ON EMBEDDINGS, TRACES AND MULTIPLIERS IN HARMONIC FUNCTION SPACES

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ABSTRACT. This paper is devoted to certain applications of classical Whitney decomposition of the upper half space $\mathbb{R}^{n+1}_+$ to various problems in harmonic function spaces in the upper half space. We obtain sharp new assertions on embeddings, distances and traces for various spaces of harmonic functions. New sharp theorems on multipliers for harmonic function spaces in the unit ball are also presented.

1. INTRODUCTION, PRELIMINARIES AND AUXILIARY RESULTS

The role of Whitney decomposition of an open set $\Omega \subset \mathbb{R}^n$ in Analysis is well known, it has many applications, for example in the theory of singular integral operators, see [8] and [18]. The aim of this work is to present new applications of Whitney decomposition of the upper half space. In the first section we introduce notations we use in this paper and collect various auxiliary results. The second section is devoted to embedding theorems for spaces of harmonic functions in the upper half space. The third section contains results on trace problems and distance estimates and in the last one we turn from the upper half space to the unit ball $\mathbb{B}$ in $\mathbb{R}^n$ and characterize multipliers between certain spaces of harmonic functions on $\mathbb{B}$. Most of our results rely on Lemma 1.1, which provides a Whitney type decomposition of the upper half space, and Lemma 1.2.

We set $\mathbb{R}_+^{n+1} = \{(x,t) : x \in \mathbb{R}^n, t > 0\} \subset \mathbb{R}^{n+1}$. For $z = (x,t) \in \mathbb{R}_+^{n+1}$ we set $\overline{z} = (x,-t)$. We denote the points in $\mathbb{R}_+^{n+1}$ usually by $z = (x,t)$ or $w = (y,s)$. The Lebegue measure is denoted by $dm(z) = dz = dxdt$ or $dm(w) = dw = dyds$. We also use standard weighted measures $dm_\lambda(z) = t^\lambda dxdt$, $\lambda \in \mathbb{R}$. Given two positive quantities A and B, we write $A \asymp B$ if there are two constants $c, C > 0$ such that $cA \leq B \leq CA$.

The space of all harmonic functions in a domain Ω is denoted by $h(\Omega)$. The space of all functions $f(z_1, \ldots, z_m)$ on $(\mathbb{R}_+^{n+1})^m$ which are harmonic in each of variables $z_1, \ldots, z_m$ is denoted by $\tilde{h}((\mathbb{R}_+^{n+1})^m)$. Weighted harmonic Bergman spaces on $\mathbb{R}_+^{n+1}$ are defined, for $0 < p < \infty$ and $\lambda > -1$, by

$$A_\lambda^p = A_\lambda^p(\mathbb{R}_+^{n+1}) = \left\{ f \in h(\mathbb{R}_+^{n+1}) : \|f\|_{A_\lambda^p} = \left(\int_{\mathbb{R}_+^{n+1}} |f(z)|^p dm_\lambda(z) \right)^{1/p} < \infty \right\},$$

these spaces are complete metric spaces and for $1 \leq p < \infty$ they are Banach spaces.

For $\overrightarrow{s} = (s_1, \ldots, s_m)$, where $s_j > -1$, and $0 < p < \infty$ we set

$$A_{\overrightarrow{s}}^p = h((R_+^{n+1})^m) \cap L^p((\mathbb{R}_+^{n+1})^m, dm_{s_1}(z_1) \ldots dm_{s_m}(z_m)).$$

For $p = \infty$ and $s_j > 0$, $1 \leq j \leq m$, we define $A_{\overrightarrow{s}}^\infty$ as the space of all $f \in h((\mathbb{R}_+^{n+1})^m)$ such that

$$\|f\|_{A_{\overrightarrow{s}}^\infty} = \sup_{z_j \in \mathbb{R}_+^{n+1}} |f(z_1, \ldots, z_m)| t_1^{s_1} \ldots t_m^{s_m} < \infty,$$

for $m = 1$ and $s_1 = \alpha$ we use simpler notation A_α^∞. Again, the spaces $A_{\overrightarrow{s}}^p$ are complete metric spaces, for $1 \leq p \leq \infty$ they are Banach spaces.

Finally, if X is a space of functions harmonic on $(\mathbb{R}_+^{n+1})^m$, then we set $\tilde{h}X = X \cap \tilde{h}((\mathbb{R}_+^{n+1})^m)$.

Definition 1.1. For a function $f : (\mathbb{R}_+^{n+1})^m \to \mathbb{C}$ we define $\mathrm{Tr} f : \mathbb{R}_+^{n+1} \to \mathbb{C}$ by $\mathrm{Tr} f(z) = f(z, \ldots, z)$.

Let $X \subset h(\mathbb{R}_+^{n+1})^m$. The trace of X is Trace $X = \{\mathrm{Tr} f : f \in X\}$.

The problem of characterizing traces of various analytic spaces received much attention, for some results and further references see [15][16][17].

We denote the Poisson kernel for $\mathbb{R}_+^{n+1}$ by $P(x,t)$, i.e.

$$P(x,t) = c_n \frac{t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}, \qquad x \in \mathbb{R}^n, \quad t > 0.$$

For $l \in \mathbb{N}_0$ a Bergman kernel $Q_l(z, w)$, where $z = (x, t) \in \mathbb{R}_+^{n+1}$ and $w = (y, s) \in \mathbb{R}_+^{n+1}$, is defined by

$$Q_l(z, w) = \frac{(-2)^{l+1}}{l!} \frac{\partial^{l+1}}{\partial t^{l+1}} P(x - y, t + s).$$

The following theorem from [7] gives a Bergman type integral representation for functions in A_α^p spaces and justifies the above terminology.

Theorem 1.1. *Let* $0 < p < \infty$ *and* $\alpha > -1$. *If* $0 < p \leq 1$ *and* $l \geq \frac{\alpha+n+1}{p} - (n + 1)$ *or* $1 \leq p < \infty$ *and* $l > \frac{\alpha+1}{p} - 1$, *then*

$$(1.1) \qquad f(z) = \int_{\mathbb{R}_+^{n+1}} f(w) Q_l(z, w) s^l dy ds, \qquad f \in A_\alpha^p, \quad z \in \mathbb{R}_+^{n+1}.$$

The following elementary estimate of this kernel is contained in [7]

$$(1.2) \qquad |Q_l(z, w)| \leq C |z - \overline{w}|^{-(l+n+1)}, \qquad z = (x, t), \; w = (y, s) \in \mathbb{R}_+^{n+1}.$$

Next we formulate two lemmata which are at the core of proofs of some of our main results. The first one provides the above mentioned Whitney type decomposition of the upper half space. The second one is based on subharmonic behavior of $|f|^p$ for harmonic f and $0 < p < \infty$.

Lemma 1.1 ([18]). *There exists a collection* $\{\Delta_k\}_{k=1}^\infty$ *of closed cubes in* $\mathbb{R}_+^{n+1}$ *with sides parallel to coordinate axes such that*

$1°$. $\cup_{k=1}^\infty \Delta_k = \mathbb{R}_+^{n+1}$ *and* $\operatorname{diam} \Delta_k \asymp \operatorname{dist}(\Delta_k, \partial \mathbb{R}_+^{n+1})$.

