**K-DOMINATION ON HEXAGONAL CACTUS CHAINS**

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**Abstract.** In this paper we use the concept of $k$-domination, where $k \geq 1$. We determine minimum $k$-dominating sets and $k$-domination numbers of three special types of hexagonal cactus chains. Those are para-, meta- and ortho-chains.

For an arbitrary hexagonal chain $G_h$ of length $h \geq 1$ we establish the lower and the upper bound for $k$-domination number $\gamma_k$. As a consequence, we find the extremal chains due to $\gamma_k$.

1. Introduction and terminology

We will first give some mathematical definitions. For any graph $G$ we denote the vertex-set and edge-set of $G$ by $V(G)$ and $E(G)$ respectively. A subset $D$ of $V(G)$ is called a $k$-dominating set, if for every vertex $y$ not in $D$, there exists at least one vertex $x$ in $D$, such that distance between them is $\leq k$. For convenience we also say that $D$ $k$-dominates $G$. The $k$-domination number is the cardinality of the smallest $k$-dominating set. 1-domination number is also called domination number and 1-domination set is called dominating set.

A dominating set $D$ of a graph $G$ is perfect if each vertex of $G$ is dominated with exactly one vertex in $D$. A perfect dominating set of $G$ is necessarily a minimum dominating set of it as well.

Chemical structures are conveniently represented by graphs, where atoms correspond to vertices and chemical bonds correspond to edges [1], [2]. However, this representation does not just provide the visual insight of the molecular structures, but inherits many useful information about chemical properties of molecules. It has been shown in QSAR and QSPR studies that many physical and chemical properties of molecules are well correlated with graph theoretical invariants that are termed...
topological indices or molecular descriptors [3]. One of such graph theoretical invariants is domination number. It has been shown that this number discriminates well even the slightest changes in trees and hence it is very suitable for the analysis of the RNA structures [4]. From previous it follows that domination number is just the simplest variant of \( k \)-domination numbers well known in mathematics [5].

A cactus graph is connected graph in which no edge lies in more than one cycle. The study of these objects started in 1950’s under the name of Husimi trees. In papers by Husimi [6] and Riddell [7] those graphs were used in studies of cluster integrals in the theory of condensation in statistical mechanics [8]. Later they found applications in the theory of electrical and communication networks [9] and in chemistry [10].

In this paper, we analyze \( k \)-domination of hexagonal cactus chains which are generalizations of chain benzenoids. Usage of topological indices for the analysis of graphite samples has already shown to be useful and there is quite a substantial amount of the literature covering connection between benzenoids and topological indices. In some papers was investigated \( k \)-domination on the cartesian product of two paths which is equivalent to rectangular square grid [11]. The matching-related properties of hexagonal cacti were investigated in a series of papers by Farrell [12], [13], [14] and their matching and independence polynomials were studied in a recent paper by one of the present authors [15].

A hexagonal cactus \( G \) is a cactus graph consisting only of cycles with 6 vertices, i.e. hexagons. A vertex shared by two or more hexagons is called a cut-vertex. If every hexagon in \( G \) has at most two cut-vertices, and every cut-vertex is shared by exactly two hexagons, we call \( G \) a hexagonal cactus chain. The number of hexagons in \( G \) is called the length of a chain. With \( G_h \) we denote a hexagonal cactus chain of length \( h \) and write \( G_h = C^1C^2\ldots C^h \), where \( C^i \) are consecutive hexagons, \( i = 1, \ldots, h \). Let \( c_i = \min\{d(y,w) : y \in C^i, w \in C^{i+2}\}, i = 1, 2, \ldots, h - 2 \). We say that \( c_i \) is the distance between hexagons \( C^i \) and \( C^{i+2} \).

An example of a hexagonal cactus chain is given in Figure 1.

![Hexagonal cactus chain of length 7.](image)

Every \( G_h, h \geq 2 \), has exactly two hexagons with only one cut-vertex. Such hexagons are called terminal hexagons. All other hexagons in a chain are called internal hexagons. An internal hexagon in \( G_h \) is called an ortho-hexagon if its cut-vertices are adjacent, a meta-hexagon if the distance between its cut-vertices is 2, and a para-hexagon if the distance between its cut-vertices is 3. A hexagonal cactus chain is said to be uniform if all its internal hexagons are of the same type. So a chain \( G_h \) is called an ortho-chain if all its internal hexagons are ortho-hexagons. In the same
way we define meta- and para-chain. An ortho-chain of length $h$ is denoted by $O_h$, a meta-chain by $M_h$, and a para-chain by $L_h$. Notice that for $O_h$ we have $c_i = 1$, for $M_h c_i = 2$ and for $L_h c_i = 3$, $i = 1, \ldots, h - 2$. See Figure 2.

