w_{i_{r'}}$, $i_r < i_{r'} < i_{r+1}$, such that $w_{i_{r'}} \in N_k[x_2^{j_{\lceil \frac{k}{2} \rceil}}]$ and $w_{i_{r'}}$ is k-dominating vertex. This implies $|D| > p$ which is a contradiction to the assumption that D is a minimum k-dominating set of cardinality p.

For k even, we have $x_3^{j_{\frac{k}{2}}}, x_1^{j_{\frac{k}{2}+1}} \notin N_k[w_{i_r}] \cap N_k[w_{i_{r+1}}]$. But then we need at least one more k-dominating vertex $w_{i_{r'}}$ between w_{i_r} and $w_{i_{r+1}}$. Again, we obtain a contradiction to the assumption that $|D| = p$.

If $i_r \equiv 0(mod 2)$, then there are at least $k-1$ hexagons between w_{i_r} and $w_{i_{r+1}}$. In a similar way, we conclude that there must be at least one k-dominating vertex $w_{i_{r'}}$ between these two vertices, and this is a contradiction to $|D| = p$.

(ii) Let $d(w_1, w_{i_1}) \geq k$. Then $d(w_{i_1}, w_3) \geq k-2$ and $x_1^1 \notin N_k[w_{i_1}]$, since $d(x_1^1, w_3) = 3$. Therefore, $d(w_1, w_{i_1}) \leq k-1$. In a similar way, we prove $d(w_{i_p}, w_{2h+1}) \leq k-1$. □

Let D be a minimum k-dominating set as in Lemma 4.2. Then, D $(k-1)$-dominates P_{2h+1} and $|D| \geq \gamma_{k-1}(P_{2h+1}) = t$. We proved that $|D| \geq t$ for any minimum k-dominating set D of M_h. From this it follows that $\gamma_k(M_h) = t$.

At last, we consider O_h. We set $t = \left\lceil \dfrac{h+1}{2k-3} \right\rceil$ and

$$S = \{w_{(2k-3)i+k-1} : i = 0, 1, \ldots, t-1\}.$$

With the condition $(2k-3)(t-1)+k-1 \leq h+1$ set S is a k-dominating set of O_h.

Otherwise, $S' = (S \setminus \{w_{(2k-3)(t-1)+k-1}\}) \cup \{w_{h+1}\}$ is k-dominating set of O_h.

Since $|S| = |S'| = t$, we conclude $\gamma_k(O_h) \leq t$.

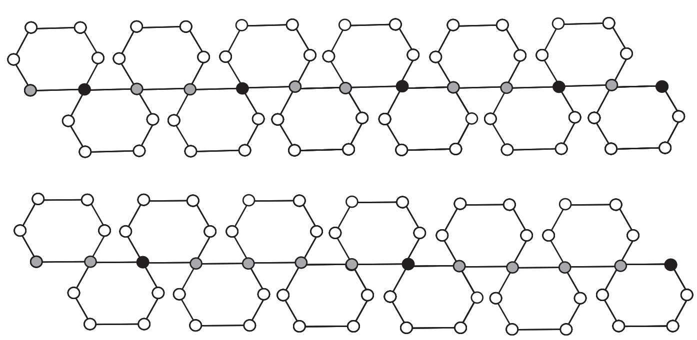

Figure 10. Minimum k-dominating sets of O_{12} with $k = 3$ and $k = 4$.

Lemma 4.3. *Let P_{h+1} be an induced subgraph of O_h with vertex-set*

$$V(P_{h+1}) = \{w_i : i = 0, 1, \ldots, h+1\}.$$

There exists a minimum k-dominating set D of O_h such that $D \subset V(P_{h+1})$.

With the notation $D = \{w_{i_1}, w_{i_2}, \ldots, w_{i_p}\}$, $1 \le i_1 < i_2 < \ldots < i_p \le h+1$, D has the following properties:

(i) $d(w_{i_j}, w_{i_{j+1}}) \le 2k-3$, $\forall j = 1, \ldots, p-1$.

(ii) $d(w_1, w_{i_1}) \le k-2$ *and* $d(w_{i_p}, w_{h+1}) \le k-2$.

Proof. We use the same approach as in Lemma 4.2. □

If D satisfy Lemma 4.3, then D $(k-2)$-dominates P_{h+1}. Therefore $|D| \ge \gamma_{k-2}(P_{h+1}) = t$. We conclude $\gamma_k(O_h) = t$.