$2°$. *The interiors of the cubes* Δ_k *are pairwise disjoint.*

$3°$. *If* Δ_k^* *is a cube with the same center as* Δ_k, *but enlarged* $5/4$ *times, then the collection* $\{\Delta_k^*\}_{k=1}^\infty$ *forms a finitely overlapping covering of* $\mathbb{R}_+^{n+1}$, *i.e. there is a constant* $C = C_n$ *such that* $\sum_k \chi_{\Delta_k^*} \leq C$.

Lemma 1.2 ([6]). *Let* Δ_k *and* Δ_k^* *be the cubes from the previous lemma and let* (ξ_k, η_k) *be the center of* Δ_k. *Then, for* $0 < p < \infty$ *and* $\alpha > 0$, *we have*

$$(1.3) \qquad \eta_k^{\alpha p-1} \max_{\Delta_k} |f|^p \leq \frac{C}{|\Delta_k^*|} \int_{\Delta_k^*} t^{\alpha p-1} |f(x, t)|^p dx dt, \qquad f \in h(\mathbb{R}_+^{n+1}), \quad k \geq 1.$$

We will also use the following three technical estimates.

Lemma 1.3 ([18]). *Let* Δ_k *and* Δ_k^* *are as in the previous lemma, let* $\zeta_k = (\xi_k, \eta_k)$ *be the center of the cube* Δ_k. *Then we have*

$$(1.4) \qquad m_\lambda(\Delta_k) \asymp \eta_k^{n+1+\lambda} \asymp m_\lambda(\Delta_k^*), \qquad \lambda \in \mathbb{R},$$

$$(1.5) \qquad |\overline{w} - z| \asymp |\overline{\zeta}_k - z|, \qquad w \in \Delta_k^*, \quad z \in \mathbb{R}_+^{n+1},$$

$$(1.6) \qquad t \asymp \eta_k, \qquad (x, t) \in \Delta_k^*.$$

Lemma 1.4 ([1]). *For $\delta > -1$, $\gamma > n + 1 + \delta$ and $m \in \mathbb{N}_0$ we have*

$$\int_{\mathbb{R}_+^{n+1}} |Q_m(z, w)|^{\frac{\gamma}{n+m+1}} s^\delta dy ds \le C t^{\delta - \gamma + n + 1}, \qquad z = (x, t) \in \mathbb{R}_+^{n+1}.$$

Lemma 1.5 ([10]). *If $\alpha > -1$ and $n + \alpha < 2\gamma - 1$, then*

$$(1.7) \qquad \int_{\mathbb{R}_+^{n+1}} \frac{t^\alpha dz}{|z - \overline{w}|^{2\gamma}} \le C s^{\alpha + n + 1 - 2\gamma}, \qquad w = (y, s) \in \mathbb{R}_+^{n+1}.$$

2. Embedding theorems in spaces of harmonic functions in the upper half space

In this section we provide, using Whitney decomposition, various extensions of some embedding results from [9]. In particular, we prove embedding theorems for $B_\alpha^{p,q}$ and $F_\alpha^{p,q}$ mixed norm classes.

Our first result relates Carleson type condition to the trace operator, as a consequence we obtain a generalization of an embedding result from [9].

Theorem 2.1. *Let $0 < p < \infty$, $s_1, \ldots, s_m > -1$ and let μ be a positive Borel measure on $\mathbb{R}_+^{n+1}$. Then the following conditions are equivalent.*

$1°$. *The measure μ satisfies a Carleson type condition*

$$(2.1) \qquad \frac{\mu(\Delta_k)}{|\Delta_k|^{m + \frac{1}{n+1} \sum_{j=1}^m s_j}} \le C, \qquad k \ge 1.$$

$2°$. *Trace $A_{\vec{s}}^p \hookrightarrow L^p(\mathbb{R}_+^{n+1}, d\mu)$, i.e. the trace operator is bounded from $A_{\vec{s}}^p$ to $L^p(\mathbb{R}_+^{n+1}, d\mu)$.*

$3°$. *Trace $hA_{\vec{s}}^p \hookrightarrow L^p(\mathbb{R}_+^{n+1}, d\mu)$, i.e. the trace operator is bounded from $hA_{\vec{s}}^p$ to $L^p(\mathbb{R}_+^{n+1}, d\mu)$.*

$4°$. *If $f_j \in A_{s_j}^p$ for $1 \le j \le m$, then*

$$(2.2) \qquad \int_{\mathbb{R}_+^{n+1}} \prod_{j=1}^m |f_j(z)|^p d\mu(z) \le C \prod_{j=1}^m \|f_j\|_{A_{s_j}^p}^p.$$

Proof. Let us show $1° \Rightarrow 2°$. Let us choose $f \in A^p_{\vec{s}}$. We use partition of $\mathbb{R}^{n+1}_+$ into cubes Δ_k centered at $\zeta_k = (\xi_k, \eta_k)$. Using Lemma 1.1 we obtain

$$\|\operatorname{Tr} f\|^p_{L^p(\mu)} = \int_{\mathbb{R}^{n+1}_+} |f(z, \dots, z)|^p d\mu(z) = \sum_{k=1}^{\infty} \int_{\Delta_k} |f(z, \dots, z)|^p d\mu(z)$$

$$\leq \sum_{k=1}^{\infty} \mu(\Delta_k) \sup_{z \in \Delta_k} |f(z, \dots, z)|^p.$$

Let us fix $z \in \Delta_k$. Let $B_k(z) \subset (\mathbb{R}^{n+1}_+)^m$ be the ball centered at $(z, \dots, z)$ with radius equal $1/8$ of the side length of Δ_k. Since $|B_k| \asymp |\Delta_k|^m \asymp \eta_k^{m(n+1)}$ and $B_k(z) \subset (\Delta_k^*)^m$ we obtain, using Lemma 1.2

$$|f(z, \dots, z)|^p \leq \frac{C}{|B_k(z)|} \int_{B_k(z)} |f(w_1, \dots, w_m)|^p dm(w_1) \dots dm(w_m)$$

$$\leq \frac{C}{\eta_k^{m(n+1)}} \int_{(\Delta_k^*)^m} |f(w_1, \dots, w_m)|^p dm(w_1) \dots dm(w_m).$$

Since the last estimate is valid for every $z \in \Delta_k$ we obtain, using (1.6)

$$\|\operatorname{Tr} f\|^p_{L^p(\mu)} \leq C \sum_{k=1}^{\infty} \frac{\mu(\Delta_k)}{\eta_k^{m(n+1)}} \int_{(\Delta_k^*)^m} |f(w_1, \dots, w_m)|^p dm(w_1) \dots dm(w_m)$$

$$\leq C \sum_{k=1}^{\infty} \frac{\mu(\Delta_k)}{\eta_k^{m(n+1)+\sum_{j=1}^m s_j}} \int_{(\Delta_k^*)^m} |f(w_1, \dots, w_m)|^p dm_{s_1}(w_1) \dots dm_{s_m}(w_m).$$

Now condition (2.1) and finite overlapping property of Δ_k^* combine to give

$$\|\operatorname{Tr} f\|^p_{L^p(\mu)} \leq C \sum_{k=1}^{\infty} \int_{(\Delta_k^*)^m} |f(w_1, \dots, w_m)|^p dm_{s_1}(w_1) \dots dm_{s_m}(w_m)$$

$$\leq C \int_{(\mathbb{R}^{n+1}_+)^m} |f(w_1, \dots, w_m)|^p dm_{s_1}(w_1) \dots dm_{s_m}(w_m),$$

which implies $2°$.

