

## RICCI AND CASORATI PRINCIPAL DIRECTIONS OF $\delta(2)$ CHEN IDEAL SUBMANIFOLDS

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*Dedicated to Professor Bang–Yen Chen at the occasion of his 70th anniversary*

ABSTRACT. We show that for  $\delta(2)$  Chen ideal submanifolds in Euclidean spaces the (intrinsic) Ricci principal directions and the (extrinsic) Casorati principal directions coincide.

### 1. $\delta(2)$ CHEN IDEAL SUBMANIFOLDS OF EUCLIDEAN SPACES

Let  $M^n$  be an  $n$ -dimensional Riemannian submanifold of an  $(n + m)$ -dimensional Euclidean space  $E^{n+m}$ , ( $n \geq 2, m \geq 1$ ) and let  $g, \nabla$  and  $\tilde{g}, \tilde{\nabla}$  be the Riemannian metric and the corresponding Levi–Civita connection on  $M^n$  and on  $E^{n+m}$ , respectively. Tangent vector fields on  $M^n$  will be written as  $X, Y, \dots$  and normal vector fields on  $M^n$  in  $E^{n+m}$  will be written as  $\xi, \eta, \dots$ . The formulae of Gauss and Weingarten, concerning the decomposition of the vector fields  $\tilde{\nabla}_X Y$  and  $\tilde{\nabla}_X \xi$ , respectively, into their tangential and normal components along  $M^n$  in  $E^{n+m}$ , are given by  $\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$  and  $\tilde{\nabla}_X \xi = -A_\xi(X) + \nabla_X^\perp \xi$ , respectively, whereby  $h$  is the second fundamental form and  $A_\xi$  is the shape operator or Weingarten map of  $M^n$  with respect to the normal vector field  $\xi$ , such that  $\tilde{g}(h(X, Y), \xi) = g(A_\xi(X), Y)$ , and  $\nabla^\perp$  is the connection in the normal bundle. The mean curvature vector field  $\vec{H}$  is defined by  $\vec{H} = \frac{1}{n} \text{tr } h$  and its length  $\|\vec{H}\| = H$  is the (extrinsic) mean curvature of  $M^n$  in  $E^{n+m}$ . A submanifold  $M^n$  in  $E^{n+m}$  is totally geodesic when  $h = 0$ , totally umbilical when  $h = g\vec{H}$ , minimal when  $H = 0$  and pseudo-umbilical when  $\vec{H}$  is an umbilical

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normal direction. Let  $\{E_1, \dots, E_n, \xi_1, \dots, \xi_m\}$  be any *adapted orthonormal* local frame field on the submanifold  $M^n$  in  $E^{n+m}$ , denoted for short also as  $\{E_i, \xi_\alpha\}$ , whereby  $i \in \{1, 2, \dots, n\}$  and  $\alpha \in \{1, 2, \dots, m\}$ . By the *equation of Gauss*, the  $(0, 4)$  *Riemann–Christoffel curvature tensor* of a submanifold  $M^n$  in  $E^{n+m}$  is given by  $R(X, Y, Z, W) = \tilde{g}(h(Y, Z), h(X, W)) - \tilde{g}(h(X, Z), h(Y, W))$ . The  $(0, 2)$  *Ricci curvature tensor* of  $M^n$  is defined by  $S(X, Y) = \sum_i R(X, E_i, E_i, Y)$  and the metrically corresponding  $(1, 1)$  tensor or *Ricci operator* will also be denoted by  $S$ :  $g(S(X), Y) = S(X, Y)$ . Since  $S$  is *symmetric* there exists on  $M^n$  an *orthonormal set of eigenvector fields*  $R_1, \dots, R_n$  which determine the (intrinsic) *Ricci principal directions* of the Riemannian manifold  $M^n$ , and the corresponding *eigenfunctions*  $Ric_1, \dots, Ric_n$  are the *Ricci curvatures* of  $M^n$ :  $S(R_i) = Ric_i R_i$ . A Riemannian manifold  $M^n$  is an *Einstein space* when  $S = Ric g$ , or still when *all Ricci curvatures are equal*  $Ric_1 = \dots = Ric_n = Ric$ ,  $M^n$  is a *quasi–Einstein space* when it has a *Ricci curvature of multiplicity*  $\geq n - 1$  and  $M^n$  is a *2–quasi–Einstein space* when it has a *Ricci curvature of multiplicity*  $\geq n - 2$ . The *scalar curvature* of a Riemannian manifold  $M^n$  is defined by  $\tau = \sum_{i < j} K(E_i \wedge E_j)$  whereby  $K(E_i \wedge E_j) = R(E_i, E_j, E_j, E_i)$  is the *sectional curvature* for the plane section  $\pi = E_i \wedge E_j$ , ( $i \neq j$ ). By the *equation of Ricci*, the *normal curvature tensor* of a submanifold  $M^n$  in  $E^{n+m}$  is given by  $R^\perp(X, Y, \xi, \eta) = g([A_\xi, A_\eta](X), Y)$ , whereby  $[A_\xi, A_\eta] = A_\xi A_\eta - A_\eta A_\xi$ , which, as already observed by Cartan [1], implies that *the normal connection is flat or trivial* if and only if *all shape operators  $A_\xi$  are simultaneously diagonalisable*.