![Image of ortho-chain, meta-chain, and para-chain]

Figure 2: a) para-chain, b) ortho-chain, c) meta-chain.

The open $k$-neighborhood $N_k(v)$ of $v \in V(G)$ is the set of vertices in $V(G) \setminus \{v\}$ at distance at most $k$ from $v$.

In this paper we first deal with $k$-domination on $L_h$, $M_h$ and $O_h$, where $h \geq 1$. Then we extend these investigations to general hexagonal cactus chains and find the extremal ones.

2. Domination number of uniform hexagonal cactus chains

In this section we consider the 1-domination or just domination for three uniform hexagonal cactus chains $L_h$, $M_h$ and $O_h$. We start by labeling their vertices in the way shown in Figure 3.

![Image of labeled vertices in uniform chains]

Figure 3. Labeling of vertices in uniform chains

Before we present main results, we will need the following proposition:

**Proposition 2.1.** [20] Let $P_n$ be a path and $C_n$ be a cycle with $n$ vertices. Then

$$\gamma_k(P_n) = \gamma_k(C_n) = \left\lceil \frac{n}{2k + 1} \right\rceil.$$

The case of $L_h$ is the simplest and we treat it first.
Theorem 2.1. \(\gamma(L_h) = h + 1\).

Proof. Let \(D_{Lh} = \left\{ w_{3i-2} : i = 1, \ldots, \left\lfloor \frac{h}{3} \right\rfloor \right\}\). The set \(D_{Lh}\) is a dominating set of \(L_h\) and therefore \(\gamma(L_h) \leq |D_{Lh}| = h + 1\).

To prove that \(\gamma(L_h) \geq h + 1\), we need the following lemma:

Lemma 2.1. Let \(h \geq 2\). If \(D\) is a minimum dominating set of \(L_h\), then
\[
\left\{ w_{3i-2} : i = 1, \ldots, \left\lfloor \frac{h}{3} \right\rfloor \right\} \subseteq D.
\]

Proof. Let \(D\) be a minimum dominating set of \(L_h\) such that \(w_{3t-2} \notin D\) for some fixed \(t \in \{2, \ldots, \left\lfloor \frac{h}{3} \right\rfloor - 1\}\). Then at least one of the vertices from the set
\[
\left\{ w_{3t-3}, w_{3t-1}, x_{2t}, x_{2t+1} \right\}
\]
is in \(D\). Let \(x_t \in D\) and \(S = D \cap C^tC^{t+1}\). Then \(C^t\) contains one more dominating vertex and \(C^{t+1}\) contains two dominating vertices. We have \(|S| = 4\). Let \(S' = \{w_{3t-5}, w_{3t-2}, w_{3t+1}\}\). We define \(D' = (D \setminus S) \cup S'\). Set \(D'\) also dominates \(L_h\) and \(|D'| < |D|\). This is a contradiction to the assumption that \(D\) is a minimum dominating set of \(L_h\). Therefore, we have \(w_{3t-2} \in D\), for any minimum dominating set \(D\) of \(L_h\), \(t = 2, \ldots, \left\lfloor \frac{h}{3} \right\rfloor - 1\).

The similar approach is used for the case \(t \in \{1, \left\lfloor \frac{h}{3} \right\rfloor\}\).

From Lemma 2.1 we conclude that \(\gamma(L_h) \geq h + 1\) and \(D_{Lh}\) is unique (see Figure 4). This completes the proof of Theorem 2.1. \(\square\)

Figure 4. Minimum dominating set of \(L_h\).

Corollary 2.1. \(D_{Lh} \subset D_{Lh+1}\) and \(\gamma(L_{h+1}) = \gamma(L_h) + 1\), \(\forall h \geq 1\). \(\square\)

The remaining two cases are similar and we treat them together.

Theorem 2.2. \(\gamma(M_h) = \gamma(O_h) = \left\lceil \frac{3h}{2} \right\rceil\).