The implication $2° \Rightarrow 3°$ being trivial, we turn to $3° \Rightarrow 4°$. Let us choose $f_j \in A^p_{s_j}$, $j = 1, \dots, m$. Using Fubini's theorem we see that $f(z_1, \dots, z_m) = f_1(z_1) \cdots f_m(z_m)$ is in $hA^p_{\vec{s}}$ with $\|f\|_{A^p_{\vec{s}}} = \prod_{j=1}^m \|f_j\|_{A^p_{s_j}}$ and (2.2) follows immediately from $3°$.

Finally, we prove $4° \Rightarrow 1°$. Let us fix a cube Δ_k and set $f_j(z) = |z - \overline{\zeta}_k|^{-n+1}$ for $1 \leq j \leq m$. Clearly $f_j \in A^p_{s_j}$ and, by Lemma 1.5, $\|f_j\|^p_{A^p_{s_j}} \leq C \eta_k^{s_j+n+1-p(n-1)}$. Since $\prod_{j=1}^m f_j(z) = |z - \overline{\zeta}_k|^{m(1-n)}$ we have, using Lemma 1.3,

$$\mu(\Delta_k)\eta_k^{mp(1-n)} \leq C \|\prod_{j=1}^m f_j\|^p_{L^p(\mu)} \leq C \prod_{j=1}^m \|f_j\|^p_{A^p_{s_j}} \leq C \eta_k^{mp(1-n)+m(n+1)+\sum_{j=1}^m s_j},$$

which gives (2.1) and completes the proof. □

Corollary 2.1. *Assume $0 < p < \infty$, $\alpha > -1$ and let μ be a positive Borel measure on $\mathbb{R}^{n+1}_+$. Then the following conditions are equivalent.*

1°. The measure μ satisfies a Carleson type condition

$$(2.3) \qquad \frac{\mu(\Delta_k)}{|\Delta_k|^{1+\frac{\alpha}{n+1}}} \leq C, \qquad k \geq 1.$$

2°. A^p_α is continuously embedded into $L^p(\mathbb{R}^{n+1}_+, d\mu)$.

The above result was proved, in the special case $\alpha = 0$ and $1 \leq p < \infty$, in [9].

For a measurable function $f(x, t)$ defined on $\mathbb{R}^{n+1}_+$ we define

$$M_p(f, t) = \|f(\cdot, t)\|_{L^p(dx)}, \qquad t > 0, \quad 0 < p \leq \infty.$$

The following spaces of harmonic functions were considered, for $0 < p < \infty$, $0 < q < \infty$ and $\alpha > 0$, in [4]

$$B^{p,q}_\alpha = \left\{ f \in h(\mathbb{R}^{n+1}_+) : \|f\|_{B^{p,q}_\alpha} = \left(\int_0^\infty M^p_q(f, t) t^{\alpha p - 1} dt \right)^{1/p} < \infty \right\},$$

in fact one can consider these spaces for $p = \infty$ or $q = \infty$, see [4]. These spaces have obvious (quasi)-norms, with respect to these (quasi)-norms they are Banach spaces or complete metric spaces.

Let us set, for $w \in \mathbb{R}^{n+1}_+$ and $l \geq 0$,

$$(2.4) \qquad f_{w,l}(z) = \frac{\partial^l}{\partial t^l} \frac{1}{|z - \overline{w}|^{n-1}}.$$

This function is harmonic in $\mathbb{R}^{n+1}_+$ and, for every $l \geq 0$, we have

$$(2.5) \qquad f_{w,l}(z) = \frac{1}{|z - \overline{w}|^{n-1+l}} P_l((t + s)|z - \overline{w}|^{-1}),$$

where P_l is a polynomial with integer coefficients of degree l. The last statement follows by induction on l using identity $\partial_t |z - \overline{w}| = (t + s)|z - \overline{w}|^{-1}$. In particular we have

$$(2.6) \qquad |f_{w,l}(z)| \leq C|z - \overline{w}|^{-n-l+1}, \qquad z, w \in \mathbb{R}^{n+1}_+.$$

Using (2.6) one derives estimate

$$(2.7) \qquad M_p(f_{w,l}, t) \leq C(t + s)^{\frac{n}{p} - (n-1+l)}, \qquad p(n - 1 + l) > n,$$

and this estimate yields

$$(2.8) \qquad \|f_{w,l}\|_{B_\alpha^{q,p}} \le Cs^{\frac{n}{p}-(n-1+l)+\alpha}, \qquad p(n-1+l) > n + \alpha p.$$

Next, for $w \in \mathbb{R}_+^{n+1}$, let Q_w be the closed cube centered at w with sides parallel to the coordinate axes and with side length equal to s. Since the polynomial P_l has finitely many zeroes it is easy to show that there are constants $\delta > 0$ and $c > 0$ such that

$$(2.9) \qquad |T_w| = c|Q_w|, \qquad T_w = \{z \in Q_w : |P_l((t+s)|z - \overline{w}|^{-1})| > \delta\}.$$

Lemma 2.1. *Let μ be a positive Borel measure on $\mathbb{R}_+^{n+1}$ and let $\theta > 0$. Assume $\mu(T_w) \le Cs^\theta$ for every $w = (y, s) \in \mathbb{R}_+^{n+1}$. Then we have $\mu(Q_w) \le Cs^\theta$ for every $w = (y, s) \in \mathbb{R}_+^{n+1}$.*

Proof. Let us set $T_w^o = T_w \cap Q_w^o$, where Q_w^o denotes the interior of the cube Q_w. We note that all pairs (Q_w, T_w^o) are similar to each other, either by translation by a vector parallel to the boundary $\partial \mathbb{R}_+^{n+1}$ or by a homothecy with center on $\partial \mathbb{R}_+^{n+1}$. Let us fix a cube Q_w. Using noted similarity it is easy to conclude that $Q_w \subset \cup\{T_{w'}^o : w' = (y', s') \in \mathbb{R}_+^{n+1}, s/2 \le s' \le 2s\}$. Now we have a finite subcover $Q_w \subset \cup_{j=1}^N T_{w_j}^o$ where N does not depend on w and therefore

$$\mu(Q_w) \le \sum_{j=1}^N \mu(T_{w_j}^o) \le C \sum_{j=1}^N s_j^\theta \le Cs^\theta.$$

$\square$

The above discussion is a preparation for the following embedding theorem for $B_\alpha^{p,q}$ spaces. Its analogue for analytic functions in the unit ball appeared in [14].