The function  $\inf K : M^n \rightarrow \mathbb{R}$  is defined by  $(\inf K)(p) = \inf\{K(p, \pi) \mid \pi \text{ is a plane section of } T_p(M^n)\}$ . In [2], B.-Y. Chen introduced the  $\delta(2)$ –*curvature* as  $\delta(2) = \tau - \inf K$ , which clearly is a *Riemannian scalar invariant* of the manifold  $(M^n, g)$ . Later B.-Y. Chen introduced many further new scalar Riemannian invariants, together with  $\delta(2)$  called his *delta–curvatures*  $\delta(n_1, n_2, \dots, n_k)$ ; (cfr. [3][4][5][6]). And, for all submanifolds  $M^n$  of Euclidean spaces  $E^{n+m}$ , or of arbitrary Riemannian ambient spaces  $\tilde{M}^{n+m}$  for that matter, B.-Y. Chen established *optimal pointwise inequalities between these intrinsic delta–curvatures of  $M^n$  and the squared mean curvature  $H^2$* , and some number determined by the curvature of the ambient space  $\tilde{M}^{n+m}$ , which is zero for Euclidean spaces. Such inequalities can be considered as imposing definite lower bounds, basically dictated by these delta–curvatures, to the extrinsic squared mean curvature or surface tension  $H^2$  which results from the kind of shape of the submanifold  $M^n$  in the ambient space  $\tilde{M}^{n+m}$ . From this point of view, the submanifolds  $M^n$  which actually do realise such lower bound for their surface tension are called *Chen ideal submanifolds*.

For surfaces  $M^2$  in  $E^3$ , the *Euler inequality*  $K \leq H^2$ , whereby  $K$  is the *Gauss curvature* of  $M^2$  at once follows from the fact that  $K = k_1 k_2$  and  $H = \frac{1}{2}(k_1 + k_2)$ , whereby  $k_1$  and  $k_2$  are the *principal curvatures* of  $M^2$  in  $E^3$ , and, moreover,  $K = H^2$  if and only if  $M^2$  is *totally umbilical* i.e. if  $k_1 = k_2$ , or still, by a Theorem of Meusnier, if  $M^2$  is (part of) a *plane  $E^2$*  or of a *round sphere  $S^2$*  in  $E^3$ . The inequalities of Chen do generalise this Euler inequality for the submanifolds  $M^n$  in general ambient

Riemannian manifolds  $\tilde{M}^{n+m}$ , and, in particular, for ambient Euclidean spaces they take the following form, (for more details, cfr. [6]).

**Theorem A.** *For any submanifold  $M^n$  in  $E^{n+m}$ ,  $\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k).H^2$ ,  $\forall (n_1, \dots, n_k) \in S(n)$ , and equality holds at a point  $p$  if and only if, with respect to some suitable adapted orthonormal frame  $\{E_i, \xi_\alpha\}$  around  $p$  along  $M^n$  in  $E^{n+m}$ , the shape operators of  $M^n$  in  $E^{n+m}$  are given*

$$A_\alpha = \begin{pmatrix} A_1^\alpha & \dots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & A_k^\alpha & \\ & & 0 & \mu_\alpha I \end{pmatrix},$$

whereby  $I$  is an identity matrix and  $A_1^\alpha, \dots, A_k^\alpha$  are symmetric  $n_1 \times n_1, \dots, n_k \times n_k$  matrices, respectively, for which  $\text{tr } A_1^\alpha = \dots = \text{tr } A_k^\alpha = \mu_\alpha : M^n \rightarrow \mathbb{R}$ .

The next result is the special case of Theorem A for  $k = 1$  and  $n_1 = 2$  [2].

**Theorem B.** *For any submanifold  $M^n$  in  $E^{n+m}$ ,*

$$\delta(2) \leq \{[n^2(n-2)]/[2(n-1)]\}.H^2, \quad (*)$$

and equality holds at a point  $p$  of  $M^n$  if and only if, with respect to some suitable adapted orthonormal frame  $\{E_i, \xi_\alpha\}$  around  $p$  along  $M^n$  in  $E^{n+m}$ , the shape operators of  $M^n$  in  $E^{n+m}$  are given by

$$A_\alpha = \begin{pmatrix} A_1^\alpha & 0 \\ 0 & \mu_\alpha I \end{pmatrix},$$

whereby  $I$  is an identity matrix and  $A_1^\alpha$  is a symmetric  $2 \times 2$  matrix for which  $\text{tr } A_1^\alpha = \mu_\alpha : M^n \rightarrow \mathbb{R}$ .