Proof. Let us consider the set
\[
D_{Mh} = \left\{ w_{4i-1}, x_{1}^{2i-1} : i = 1, \ldots, \left\lfloor \frac{h}{2} \right\rfloor \right\} \cup \left\{ x_{3}^{2i} : i = 1, \ldots, \left\lfloor \frac{h}{2} \right\rfloor \right\}.
\]
The set \(D_{Mh}\) is a dominating set of \(M_h\) and hence \(\gamma(M_h) \leq |D_{Mh}| = \left\lceil \frac{3h}{2} \right\rceil\).

Similarly, the set
\[
D_{O_h} = \left\{ w_{2i}, x_{2}^{2i-1} : i = 1, \ldots, \left\lfloor \frac{h}{2} \right\rfloor \right\} \cup \left\{ x_{3}^{2i} : i = 1, \ldots, \left\lfloor \frac{h}{2} \right\rfloor \right\}
\]
is a dominating set of $O_h$ and hence $\gamma(O_h) \leq |D_{O_h}| = \left\lceil \frac{3h}{2} \right\rceil$.

The sets $D_{M_h}$ and $D_{O_h}$ are presented in Figure 5.

![Figure 5](image_url)

**Figure 5.** Minimum dominating sets of (a) $M_h$ and (b) $O_h$.

Let us prove that $D_{M_h}$ is a dominating set of minimum cardinality. Let $\gamma(M_h) < \left\lceil \frac{3h}{2} \right\rceil$. We split $M_h$ into subchains $C^{2i-1}C^{2i}$, $i = 0, 1, \ldots, \left\lfloor \frac{h}{2} \right\rfloor$ and the last chain $C^h$, if $h$ odd. If $h$ is even, then by the Pigeonhole Principle there exists $i$ such that the subchain $C^{2i-1}C^{2i}$ contains fewer than three vertices from $D_{M_h}$. This is impossible because $C^{2i-1}C^{2i}$ is isomorphic to $L_2$ and from Theorem 2.1 we have $\gamma(L_2) = 3$. If $h$ is odd, then either one of considered subchains contains fewer than three vertices from $D_{M_h}$ or $C^h$ contains at most one such vertex. Both cases are impossible since two hexagons cannot be dominated with fewer than three vertices, and one hexagon cannot be dominated with fewer than two vertices. We conclude $\gamma(M_h) \geq \left\lceil \frac{3h}{2} \right\rceil$.

The proof of $\gamma(O_h) \geq \left\lceil \frac{3h}{2} \right\rceil$ is similar and we omit the details.

\[\square\]

**Remark 2.1.** For $h$ even, dominating sets $D_{M_h}$ and $D_{O_h}$ are unique.

**Corollary 2.2.** $D_{M_h} \subset D_{M_{h+1}}$ and $D_{O_h} \subset D_{O_{h+1}}$, $\forall h \geq 1$.

**Corollary 2.3.** Let $G_h \in \{M_h, O_h\}$. Then $\gamma(G_1) = 2$, $\gamma(G_2) = 3$ and for $h \geq 3$

\[
\gamma(G_h) = \begin{cases} 
\gamma(G_{h-1}) + 1, & \text{for } h \text{ even,} \\
\gamma(G_{h-1}) + 2, & \text{for } h \text{ odd.}
\end{cases}
\]

**Proof.** Let us prove the recurrence for $M_h$. We proved $\gamma(M_2) = 3$. By adding one new hexagon, there are 5 vertices to consider. Since $D_{M_2}$ is unique, we cannot rearrange dominating vertices in $M_2$ so that $w_5$ is dominating vertex. Therefore, we need two more dominating vertices in the last hexagon and we have $\gamma(M_3) = \gamma(M_2) + 2$. For $M_4$, the dominating vertices in $C^3$ can be arranged so that $w_7$ is dominating vertex. Since $w_7$ is a cut-vertex, it dominates 3 vertices in $C^4$ and we need only one more dominating vertex. We have $\gamma(M_4) = \gamma(M_3) + 1$. The same procedure is followed for $M_5$ and $M_6$. Inductively, we conclude that for $h$ even $\gamma(M_h) = \gamma(M_{h-1}) + 1$, and for $h$ odd $\gamma(M_h) = \gamma(M_{h-1}) + 2$. The proof for $O_h$ is essentially the same.

\[\square\]
3. Domination number of arbitrary hexagonal chains

In the following we deal with an arbitrary hexagonal cactus chain $G_h$ and present some results about its domination number. With $D_h$ we denote the minimum dominating set of $G_h$.