Theorem 2.2. *Let $0 < p \le q < \infty$, $\alpha > 0$ and let μ be a Borel measure on $\mathbb{R}_+^{n+1}$. Then the following conditions are equivalent.*

1°. The measure μ satisfies the following condition:

$$(2.10) \qquad \mu(\Delta_k) \le C\eta_k^{n\frac{q}{p}+\alpha q}, \qquad k \ge 1,$$

where (ξ_k, η_k) is the center of Δ_k.

2°. We have continuous embedding $B_\alpha^{q,p} \hookrightarrow L^q(\mathbb{R}_+^{n+1}, d\mu)$.

Proof. Let us assume (2.10) is satisfied. We have $B_\alpha^{q,p} \hookrightarrow B_{\alpha+\frac{n}{p}-\frac{n}{q}}^{q,q} = A_\theta^q$, where $\theta = \alpha q + \frac{q}{p}n - n - 1$, for the above embedding see Lemma 5 from [4]. Now we use Corollary 2.1 to obtain $A_\theta^q \hookrightarrow L^q(\mathbb{R}_+^{n+1}, d\mu)$.

Now assume 2° is satisfied. Let us choose $l \in \mathbb{N}_0$ such that $p(n - 1 + l) > n + \alpha p$. Then, using (2.9), (2.8) and (2.5), we have

$$(2.11) \qquad \mu(T_w)s^{(1-n-l)q} \leq C\|f_{w,l}\|_{L^q(d\mu)}^q \leq C\|f_{w,l}\|_{B_\alpha^{q,p}}^q \leq Cs^{q[\frac{n}{p} - (n-1+l)q] + \alpha q},$$

which gives $\mu(T_w) \leq Cs^{q\frac{n}{p} + \alpha q}$. However, Lemma 2.1 shows that the last estimate is equivalent to (2.10). □

In the next theorem we consider harmonic Triebel-Lizorkin spaces $F_\alpha^{p,q}$ consisting of all functions $f \in h(\mathbb{R}_+^{n+1})$ satisfying

$$\|f\|_{F_\alpha^{p,q}}^p = \int_{\mathbb{R}^n} \left(\int_0^\infty |f(x,t)|^q t^{\alpha q - 1} dt \right)^{p/q} dx < \infty.$$

These spaces are complete metric spaces, for $\min(p,q) \geq 1$ these spaces are Banach spaces.

The following theorem has a counterpart for analytic functions in the unit ball in $\mathbb{C}^n$, see [14].

Theorem 2.3. *Let $0 < \tau \leq p < \infty$. Assume μ is a Borel measure on $\mathbb{R}_+^{n+1}$. Then the following conditions are equivalent.*

1°. The measure μ satisfies a Carleson type condition

$$(2.12) \qquad\qquad \mu(\Delta_k) \leq C\eta_k^{n+\alpha p}, \qquad k \geq 1,$$

where (ξ_k, η_k) is the center of Δ_k.

2°. The space $F_\alpha^{p,\tau}$ is continuously embedded into $L^p(\mathbb{R}_+^{n+1}, \mu)$.

Proof. The sufficiency of condition (2.12) follows from the following chain of embeddings

$$F_\alpha^{p,\tau} \hookrightarrow F_\alpha^{p,p} = B_\alpha^{p,p} = A_\theta^p \hookrightarrow L^p(\mathbb{R}^{n+1}, d\mu), \qquad \theta = \alpha p - 1,$$

for the first one see [4] and for the last one see Corollary 2.1.

Next we prove necessity of the condition (2.12). Let us choose $l \in \mathbb{N}_0$ satisfying $p(n - 1 + l) > n + \alpha p$ and as test functions consider $f_{w,l}$ from (2.4). Our condition on l ensures that $f_{w,l} \in F_\alpha^{p,\tau}$, $w = (y,s) \in \mathbb{R}_+^{n+1}$. Clearly the norm $\|f_{w,l}\|_{F_\alpha^{p,\tau}}$ depends on s but not on y, hence we can assume $w = (0,s)$, $s > 0$. A simultaneous change of variables $x = \lambda\xi$, $t = \lambda r$ where $\xi \in \mathbb{R}^n$, $r > 0$, gives the following identity

$$\|f_{(0,s),l}\|_{F_\alpha^{p,\tau}}^p = \lambda^{n - p(n-1+l-\alpha)} \|f_{(0,s/\lambda),l}\|_{F_\alpha^{p,\tau}}^p, \qquad \lambda > 0, s > 0.$$

This identity gives the following norm evaluation

$$\|f_{w,l}\|^p_{F^{p,\tau}_\alpha} = \|f_{(0,s),l}\|^p_{F^{p,\tau}_\alpha} = \|f_{(0,1),l}\|^p_{F^{p,\tau}_\alpha} s^{n-p(n-1+l-\alpha)}, \qquad w = (y,s) \in \mathbb{R}^{n+1}_+.$$

Therefore, we obtain the following inequalities, analogous to (2.11)

$$\mu(T_w)s^{(1-n-l)p} \le C\|f_{w,l}\|^p_{L^p(d\mu)} \le C\|f_{w,l}\|^p_{F^{p,\tau}_\alpha} \le Cs^{-p(n-1+l-\alpha)+n}.$$

This gives $\mu(T_w) \le Cs^{n+\alpha p}$ which is, by Lemma 2.1, equivalent to (2.12). $\qquad\square$

Remark 2.1. We note that in Theorems 2.2 and 2.3 it is tempting to use $|z - \overline{w}|^{-n+1}$ as test functions, they have useful property of being positive. But this works only for certain values of α and p. The problem is that this test function is not always in the required space. If it happens to be, the proof can be simplified. But in general we take derivatives and improve "size" of our test functions. However, taking derivatives in principle destroys positivity, the functions $f_{w,l}$ can have zeros and this makes proofs technically more complicated and explains the need for Lemma 2.1.

3. Traces and distances in spaces of harmonic functions in the upper half space

In this section, using Whitney decomposition, we present complete analogues of previously known results on traces in analytic classes in polydisk and polyball [15][16][17] and complement our previous results on distances [2][1].

Theorem 3.1. *Let* $s_1, \ldots, s_m > -1$ *and set* $\lambda = \sum_{j=1}^m s_j$. *Then*

$$(3.1) \qquad\qquad\qquad A^\infty_\lambda \subset \text{Trace } \tilde{h}A^\infty_{\vec{s}}.$$

Conversely, if $f \in A^\infty_{\vec{s}}$ *and if* Tr f *is harmonic, then* Tr $f \in A^\infty_\lambda$.