If we put $c = d(u_{i-1}, u_i)$, $\forall i = 1, \ldots, h$, then domination numbers of all three considered chains can be joined into a single formula. If $G_h \in \{L_h, M_h, O_h\}$, then $\gamma_k(G_h) = \left\lceil \frac{ch+1}{2(k+c)-5} \right\rceil$ with c being the distance between the nearest two cut vertices in G_h. □

Corollary 4.1. *For $k \ge 3$ and $G_h \in \{L_h, M_h, O_h\}$ we have*

$$\gamma_k(G_h) = \gamma_{k+c-3}(P_{ch+1}),$$

with $c = 3$ if $G_h = L_h$, $c = 2$ if $G_h = M_h$ and $c = 1$ if $G_h = O_h$. Moreover, minimum k-dominating set of G_h is the minimum $k+c-3$-dominating sets of induced subpath P_{ch+1}. □

5. k-domination on arbitrary hexagonal cactus chains

In the following with G_h, $h \ge 2$, we denote a hexagonal cactus chain obtained by adding one new hexagon to C^{h-1} in G_{h-1}. With D_h we denote the minimum 2-dominating set of G_h.

Theorem 5.1. *Let G_h be an arbitrary hexagonal cactus chain of length $h \ge 1$. Then $\gamma_2(G_h) = h+1$.*

Proof. Let $D = \{u_i : i = 1, \ldots, h+1\}$ be the set of vertices of G_h such that $C^i \cap C^{i+1} = \{u_{i+1}\}$, $i = 1, \ldots, h-1$ and $d(u_1, u_2) = d(u_h, u_{h+1}) = 1$. Then D is 2-dominating set of G_h and since $|D| = h+1$, D has the smallest cardinality among all 2-dominating sets of G_h. □

Theorem 5.2. *Let G_h be arbitrary hexagonal cactus chain of length $h \ge 1$ and let $k \ge 3$. Then*

$$\gamma_k(O_h) \le \gamma_k(G_h) \le \gamma_k(L_h).$$

Proof. For $h = 1, 2$ it is obvious. Let $h \ge 3$. With u_{j-1} and u_j we denote cut-vertices contained in C^j, $j = 2, \ldots, h-1$, and with P_s we denote the shortest path that contains all cut-vertices of G_h with additional 3 vertices from C^1 and 3 vertices from C^h.

By merging Lemma 4.1 and the first part of Lemma 4.2 and 4.3 we obtain that there exists a minimum k-dominating set D of G_h such that $D \subset V(P_s)$. Let D' be a minimum k-dominating set of G_h such that $v \in C^j$, $v \notin V(P_s)$ and $v \in D'$.

If C^j is para-hexagon, then we consider D constructed from D' by replacing v with $v' \in V(P_s)$, so that v' and v are at the same distance from their nearest cut-vertex. We have $N_k[v] = N_k[v']$. If C^j is meta- or ortho-hexagon, then we can take D instead of D' by replacing v with its nearest cut-vertex. If u is the cut-vertex that is nearest to v, then $u = u_{j-1}$ or $u = u_j$. Obviously, $N_k[v] \subset N_k[u]$ so D' also k-dominates G_h. Since $|D| = |D'|$, D is minimum k-dominating set of G_h. Since j is an arbitrary element from $\{2, \ldots, h-1\}$, we proved the existence of dominated set D, such that $D \subset V(P_s)$.

For any vertex $w \in V(P_s)$, the distance from w to any other hexagon in G_h is the smallest if $G_h = O_h$. This means that $N_k[w]$ contains the largest possible number of vertices for $G_h = O_h$. Therefore, $\gamma_k(O_h) \leq \gamma_k(G_h)$. Distance from w to any other hexagon is the greatest if $G_h = L_h$ which implies that $N_k[w]$ is minimum. Therefore, $\gamma_k(G_h) \leq \gamma_k(L_h)$. □

6. Conclusions

By studying the k-domination on hexagonal cactus chains, we obtained the results that show the extremality of uniform chains: para-, meta- and ortho-chain with given k. For $k = 2$ all hexagonal cactus chains have the same 2-domination number. For $k = 1$ para-chain has the smallest, and for $k \geq 3$, the largest k-domination number. Ortho-chain have the largest 1-domination number (and so does the meta-chain) and the smallest k-domination number, $k \geq 3$.

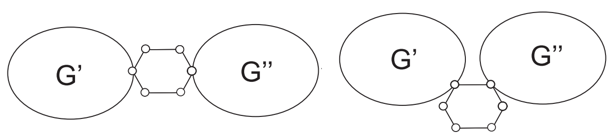

Figure 11. Subchains G' and G'' with only one para- and ortho-hexagon in between, respectively.

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[1] Department of Mathematics,
University of Osijek,
Croatia
E-mail address: `smajstor@mathos.hr`

[2] The Faculty of Civil Engineering,
University of Zagreb,
Croatia
E-mail address: `doslic@grad.hr`

[3] Department of Mathematics,
University of Osijek,
Croatia
E-mail address: `antoaneta.klobucar@os.t-com.hr`