**Theorem 3.1.** Let $G_h$ be a hexagonal cactus chain of length $h$. Then:

1) either $\gamma(G_h) = \gamma(G_{h-1}) + 1$ or $\gamma(G_h) = \gamma(G_{h-1}) + 2$, $h \geq 2$; 
2) if $\gamma(G_{h-1}) = \gamma(G_{h-2}) + 2$, then $\gamma(G_h) = \gamma(G_{h-1}) + 1$, $h \geq 3$.

**Proof.** 1) Let $G_{h-1}$ be an arbitrary hexagonal cactus chain of length $h-1$ with a minimum dominating set $D_{h-1}$, and let $u$ be a cut-vertex as in Figure 6. Adding one new hexagon to $G_{h-1}$ results in 5 new vertices. We consider the following cases:

1° $u \in D_{h-1}$. Vertex $u$ dominates three vertices in $C^h$. For the remaining three vertices in $C_h$ we need at most one dominating vertex. See Figure 6a.

We have $|D_h| \leq |D_{h-1}| + 1$ and $u \in D_h$.

For any dominating set $D_h$ of $G_h$ we have $|D_h \cap C^j| = 2$, $j = 1, h$. To prove this, notice that the vertices from $C^2 \setminus \{u_1\}$, where $\{u_1\} \in C^1 \cap C^2$, can dominate at most one vertex from $C^1$. For the remaining 5 vertices in $C^1$ we need at least 2 dominating vertices. Vertices from $C^2$ can dominate at most three vertices in $C^1$. But then $u_1 \in D_h$ and we need at least one more vertex to dominate $C^1$. Since $u_1 \in C^1$, it follows that $|D_h \cap C^1| \geq 2$. Since the domination number of a hexagon is 2, we have $|D_h \cap C^1| \leq 2$, that is, $|D_h \cap C^1| = 2$. For $C^h$ the proof is essentially the same.

From $u \in D_{h-1} \Rightarrow u \in D_h$ and $|D_h \cap C^h| = 2$, we conclude that $|D_{h-1}| \leq |D_h| - 1$, that is, $|D_h| \geq |D_{h-1}| + 1$. We obtain $|D_h| = |D_{h-1}| + 1$ and $\gamma(G_h) = \gamma(G_{h-1}) + 1$.

2° $u \notin D_{h-1}$. If there exists another minimum dominating set $D'_{h-1}$ such that $u \in D'_{h-1}$, then we consider $D'_{h-1}$ instead of $D_{h-1}$, and continue as in previous case. Otherwise, $u$ is dominated with at least one vertex from $C^{h-1}$. Then $|D_h| \leq |D_{h-1}| + 2$, since we have 5 undominated vertices in $C^h$. From $|D_h \cap C^h| = 2$ it follows that $|D_{h-1}| \leq |D_h| - 2$. We conclude $|D_h| = |D_{h-1}| + 2$ and $\gamma(G_h) = \gamma(G_{h-1}) + 2$. The case is shown in Figure 6b.

2) If $\gamma(G_{h-1}) = \gamma(G_{h-2}) + 2$ for some $h \geq 3$, then at least 4 vertices in $C^{h-1}$ are not dominated with $D_{h-2}$. That means that $u \notin D_{h-2}$, where $\{u\} = C^{h-2} \cap C^{h-1}$. From part 1) of the theorem we conclude $|D_{h-1}| = |D_{h-2}| + 2$, and dominating vertices in $C_{h-1}$ can be chosen arbitrarily, as long as not both of them are adjacent to $u$. If we attach one more hexagon to $C^{h-1}$, we can set $v \in D_{h-1}$, where $\{v\} \in C^{h-1} \cap C^h$. Now we have case 1° in 1). It follows that $\gamma(G_h) = \gamma(G_{h-1}) + 1$. See Figure 6c. \[
\]

**Figure 6.**
We close this section by showing that the uniform chains are extremal among all hexagonal cactus chains with respect to the dominating number $\gamma$.

**Theorem 3.2.** Let $G_h$ be a hexagonal cactus chain of length $h \geq 1$. Then

$$\gamma(L_h) \leq \gamma(G_h) \leq \gamma(M_h) = \gamma(O_h).$$

**Proof.** The left inequality follows from Theorem 3.1(1) and Corollary 2.1, while the right inequality follows from Theorem 3.1 and Corollary 2.3. \qed

4. $k$-Domination Numbers of Uniform Hexagonal Chains, $k \geq 2$

**Theorem 4.1.** Let $G_h \in \{L_h, M_h, O_h\}$ and let $c$ be the distance between the nearest two cut-vertices in $G_h$. Then

$$\gamma_k(G_h) = \begin{cases} h + 1, & \text{for } k = 2 \\ \left\lceil \frac{ch + 1}{2(k + c)} \right\rceil, & \text{for } k \geq 3 \end{cases},$$

with $G_h$ being $O_h$, $M_h$ and $L_h$ when $c$ is equal to 1, 2 and 3, respectively.