Proof. The second part of theorem follows from definitions. Let $g \in A^\infty_\lambda$. We choose a non negative integer $k > \lambda - 1$ and define

$$f(z_1, \ldots, z_m) = \int_{\mathbb{R}^{n+1}_+} Q_k \left(\frac{z_1 + \cdots + z_m}{m}, w \right) g(w)s^k dw, \qquad z_j \in \mathbb{R}^{n+1}_+.$$

It is immediate from Theorem 1.1 that Tr $f = g$. Since the kernel Q_k is harmonic in the variables $z_1, \ldots, z_m$ it is clear that $f \in \tilde{h}((\mathbb{R}^{n+1}_+)^m)$. Using classical inequality

between arithmetic and geometric mean we obtain

$$\begin{aligned}
|f(z_1,\ldots,z_m)| &\leq \|g\|_{A_\lambda^\infty} \int_{\mathbb{R}_+^{n+1}} \left| Q_k\left(\frac{z_1+\cdots+z_m}{m}, w\right) \right| s^{k-\lambda} dw \\
&\leq C\|g\|_{A_\lambda^\infty} \int_0^\infty s^{k-\lambda} \int_{\mathbb{R}^n} \frac{dy}{|\frac{z_1+\cdots+z_m}{m} - \overline{w}|^{k+n+1}} ds \\
&\leq C\|g\|_{A_\lambda^\infty} \int_0^\infty \frac{s^{k-\lambda} ds}{(\frac{t_1+\cdots+t_m}{m} + s)^{k+1}} \\
&\leq C\|g\|_{A_\lambda^\infty} \int_0^\infty \frac{s^{k-\sum_{j=1}^m s_j}}{\left(\prod_{j=1}^m (t_j + s)\right)^{(k+1)/m}} ds \\
&= C\|g\|_{A_\lambda^\infty} \int_0^\infty \prod_{j=1}^m \frac{s^{k/m - s_j}}{(t_j + s)^{\frac{k+1}{m}}} ds.
\end{aligned}$$

Next we use Hölder's inequality for m functions to obtain

$$|f(z_1,\ldots,z_m)| \leq C \left(\prod_{j=1}^m \int_0^\infty \frac{s^{k-ms_j}}{(t_j+s)^{k+1}} ds\right)^{1/m} \leq C t_1^{-s_1} \ldots t_m^{-s_m},$$

and the proof is finished. $\square$

For any two m-tuples ($m \geq 1$) of reals $a = (a_1,\ldots,a_m)$ and $b = (b_1,\ldots,b_m)$ we define an integral operator

$$(3.2) \qquad (S_{a,b}f)(z_1,\ldots,z_m) = \prod_{j=1}^m t_j^{a_j} \int_{\mathbb{R}_+^{n+1}} \frac{f(w)s^{-n-1+\sum_{j=1}^m b_j}}{\prod_{j=1}^m |z_j - \overline{w}|^{a_j+b_j}} dw, \qquad z_j \in \mathbb{R}_+^{n+1}.$$

This operator is used in norm estimates of extension operators used in the proofs of trace theorems, it is well defined for $f(w) \in L^1(\mathbb{R}_+^{n+1}, s^{-n-1-\sum_{j=1}^m b_j})$. A unit ball analogue of this operator was used in [15], see also [21].

The following proposition is well known in the case $m = 1$, its analogues for $m > 1$ in polydiscs and polyballs were proved in [15], [16]. The proof relies on Lemma 1.2 and on Whitney decomposition, i.e. on Lemma 1.1.

Proposition 3.1. *Let $0 < p \leq 1$, $a, b \in \mathbb{R}^m$ and $s_1,\ldots,s_m > -1$ satisfy $pa_j > -1 - s_j$ and $pb_j > n+1+s_j$ for $j = 1,\ldots,m$. Set $\lambda = (m-1)(n+1) + \sum_{j=1}^m s_j$. Then there is a constant $C > 0$ such that*

$$(3.3) \qquad \int_{\mathbb{R}_+^{n+1}} \cdots \int_{\mathbb{R}_+^{n+1}} |(S_{a,b}f)(z_1,\ldots,z_m)|^p dm_{s_1}(z_1)\ldots dm_{s_m}(z_m) \leq C\|f\|_{A_\lambda^p}^p$$

for every $f \in A_\lambda^p(\mathbb{R}_+^{n+1})$.

Proof. We use again family Δ_k of cubes from Lemma 1.1. We have, using (1.5),

$$
|(S_{a,b}f)(z_1,\ldots,z_m)| = \prod_{j=1}^m t_j^{a_j} \sum_{k=1}^\infty \int_{\Delta_k} \frac{f(w)s^{-n-1+\sum_{j=1}^m b_j}}{\prod_{j=1}^m |z_j - \overline{w}|^{a_j+b_j}} dw
$$

$$
\leq C \sum_{k=1}^\infty \prod_{j=1}^m \frac{t_j^{a_j}}{|z_j - \overline{\zeta}_k|^{a_j+b_j}} \int_{\Delta_k} |f(w)|s^{-n-1+\sum_{j=1}^m b_j} dm(w)
$$

$$
\leq C \sum_{k=1}^\infty \prod_{j=1}^m \frac{t_j^{a_j} \eta_k^{\sum_{j=1}^m b_j}}{|z_j - \overline{\zeta}_k|^{a_j+b_j}} \sup_{\Delta_k} |f|.
$$

Since $0 < p \leq 1$ this gives

$$
(3.4) \qquad |(S_{a,b}f)(z_1,\ldots,z_m)|^p \leq C \sum_{k=1}^\infty \prod_{j=1}^m \frac{t_j^{pa_j} \eta_k^{p\sum_{j=1}^m b_j}}{|z_j - \overline{\zeta}_k|^{pa_j+pb_j}} \sup_{\Delta_k} |f|^p.
$$

We integrate this inequality with respect to $dm_{s_1}(z_1)\ldots dm_{s_m}(z_m)$ and obtain, using Lemma 1.5 and Lemma 1.2

$$
M = \int_{\mathbb{R}_+^{n+1}} \cdots \int_{\mathbb{R}_+^{n+1}} |(S_{a,b}f)(z_1,\ldots,z_m)|^p dm_{s_1}(z_1)\ldots dm_{s_m}(z_m)
$$

$$
\leq C \sum_{k=1}^\infty \eta_k^{m(n+1)+\sum_{j=1}^m s_j} \sup_{\Delta_k} |f|^p \leq C \sum_{k=1}^\infty \int_{\Delta_k^*} |f(w)|^p dm_\lambda(w).
$$

This is sufficient due to finite overlapping property of the family Δ_k^*. $\qquad\square$

Lemma 3.1. *Let $0 < p < \infty$ and $s_1,\ldots,s_m > -1$. Set $\lambda = (m-1)(n+1)+\sum_{j=1}^m s_j$. Then there is a constant $C > 0$ such that for all $f \in h((\mathbb{R}_+^{n+1})^m)$ we have*

$$
\int_{\mathbb{R}_+^{n+1}} |f(z,\ldots,z)|^p dm_\lambda(z)
$$

$$
(3.5) \qquad\qquad \leq C \int_{\mathbb{R}_+^{n+1}} \cdots \int_{\mathbb{R}_+^{n+1}} |f(z_1,\ldots,z_m)|^p dm_{s_1}(z_1)\ldots dm_{s_m}(z_m).
$$

Proof. This is an immediate consequence of Theorem 2.1, indeed it is easy to check that measure dm_λ satisfies condition (2.1). $\qquad\square$

Remark 3.1. We note that the trace results from [15] for harmonic spaces in the unit ball are true under one additional condition: namely that the trace of a considered harmonic function on a product of unit balls is harmonic on the unit ball.