And, one may further specialise to an orthonormal frame  $\{F_1, \dots, F_n, \eta_1, \dots, \eta_m\}$  such that  $\eta_1$  lies in the direction of  $\vec{H}$  and such that  $F_1, \dots, F_n$  diagonalise  $A_1$ , so that Theorem B can also be formulated as follows.

**Theorem C.** *For any submanifold  $M^n$  in  $E^{n+m}$ ,*

$$\delta(2) \leq \{[n^2(n-2)]/[2(n-1)]\}.H^2, \quad (*)$$

and equality holds at a point  $p$  if and only if, with respect to a suitable adapted orthonormal frame  $\{F_i, \eta_\alpha\}$  around  $p$  along  $M^n$  in  $E^{n+m}$ , the shape operators of  $M^n$  in  $E^{n+m}$  are given by

$$A_1 = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix}, \quad A_\gamma = \begin{pmatrix} c_\gamma & d_\gamma & 0 & \dots & 0 \\ d_\gamma & -c_\gamma & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

( $\gamma \in \{2, \dots, m\}$ ), whereby  $\mu = a + b$ , (and  $\inf K = ab - \sum_\gamma (c_\gamma^2 + d_\gamma^2)$ ).

Such frame  $\{F_i, \eta_\alpha\}$  will be called *an adapted Chen frame on  $\delta(2)$  Chen ideal submanifolds*.

According to Theorems B and C it is clear that such ideal submanifolds  $M^n$  in  $E^{n+m}$  in general admit two *mutually orthogonal and complementary distinguished (tangent) distributions*: (i) the 2D distribution  $E_1 \wedge E_2 = F_1 \wedge F_2$ , which will be called *the distribution of the Chen planes* and which is the 2D distribution on which the Riemannian manifolds  $M^n$  *point by point* do attain their *minimal sectional curvatures*, and (ii) the orthogonally complementary distribution  $E_3 \wedge \dots \wedge E_n = F_3 \wedge \dots \wedge F_n$ . For a detailed description of the *minimal  $\delta(2)$  Chen ideal submanifolds* referring to [2], we recall that for the *non-minimal* such submanifolds  $\eta_1$  determines their mean curvature vector field  $\vec{H}$ .

## 2. THE CASORATI AND THE RICCI PRINCIPAL DIRECTIONS ON $\delta(2)$ CHEN IDEAL SUBMANIFOLDS

For any submanifold  $M^n$  in some ambient Riemannian manifold  $\tilde{M}^{n+m}$ , the  $(1, 1)$  tensor field  $A^C = \sum_\alpha A_\alpha^2$  is called its *Casorati operator* and the *Casorati curvature (as such)* of  $M^n$  in  $\tilde{M}^{n+m}$  is defined by  $C = \frac{1}{n} \text{tr } A^C = \frac{1}{n} \|h\|^2$ . The Casorati operator being *symmetric* there exists on  $M^n$  an *orthonormal set of eigenvector fields*  $F_1, \dots, F_n$  which determine *the extrinsic or Casorati principal directions* of the submanifold  $M^n$  in  $\tilde{M}^{n+m}$ , and the corresponding eigenfunctions  $c_1, \dots, c_n$  (all  $\geq 0$ ), are the *extrinsic (tangential) principal curvatures* or the *(tangential) Casorati principal curvatures* of  $M^n$  in  $\tilde{M}^{n+m}$ ;  $A^C(F_i) = c_i F_i$ . For the *geometrical meanings* of these notions, which essentially go back to Jordan and Casorati, see [7][8][9][10].

A *hypersurface*  $M^n$  in a Riemannian space  $\tilde{M}^{n+1}$  is called *umbilical* when *its shape operator is proportional to the identity*, i.e. *has an eigenvalue of multiplicity  $n$* , or, still, when *all its principal curvatures are equal*. A hypersurface  $M^n$  in  $\tilde{M}^{n+1}$  is called *quasi-umbilical* when *its shape operator has an eigenvalue of multiplicity  $\geq n-1$* , (see e.g. [11]), and it is called *2-quasi-umbilical* when *its shape operator has an eigenvalue of multiplicity  $\geq n-2$*  ([12][13]). Similarly, a general submanifold  $M^n$  in some ambient Riemannian space  $\tilde{M}^{n+m}$  is called *Casorati umbilical* when *its Casorati operator is proportional to the identity*, i.e. *has an eigenvalue of multiplicity  $n$* , or, still, when *all its (tangential) Casorati principal curvatures are equal*. A submanifold  $M^n$  in  $\tilde{M}^{n+m}$  is called *Casorati quasi-umbilical* when *its Casorati operator has an eigenvalue of*

multiplicity  $\geq n - 1$ , and it is called *Casorati 2-quasi-umbilical* when its Casorati operator has an eigenvalue of multiplicity  $\geq n - 2$ .