**Proof.** Case $k = 2$.

For $L_h$ set $DL = \{w_{3i-2} : i = 1, \ldots, h + 1\}$ is 2-dominating set of $L_h$ and $\gamma_2(L_h) \leq |DL| = h + 1$.

For $M_h$ we have dominating set

$$DM = \left\{x_1^{2i-1} : i = 1, \ldots, \left\lceil \frac{h}{2} \right\rceil \right\} \cup \left\{x_2^{2i} : i = 1, \ldots, \left\lceil \frac{h}{2} \right\rceil \right\} \cup \{w_{2h+1}\},$$

and $\gamma_2(M_h) \leq |DM| = h + 1$.

For $O_h$ dominating set is $DO = \{x_2^i : i = 0, 1, \ldots, h\} \cup \{w_{h+1}\}$ and $\gamma_2(O_h) \leq |DO| = h + 1$.

\[ \text{Figure 7: Minimum 2-dominating sets of a) } L_6, \text{ b) } M_6 \text{ and c) } O_6. \]
In the following we prove that sets $DL$, $DM$ and $DO$ have minimum cardinality among all 2-dominating sets of $L_h$, $M_h$ and $O_h$, respectively.

Let $D_h$ be the minimum 2-dominating set of $L_h$. Then from Proposition 2.1 we have $|D_1| = 2$. If we consider $G_{h-1}$, $h \geq 2$, then by adding one new hexagon, we have 5 new vertices. Vertices from $D_{h-1}$ 2-dominate at most 5 vertices in $C^h$. Still, there is one not dominated vertex for which we need at least one 2-dominating vertex. Therefore, $|D_h| \geq |D_{h-1}| + 1$, $\forall h \geq 2$. We obtain $|D_h| \geq |D_1| + h - 1$, that is, $|D_h| \geq h + 1$. We proved $\gamma_2(L_h) = h + 1$. The same conclusions are obtained for $M_h$ and $L_h$.

**Case:** $k \geq 3$.

We will first consider $L_h$. Let $t = \left\lceil \frac{3h + 1}{2k + 1} \right\rceil$. We consider the set

$$S = \{w_{(2k+1)i+k+1} : i = 0, 1, \ldots, t - 1\}.$$ 

If $(2k + 1)(t - 1) + k + 1 \leq 3h + 1$, then $S$ is a $k$-dominating set for $L_h$. Otherwise,

$$S' = (S \setminus \{w_{(2k+1)(t-1)+k+1}\}) \cup \{w_{3h+1}\}$$

is $k$-dominating set of $L_h$. We have $\gamma_k(L_h) \leq |S| = |S'| = t$.

![Figure 8. Minimum k-dominating sets of L₁₁ with k = 3 and k = 4.](image)

Let us prove that $|D| \geq t$ for any $k$-dominating set $D$ of $L_h$.

**Lemma 4.1.** Let $P_{3h+1}$ be an induced subgraph of $L_h$ with vertex-set

$$V(P_{3h+1}) = \{w_i : i = 0, 1, \ldots, 3h + 1\}.$$ 

There exists a minimum $k$-dominating set $D$ of $L_h$ such that $D \subset V(P_{3h+1})$.

**Proof.** Let $D$ be the minimum $k$-dominating set of $L_h$ such that $x^j_1 \in D$, $j \in \{1, 2, \ldots, h\}$. Then $N_k[x^j_1] = N_k[w_{3j-1}]$ since both vertices $w_{3j-1}$ and $x^j_1$ $k$-dominate $C^j$, they are at the same distance from cut-vertices $w_{3j-2}$ and $w_{3j+1}$, which means that they $k$-dominate same set of vertices in other hexagons. Let $D' = D \setminus \{x^j_1\} \cup \{w_{3j-1}\}$. Set $D'$ also $k$-dominates $L_h$, and since $|D'| = |D|$, we concluded that $D'$ is a minimum $k$-dominating set of $L_h$. In a similar way we conclude that if $x^j_2 \in D$, then $N_k[x^j_2] = N_k[w_{3j}]$, $j \in \{1, \ldots, h\}$ and $D' = D \setminus \{x^j_2\} \cup \{w_{3j}\}$ is a minimum $k$-dominating set of $L_h$. We proved that for any minimum $k$-dominating set containing vertices $x^j_1$ or $x^j_2$, there exists another minimum $k$-dominating set $D$ of $L_h$ such that $D \subset V(P_{3h+1})$. □
Let $D$ be the set that satisfy Lemma 4.1. Then $D$ $k$-dominates $P_{3h+1}$. Therefore,