The theorem below was announced, without proof, in [15] for $0 < p < \infty$. Analogous results in the case of the unit ball were proved in [15]. Let us note that in the

case $n = 1$ a precise subclass for which we have characterization of traces was found in [17], this is precisely the space of all pluriharmonic functions in the unit poly disk.

Theorem 3.2. *Let* $0 < p \le 1$, $s_1, \ldots, s_m > -1$ *and set* $\lambda = (m-1)(n+1) + \sum_{j=1}^m s_j$. *Then*

$$(3.6) \qquad A_\lambda^p \subset \text{Trace } \tilde{h} A_{\vec{s}}^p \subset \text{Trace } A_{\vec{s}}^p \subset L^p(\mathbb{R}_+^{n+1}, dm_\lambda).$$

In particular, if $f \in A_{\vec{s}}^p$ *and if* $\text{Tr } f$ *is harmonic, then* $\text{Tr } f \in A_\lambda^p$.

Proof. The second inclusion in (3.6) is trivial and the last one follows from Lemma 3.1, which also establishes the last part of the theorem. It remains to prove $A_\lambda^p \subset \text{Trace } \tilde{h} A_{\vec{s}}^p$. Let $g \in A_\lambda^p$ and set

$$f(z_1, \ldots, z_m) = \int_{\mathbb{R}_+^{n+1}} Q_l\left(\frac{z_1 + \cdots + z_m}{m}, w\right) g(w) s^l dw, \qquad z_j \in \mathbb{R}_+^{n+1},$$

where $l > \lambda - 1$ is an integer. As in the previous theorem, clearly $f \in \tilde{h}((\mathbb{R}_+^{n+1})^m)$. In order to simplify notation we set $Z = z_1 + \cdots + z_m$. Now we have, using (1.6), (1.2) and (1.5),

$$\begin{aligned} |f(z_1, \ldots, z_m)| &\le \sum_{k=1}^\infty \int_{\Delta_k} |Q_l(Z/m, w)| \, |g(w)| s^l dw \\ &\le C \sum_{k=1}^\infty \eta_k^l \int_{\Delta_k} |Q_l(Z/m, w)| dw \sup_{\Delta_k} |g| \\ &\le C \sum_{k=1}^\infty \frac{\eta_k^{n+l+1}}{|Z/m - \overline{\zeta}_k|^{n+1+l}} \sup_{\Delta_k} |g|. \end{aligned}$$

Since $0 < p \le 1$ this estimate and Lemma 1.2 give

$$\begin{aligned} |f(z_1, \ldots, z_m)|^p &\le C \sum_{k=1}^\infty \frac{\eta_k^{p(n+l+1)}}{|Z/m - \overline{\zeta}_k|^{p(n+1+l)}} \sup_{\Delta_k} |g|^p \\ &\le C \sum_{k=1}^\infty \frac{\eta_k^{p(n+l+1)-\lambda-n-1}}{|Z/m - \overline{\zeta}_k|^{p(n+1+l)}} \|g\|_{L^p(\Delta_k^*, dm_\lambda)}^p. \end{aligned}$$

Now we set $z_j = (x_j, t_j)$, $1 \le j \le m$ and obtain

$$\begin{aligned} I_k(t_1, \ldots, t_m) &= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \frac{dx_1 \ldots dx_m}{\left| \frac{(x_1 + \cdots + x_m, t_1 + \cdots + t_m)}{m} - \overline{\zeta}_k \right|^{p(n+l+1)}} \\ &\le C \left(\frac{t_1 + \cdots + t_m}{m} + \eta_k \right)^{-p(n+l+1)+mn}. \end{aligned}$$

Inequality between arithmetic and geometric mean and the above estimate imply

$$M = \int_{\mathbb{R}_+^{n+1}} \cdots \int_{\mathbb{R}_+^{n+1}} |f(z_1, \ldots, z_m)|^p dm_{s_1}(z_1) \ldots dm_{s_m}(z_m)$$

$$\leq C \sum_{k=1}^{\infty} \eta_k^{p(n+l+1)-\lambda-n-1} \int_0^{\infty} \cdots \int_0^{\infty} \frac{t_1^{s_1} \ldots t_m^{s_m} dt_1 \ldots dt_m}{\left(\frac{t_1+\cdots t_m}{m} + \eta_k\right)^{p(n+l+1)-mn}} \|g\|^p_{L^p(\Delta_k^*, dm_\lambda)}$$

$$\leq C \sum_{k=1}^{\infty} \eta_k^{p(n+l+1)-\lambda-n-1} \prod_{j=1}^{m} \int_0^{\infty} \frac{t_j^{s_j} dt_j}{(t_j + \eta_k)^{\frac{p(n+l+1)-mn}{m}}} \|g\|^p_{L^p(\Delta_k^*, dm_\lambda)}$$

$$\leq C \sum_{k=1}^{\infty} \|g\|^p_{L^p(\Delta_k^*, dm_\lambda)} \leq C \|g\|_{A_\lambda^p}$$

due to finite overlapping property of the family Δ_k^*. $\qquad \square$

Remark 3.2. As already noted in [16], the above results on traces can be extended to certain mixed norm spaces of harmonic functions.

We note that A_α^p is continuously embedded into $A_{\frac{\alpha+n+1}{p}}^\infty$, see [4]. Hence it is a natural problem to look for estimates of $\mathrm{dist}_{A_{\frac{\alpha+n+1}{p}}^\infty}(f, A_\alpha^p)$ for $f \in A_{\frac{\alpha+n+1}{p}}^\infty$. The case $1 < p < \infty$ was treated in [1], the theorem below covers the remaining case $0 < p \leq 1$.

For $\epsilon > 0$, $\lambda > 0$ and $f \in h(\mathbb{R}_+^{n+1})$ we set

$$V_{\epsilon,\lambda}(f) = V_{\epsilon,\lambda} = \left\{(x,t) \in \mathbb{R}_+^{n+1} : |f(x,t)|t^\lambda \geq \epsilon\right\}.$$

Theorem 3.3. *Let* $0 < p \leq 1$, $\alpha > -1$, $\lambda = \frac{\alpha+n+1}{p}$, $m \in \mathbb{N}_0$ *and* $m > \max(\lambda - 1, \frac{\alpha}{p})$. *Set, for* $f \in A_{\frac{\alpha+n+1}{p}}^\infty$,

$$(3.7) \qquad\qquad\qquad d_1(f) = \mathrm{dist}_{A_{\frac{\alpha+n+1}{p}}^\infty}(f, A_\alpha^p),$$

$$(3.8) \qquad d_2(f) = \inf\left\{\epsilon > 0 : \int_{\mathbb{R}_+^{n+1}} \left(\int_{V_{\epsilon,\lambda}} |Q_m(z,w)|s^{m-\lambda}dyds\right)^p t^\alpha dxdt < \infty\right\}.$$

Then $d_1(f) \asymp d_2(f)$.