From Theorem C it follows that

$$A_1^2 = \begin{pmatrix} a^2 & 0 & 0 & \dots & 0 \\ 0 & b^2 & 0 & \dots & 0 \\ 0 & 0 & \mu^2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \mu^2 \end{pmatrix},$$

$$A_\gamma^2 = \begin{pmatrix} c_\gamma^2 + d_\gamma^2 & 0 & 0 & \dots & 0 \\ 0 & c_\gamma^2 + d_\gamma^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$A^C = \begin{pmatrix} a^2 + \sum_\gamma (c_\gamma^2 + d_\gamma^2) & 0 & 0 & \dots & 0 \\ 0 & b^2 + \sum_\gamma (c_\gamma^2 + d_\gamma^2) & 0 & \dots & 0 \\ 0 & 0 & \mu^2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \mu^2 \end{pmatrix}.$$

This shows that for  $\delta(2)$  Chen ideal submanifolds the eigendirections  $F_1, F_2, F_3, \dots, F_n$  of the mean curvature vector field  $\vec{H}$  also are the eigendirections of the Casorati operator. In particular, it follows from these expressions of the matrices  $A_1$  and  $A^C$  that for both of them the  $(n-2)D$  subspaces  $F_3 \wedge \dots \wedge F_n$  are eigenspaces corresponding to their common eigenvalue  $\mu^2$  of multiplicity  $n - 2$ . And since  $S = nHA_1 - A^C$ , as follows by contraction of the Gauss equation for any submanifold  $M^n$  in  $E^{n+m}$ , we have the following.

**Theorem 2.1.**

- (i) On all  $\delta(2)$  Chen ideal submanifolds  $M^n$  in  $E^{n+m}$  the (extrinsic) principal (tangential) Casorati directions and the (intrinsic) principal Ricci directions do coincide.
- (ii) Every generic such submanifold is Ricci and Casorati 2-quasi-umbilical, whereby the corresponding common  $(n - 2)D$  eigenspaces are the orthogonal complements of the Chen planes on which the Riemannian manifolds  $M^n$  realise their minimal sectional curvatures.

**Remark.** Part (i) of this result had been stated already in [14]. Part (ii), as also various other kinds of results, from e.g. [2] [15] and [16], could be reflected upon in the light of some questions raised by Professor Berger (see e.g. [17]) concerning the distribution of the  $2D$ -planes on which Riemannian manifolds take extremal values.

3.  $\delta(2, 2, \dots, 2)$  CHEN IDEAL SUBMANIFOLDS

The special case of B.-Y. Chen's Theorem A for  $n_1 = \dots = n_k = 2$  and for  $k \geq 2$  is the following.

**Theorem D.** *For any submanifold  $M^n$  in  $E^{n+m}$ ,*

$$\delta(2, 2, \dots, 2) \leq \{n^2[(n-k) - 1]/[2(n-k)]\}.H^2, \quad (**)$$

and equality holds at a point  $p$ , if and only if, with respect to some suitable adapted orthonormal frame  $\{E_i, \xi_\alpha\}$  around  $p$  along  $M^n$  in  $E^{n+m}$ , the shape operators of  $M^n$  in  $E^{n+m}$  are given by

$$A_\alpha = \begin{pmatrix} b_1^\alpha & c_1^\alpha & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ c_1^\alpha & d_1^\alpha & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & . & \dots & . & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & . & \dots & . & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & b_k^\alpha & c_k^\alpha & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & c_k^\alpha & d_k^\alpha & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \mu_\alpha & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \mu_\alpha \end{pmatrix},$$

whereby  $b_1^\alpha + d_1^\alpha = \dots = b_k^\alpha + d_k^\alpha = \mu_\alpha : M^n \rightarrow \mathbb{R}$ .

The submanifolds  $M^n$  of  $E^{n+m}$  for which  $(**)$  at all of their points actually is an equality are called  $\delta(2, 2, \dots, 2)$  Chen ideal submanifolds. And since the algebraic considerations of the eigenvectors and eigenvalues of the matrices  $A_{\vec{H}}$  and  $A^C$  of such submanifolds, just like in Section 2, essentially concern the  $2 \times 2$  blocks involved, we also have the following.

**Theorem 3.1.** *On all  $\delta(2, 2, \dots, 2)$  Chen ideal submanifolds  $M^n$  in  $E^{n+m}$  the principal Casorati directions and the principal Ricci directions do coincide.*

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