$$|D| \geq \gamma_k(P_{3h+1}) = t.$$ 

For $M_h$ let $t = \left\lceil \frac{2h+1}{2k-1} \right\rceil$ and $S = \{w(2k-1)i+k : i = 0, 1, \ldots, t-1\}$.

If $(2k-1)(t-1)+k \leq 2h+1$, set $S$ is a $k$-dominating set of $M_h$. Otherwise,

$$S' = (S \setminus \{w(2k-1)(t-1)+k\}) \cup \{w_{2h+1}\}$$

is $k$-dominating set of $M_h$. Hence, $\gamma_k(M_h) \leq |S| = |S'| = t$.

**Figure 9.** Minimum $k$-dominating sets of $M_{10}$ with $k = 3$ and $k = 4$.

To prove that $\gamma_k(M_h) \geq t$, we consider the following lemma:

**Lemma 4.2.** Let $P_{2h+1}$ be an induced subgraph of $M_h$ with vertex-set

$$V(P_{2h+1}) = \{w_i : i = 1, \ldots, 2h+1\}.$$ 

There exists a minimum $k$-dominating set $D$ of $M_h$ such that $D \subset V(P_{2h+1})$. With the notation $D = \{w_{i_1}, w_{i_2}, \ldots, w_{i_p}\}$, $1 \leq i_1 < i_2 < \ldots < i_p \leq 2h+1$, $D$ has the following properties:

(i) $d(w_{i_j}, w_{i_{j+1}}) \leq 2k - 1$, $\forall j = 1, \ldots, p - 1$.

(ii) $d(w_{i_1}, w_{i_{p}}) \leq k - 1$ and $d(w_{i_p}, w_{2h+1}) \leq k - 1$.

**Proof.** Let $D$ be the minimum $k$-dominating set of $M_h$ such that $x^1_j \in D$, $j \in \{1, \ldots, h\}$. Then $N_k[x^1_j] \subseteq N_k[w_{2j-1}]$ since $x^1_j$ $k$-dominates $C^j$, $d(x^1_j, w_{2j-1}) = 1$, $d(x^1_1, w_{2j+1}) = 3$ and $w_{2j-1}$ $k$-dominates at least $N_k[x^1_j]$. Set $D' = D \setminus \{x^1_j\} \cup \{w_{2j-1}\}$ also $k$-dominates $M_h$. Since $|D| = |D'|$, $D'$ is minimum $k$-dominating set. With the similar approach we conclude that if $x^2_j \in D$, then $N_k[x^2_j] \subseteq N_k[w_{2j-1}]$ and $D' = D \setminus \{x^2_j\} \cup \{w_{2j-1}\}$ is also a minimum $k$-dominating set of $M_h$. (For this case, the choice of $D'$ can also be $D' = D \setminus \{x^2_j\} \cup \{w_{2j+1}\}$.) For $x^3_j \in D$, we choose $D' = D \setminus \{x^3_j\} \cup \{w_{2j+1}\}$. Since $j$ is an arbitrary element of the set $\{1, \ldots, h\}$, the first part of lemma is proved.

(i) Let us assume that $D = \{w_{i_1}, w_{i_2}, \ldots, w_{i_p}\}$, $1 \leq i_1 < i_2 < \ldots < i_p \leq 2h+1$, is minimum $k$-dominating set of $M_h$, and let $d(w_{i_r}, w_{i_{r+1}}) \geq 2k$ for some fixed $r \in \{1, \ldots, p\}$. If $i_r \equiv 1 (\text{mod} 2)$, $w_{i_r}$ is a cut-vertex and there are at least $k$ hexagons
between \(w_i\) and \(w_{i+r+1}\). We denote the corresponding subchain with \(C^{j_1}C^{j_2}\ldots C^{j_l}\), \(l \geq k, 1 \leq j_l \leq h\).