Proof. We begin with inequality $d_1(f) \geq d_2(f)$. Assume $d_1(f) < d_2(f)$. Then there are $\epsilon > \epsilon_1 > 0$ and $f_1 \in A_\alpha^p(\mathbb{R}_+^{n+1})$ such that $\|f - f_1\|_{A_\lambda^\infty} \leq \epsilon_1$ and

$$(3.9) \qquad\qquad \int_{\mathbb{R}_+^{n+1}} \left(\int_{V_{\epsilon,\lambda}} |Q_m(z,w)|s^{m-\lambda}dyds\right)^p t^\alpha dxdt = +\infty.$$

Since $\|f - f_1\|_{A_\lambda^\infty} \leq \epsilon_1$ we have

$$(3.10) \qquad\qquad (\epsilon - \epsilon_1)\chi_{V_{\epsilon,\lambda}}(w)s^{-\lambda} \leq |f_1(w)|, \qquad w = (y,s) \in \mathbb{R}_+^{n+1}.$$

Combining (3.9) and (3.10) we obtain

$$+\infty = \int_{\mathbb{R}_+^{n+1}} \left(\int_{\mathbb{R}_+^{n+1}} \chi_{V_{\epsilon,\lambda}}(w) |Q_m(z,w)| s^{m-\lambda} dy ds \right)^p t^\alpha dx dt$$

$$\leq C \int_{\mathbb{R}_+^{n+1}} \left(\int_{\mathbb{R}_+^{n+1}} |f_1(w) Q_m(z,w)| s^m dy ds \right)^p t^\alpha dx dt = M.$$

Our goal is to obtain a contradiction by showing that M is finite. We use cubes Δ_k and Δ_k^* from Lemma 1.1 and Lemma 1.2, let $\zeta_k = (\xi_k, \eta_k)$ be the corresponding centers. Using Lemma 1.3, Lemma 1.1 and assumption $0 < p \leq 1$ we obtain

$$I(z) = \left(\int_{\mathbb{R}_+^{n+1}} |f_1(w) Q_m(z,w)| s^m dy ds \right)^p = \left(\sum_{k=1}^\infty \int_{\Delta_k} |f_1(w) Q_m(z,w)| s^m dy ds \right)^p$$

$$\leq C \left(\sum_{k=1}^\infty \eta_k^m |\Delta_k| \max_{w \in \Delta_k} |f_1(w)| \max_{w \in \Delta_k} |Q_m(z,w)| \right)^p$$

$$\leq C \sum_{k=1}^\infty \eta_k^{mp} |\Delta_k|^p \max_{w \in \Delta_k} |f_1(w)|^p \max_{w \in \Delta_k} |Q_m(z,w)|^p.$$

Therefore, using Lemma 1.2, Lemma 1.3 and finite overlapping property of the family Δ_k^* we obtain

$$I(z) \leq C \sum_{k=1}^\infty \eta_k^{mp} |\Delta_k|^p |z - \overline{\zeta}_k|^{-p(m+n+1)} \eta_k^{1-\alpha p} \frac{1}{|\Delta_k^*|} \int_{\Delta_k^*} s^{\alpha p - 1} |f_1(w)|^p dy ds$$

$$\leq C \sum_{k=1}^\infty \eta_k^{mp} |\Delta_k|^{p-1} \int_{\Delta_k^*} \frac{|f_1(w)|^p dy ds}{|z - \overline{w}|^{p(m+n+1)}}$$

$$\leq C \sum_{k=1}^\infty \int_{\Delta_k^*} \frac{|f_1(w)|^p s^{mp+(n+1)(p-1)} dy ds}{|z - \overline{w}|^{p(m+n+1)}}$$

$$\leq C \int_{\mathbb{R}_+^{n+1}} \frac{|f_1(w)|^p s^{mp+(n+1)(p-1)} dy ds}{|z - \overline{w}|^{p(m+n+1)}}.$$

This estimate, Lemma 1.5 with $2\gamma = p(m+n+1)$ and Fubini's theorem yield

$$M \leq C \int_{\mathbb{R}_+^{n+1}} \left(\int_{\mathbb{R}_+^{n+1}} \frac{|f_1(w)|^p s^{mp+(n+1)(p-1)} dw}{|z - \overline{w}|^{p(m+n+1)}} \right) t^\alpha dz$$

$$= C \int_{\mathbb{R}_+^{n+1}} |f_1(w)|^p s^{mp+(n+1)(p-1)} \int_{\mathbb{R}_+^{n+1}} \frac{t^\alpha dz}{|z - \overline{w}|^{p(m+n+1)}} dw$$

$$\leq C \int_{\mathbb{R}_+^{n+1}} |f_1(w)|^p s^\alpha dw < \infty,$$

arriving at a contradiction. Therefore we proved $d_1(f) \geq d_2(f)$.

Next we prove $d_1(f) \leq C d_2(f)$. We choose $\epsilon > 0$ such that the integral appearing in (3.8) is finite. Using Theorem 1.1 we obtain

$$f(z) = \int_{\mathbb{R}^{n+1}_+ \setminus V_{\epsilon,\lambda}} f(w) Q_m(z,w) s^m dw + \int_{V_{\epsilon,\lambda}} f(w) Q_m(z,w) s^m dw = f_1(z) + f_2(z).$$

Since the kernel $Q_m(z,w)$ is harmonic in the second variable both f_1 and f_2 are harmonic in $\mathbb{R}^{n+1}_+$. Next, using Lemma 1.4 and definition of the set $V_{\epsilon,\lambda}$ we obtain

$$(3.11) \qquad |f_1(z)| \leq \epsilon \int_{\mathbb{R}^{n+1}_+} |Q_m(z,w)| s^{m-\lambda} dw \leq C \epsilon t^{-\lambda}, \qquad z = (x,t) \in \mathbb{R}^{n+1}_+,$$

which gives $\|f_1\|_{A_\lambda^\infty} \leq C\epsilon$. To complete the proof it suffices to show that $f_2 \in A_\alpha^p$. Since $|f(w)| \leq \|f\|_{A_\lambda^\infty} s^{-\lambda}$ we have

$$\|f_2\|_{A_\alpha^p} \leq \|f\|_{A_\lambda^\infty} \int_{\mathbb{R}^{n+1}} \left(\int_{V_{\epsilon,\lambda}} |Q_m(z,w)| s^{m-\lambda} dw \right)^p t^\alpha dz \leq C \|f\|_{A_\lambda^\infty},$$

and the proof is complete. $\qquad \square$

4. Multipliers on spaces of harmonic functions in the unit ball

In this section we prove new sharp theorems on multipliers in spaces of harmonic functions in the unit ball $\mathbb{B} = \{x \in \mathbb{R}^n : |x| < 1\}$. This topic is fairly new, it was initiated in [13] and pursued further in [1] and [3]. This section presents new results in this direction. Even the case of the unit disc was not studied extensively, for some results we refer to [12]. Of course, the topic of multipliers between analytic function spaces is a vast subject.