For \(k\) odd, we have \(x_2^{j_2} \notin N_k[w_{i+r}] \cap N_k[w_{i+r+1}]\). Then there exists a vertex \(w_{i+r}'\), \(i_r < i_r < i_{r+1}\), such that \(w_{i+r} \in N_k[x_2^{j_2}]\) and \(w_{i+r}'\) is \(k\)-dominating vertex. This implies \(|D| > p\) which is a contradiction to the assumption that \(D\) is a minimum \(k\)-dominating set of cardinality \(p\).

For \(k\) even, we have \(x_3^{j_2}, x_1^{j_2+1} \notin N_k[w_{i+r}] \cap N_k[w_{i+r+1}]\). But then we need at least one more \(k\)-dominating vertex \(w_{i+r}'\), between \(w_i\) and \(w_{i+r+1}\). Again, we obtain a contradiction to the assumption that \(|D| = p\).

If \(i_r \equiv 0(\text{mod} 2)\), then there are at least \(k - 1\) hexagons between \(w_{i_r}\) and \(w_{i_r+1}\). In a similar way, we conclude that there must be at least one \(k\)-dominating vertex \(w_{i,r}\) between these two vertices, and this is a contradiction to \(|D| = p\).

(ii) Let \(d(w_1, w_{i_1}) \geq k\). Then \(d(w_{i_1}, w_{i_2}) \geq k - 2\) and \(x_1^0 \notin N_k[w_{i_1}]\), since \(d(x_1^0, w_3) = 3\). Therefore, \(d(w_1, w_{i_1}) \leq k - 1\). In a similar way, we prove \(d(w_{i_p}, w_{2h+1}) \leq k - 1\). \(\square\)

Let \(D\) be a minimum \(k\)-dominating set as in Lemma 4.2. Then, \(D\) \((k-1)\)-dominates \(P_{2h+1}\) and \(|D| \geq \gamma_{k-1}(P_{2h+1}) = t\). We proved that \(|D| \geq t\) for any minimum \(k\)-dominating set \(D\) of \(M_h\). From this it follows that \(\gamma_k(M_h) = t\).

At last, we consider \(O_h\). We set \(t = \left\lfloor \frac{h + 1}{2k - 3} \right\rfloor\) and

\[S = \{w_{(2k-3)i+k-1} : i = 0, 1, \ldots, t - 1\} \text{.}\]

With the condition \((2k-3)(t-1)+k-1 \leq h+1\) set \(S\) is a \(k\)-dominating set of \(O_h\).

Otherwise, \(S' = (S \setminus \{w_{(2k-3)(t-1)+k-1}\}) \cup \{w_{h+1}\}\) is \(k\)-dominating set of \(O_h\).

Since \(|S| = |S'| = t\), we conclude \(\gamma_k(O_h) \leq t\).

Figure 10. Minimum \(k\)-dominating sets of \(O_{12}\) with \(k = 3\) and \(k = 4\).

**Lemma 4.3.** Let \(P_{h+1}\) be an induced subgraph of \(O_h\) with vertex-set

\[V(P_{h+1}) = \{w_i : i = 0, 1, \ldots, h + 1\}.\]

There exists a minimum \(k\)-dominating set \(D\) of \(O_h\) such that \(D \subseteq V(P_{h+1})\).
With the notation $D = \{w_{i_1}, w_{i_2}, \ldots, w_{i_p}\}$, $1 \leq i_1 < i_2 < \ldots < i_p \leq h + 1$, $D$ has the following properties:

(i) $d(w_{i_j}, w_{i_{j+1}}) \leq 2k - 3$, $\forall j = 1, \ldots, p - 1$.
(ii) $d(w_1, w_i) \leq k - 2$ and $d(w_{i_p}, w_{h+1}) \leq k - 2$.

Proof. We use the same approach as in Lemma 4.2. \(\Box\)

If $D$ satisfy Lemma 4.3, then $D$ $(k-2)$-dominates $P_{h+1}$. Therefore $|D| \geq \gamma_{k-2}(P_{h+1}) = t$. We conclude $\gamma_k(O_h) = t$.

If we put $c = d(u_{i-1}, u_i)$, $\forall i = 1, \ldots, h$, then domination numbers of all three considered chains can be joined into a single formula. If $G_h \in \{L_h, M_h, O_h\}$, then

$\gamma_k(G_h) = \left[\frac{ch + 1}{2(k + c) - 5}\right]$ with $c$ being the distance between the nearest two cut vertices in $G_h$. \(\Box\)

Corollary 4.1. For $k \geq 3$ and $G_h \in \{L_h, M_h, O_h\}$ we have

$\gamma_k(G_h) = \gamma_{k+c-3}(P_{ch+1})$, with $c = 3$ if $G_h = L_h$, $c = 2$ if $G_h = M_h$ and $c = 1$ if $G_h = O_h$. Moreover, minimum $k$-dominating set of $G_h$ is the minimum $k + c - 3$-dominating sets of induced subpath $P_{ch+1}$. \(\Box\)

5. $k$-DOMINATION ON ARBITRARY HEXAGONAL CACTUS CHAINS

In the following with $G_h$, $h \geq 2$, we denote a hexagonal cactus chain obtained by adding one new hexagon to $C^{h-1}$ in $G_{h-1}$. With $D_h$ we denote the minimum 2-dominating set of $G_h$.

Theorem 5.1. Let $G_h$ be an arbitrary hexagonal cactus chain of length $h \geq 1$. Then $\gamma_2(G_h) = h + 1$.

Proof. Let $D = \{u_i : i = 1, \ldots, h+1\}$ be the set of vertices of $G_h$ such that $C^i \cap C^{i+1} = \{u_{i+1}\}$, $i = 1, \ldots, h - 1$ and $d(u_1, u_2) = d(u_h, u_{h+1}) = 1$. Then $D$ is 2-dominating set of $G_h$ and since $|D| = h + 1$, $D$ has the smallest cardinality among all 2-dominating sets of $G_h$. \(\Box\)

Theorem 5.2. Let $G_h$ be arbitrary hexagonal cactus chain of length $h \geq 1$ and let $k \geq 3$. Then

$\gamma_k(O_h) \leq \gamma_k(G_h) \leq \gamma_k(L_h)$.

Proof. For $h = 1, 2$ it is obvious. Let $h \geq 3$. With $u_{j-1}$ and $u_j$ we denote cut-vertices contained in $C^j$, $j = 2, \ldots, h - 1$, and with $P_s$ we denote the shortest path that contains all cut-vertices of $G_h$ with additional 3 vertices from $C^1$ and 3 vertices from $C^h$.

By merging Lemma 4.1 and the first part of Lemma 4.2 and 4.3 we obtain that there exists a minimum $k$-dominating set $D$ of $G_h$ such that $D \subset V(P_s)$. Let $D'$ be a minimum $k$-dominating set of $G_h$ such that $v \in C^j$, $v \notin V(P_s)$ and $v \in D'$. \(\Box\)
If $C^j$ is para-hexagon, then we consider $D$ constructed from $D'$ by replacing $v$ with $v' \in V(P_s)$, so that $v'$ and $v$ are at the same distance from their nearest cut-vertex. We have $N_k[v] = N_k[v']$. If $C^j$ is meta- or ortho-hexagon, then we can take $D$ instead of $D'$ by replacing $v$ with its nearest cut-vertex. If $u$ is the cut-vertex that is nearest to $v$, then $u = u_{j-1}$ or $u = u_j$. Obviously, $N_k[v] \subset N_k[u]$ so $D'$ also $k$-dominates $G_h$. Since $|D| = |D'|$, $D$ is minimum $k$-dominating set of $G_h$. Since $j$ is an arbitrary element from $\{2, \ldots, h-1\}$, we proved the existence of dominated set $D$, such that $D \subset V(P_s)$.

For any vertex $w \in V(P_s)$, the distance from $w$ to any other hexagon in $G_h$ is the smallest if $G_h = O_h$. This means that $N_k[w]$ contains the largest possible number of vertices for $G_h = O_h$. Therefore, $\gamma_k(O_h) \leq \gamma_k(G_h)$. Distance from $w$ to any other hexagon is the greatest if $G_h = L_h$ which implies that $N_k[w]$ is minimum. Therefore, $\gamma_k(G_h) \leq \gamma_k(L_h)$. □

6. Conclusions

By studying the $k$-domination on hexagonal cactus chains, we obtained the results that show the extremality of uniform chains: para-, meta- and ortho-chain with given $k$. For $k = 2$ all hexagonal cactus chains have the same 2-domination number. For $k = 1$ para-chain has the smallest, and for $k \geq 3$, the largest $k$-domination number. Ortho-chain have the largest 1-domination number (and so does the meta-chain) and the smallest $k$-domination number, $k \geq 3$.

![Subchains G' and G'' with only one para- and ortho-hexagon in between, respectively.](image)

**Figure 11.** Subchains $G'$ and $G''$ with only one para- and ortho-hexagon in between, respectively.

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**References**


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