We denote spherical harmonics of order k by $Y_j^{(k)}$, $1 \leq j \leq d_k$, see [19] for details on spherical harmonics. Let us recall, for reader's convenience, needed definitions.

Definition 4.1. For a double indexed sequence of complex numbers

$$c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$$

and a harmonic function $f(rx') = \sum_{k=0}^\infty r^k \sum_{j=1}^{d_k} b_k^j(f) Y_j^{(k)}(x')$ we define

$$(c * f)(rx') = \sum_{k=0}^\infty \sum_{j=1}^{d_k} r^k c_k^j b_k^j(f) Y_j^{(k)}(x'), \qquad rx' \in \mathbb{B},$$

if the series converges in $\mathbb{B}$. Similarly we define convolution of $f, g \in h(\mathbb{B})$ by

$$(f * g)(rx') = \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} r^k b_k^j(f) b_k^j(g) Y_j^{(k)}(x'), \qquad rx' \in \mathbb{B},$$

it is easily seen that $f * g$ is defined and harmonic in $\mathbb{B}$.

Definition 4.2. Let X and Y be subspaces of $h(\mathbb{B})$. We say that a double indexed sequence c is a multiplier from X to Y if $c * f \in Y$ for every $f \in X$. The vector space of all multipliers from X to Y is denoted by $M_H(X, Y)$.

We are looking for sufficient and/or necessary condition for a double indexed sequence c to be in $M_H(X, Y)$, for certain spaces X and Y of harmonic functions. We associate to such a sequence c a harmonic function

$$(4.1) \qquad g_c(x) = g(x) = \sum_{k \geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x'), \qquad x = rx' \in \mathbb{B},$$

The conditions we are looking for are expressed in terms of fractional derivatives of $g = g_c$.

Definition 4.3. For $t \in \mathbb{R} \setminus \mathbb{Z}_-$ and a harmonic function $f(x) = \sum_{k=0}^{\infty} r^k b_k(f) \cdot Y^k(x')$ on $\mathbb{B}$ we define a fractional derivative of order t of f by the following formula

$$(\Lambda_t f)(x) = \sum_{k=0}^{\infty} r^k \frac{\Gamma(k + n/2 + t)}{\Gamma(k + n/2)\Gamma(t)} b_k(f) \cdot Y^k(x'), \qquad x = rx' \in \mathbb{B}.$$

It is easily seen that Λ_t maps $h(\mathbb{B})$ into $h(\mathbb{B})$.

We record the following formula

$$(4.2) \qquad (c * f)(r^2 x') = \int_{\mathbb{S}} (g_c * P_{y'})(rx') f(ry') dy' = \int_{\mathbb{S}} (g_c * P_{x'})(ry') f(ry') dy'$$

where $P_{y'}(x) = P(x, y') = c_n \frac{1 - |x|^2}{|x - y'|^n}$ is the Poisson kernel for the unit ball $\mathbb{B}$, see [1] for details and further references.

Next we recall definitions of some harmonic function spaces. For $\alpha > 0$ we set $A_\alpha^\infty = \{ f \in h(\mathbb{B}) : \|f\|_{A_\alpha^\infty} = \sup_{x \in \mathbb{B}} (1 - |x|^2)^\alpha |f(x)| < \infty \}$ and

$$A_\alpha^p = \left\{ f \in h(\mathbb{B}) : \|f\|_{A_\alpha^p} = \left(\int_{\mathbb{B}} |f(x)|^p (1 - |x|^2)^\alpha dx \right)^{1/p} < \infty \right\}, \qquad 0 < p < \infty.$$

For $0 < p < \infty$, $0 \leq r < 1$ and $f \in h(\mathbb{B})$ we set

$$M_p(f, r) = \left(\int_{\mathbb{S}} |f(rx')|^p dx' \right)^{1/p},$$

with the usual modification to cover the case $p = \infty$. The harmonic Hardy spaces are defined by $H^s = \{f \in h(\mathbb{B}) : \sup_{0 \leq r < 1} M_s(f, r) < \infty\}$, $0 < s \leq \infty$. For $0 < p \leq \infty$, $0 < q \leq \infty$, $\alpha > 0$ and $f \in h(\mathbb{B})$ we consider mixed (quasi)-norms $\|f\|_{p,q,\alpha}$ defined by

$$(4.3) \qquad \|f\|_{p,q,\alpha} = \left(\int_0^1 M_q(f, r)^p (1 - r^2)^{\alpha p - 1} r^{n-1} dr \right)^{1/p}, \qquad 0 < p < \infty,$$

and, for $p = \infty$, $\|f\|_{\infty,q,\alpha} = \sup_{0 \leq r < 1} (1 - r^2)^\alpha M_q(f, r)$. The corresponding mixed norm spaces are

$$B_\alpha^{p,q}(\mathbb{B}) = B_\alpha^{p,q} = \{f \in h(\mathbb{B}) : \|f\|_{p,q,\alpha} < \infty\}.$$

For details on these spaces we refer to [7], Chapter 7. In particular these spaces are complete metric spaces and for $\min(p, q) \geq 1$ they are Banach spaces.

Our next result uses duality arguments. For $M > -1$, $0 < p < \infty$ and $\alpha > -1$ we denote by $D_{-M} A_\alpha^p$ the space of all functions $f \in h(\mathbb{B})$ such that

$$\|f\|_{p,M;\alpha} = \left(\int_{\mathbb{B}} |\Lambda_{M+1} f(x)|^p (1 - |x|^2)^\alpha dx \right)^{1/p} < \infty.$$

It is immediate that each $f \in D_{-M} A_{M-\beta}^1$, $M > \beta - 1$, generates a continuous linear functional on A_β^∞ by the following formula

$$L_f(g) = \int_{\mathbb{B}} \Lambda_{M+1} f(x) g(x) (1 - |x|^2)^M dx, \qquad g \in A_\beta^\infty.$$

Therefore, the above pairing gives an embedding $D_{-M} A_{M-\beta}^1 \hookrightarrow (A_\beta^\infty)^*$, $M > \beta - 1$, which is used in the proof of the next theorem.

Theorem 4.1. *Let $1 < s < \infty$, $\beta > 0$ and let s' be the exponent conjugate to s. Then $c \in M_H(H^s, A_\beta^\infty)$ if and only if the function $g = g_c$ satisfies the following condition*

$$(4.4) \qquad N_{s'}(g) = \sup_{0 \leq \rho < 1} \sup_{y' \in \mathbb{S}} (1 - \rho)^\beta \left(\int_{\mathbb{S}} |(g * P_{x'})(\rho y')|^{s'} dx' \right)^{1/s'} < \infty.$$

Proof. Assume $c \in M_H(H^s, A_\beta^\infty)$, i.e. $M_c : H^s \to A_\beta^\infty$ and choose $M > \beta - 1$. Then, using the above embedding, the adjoint operator M_c^* maps $D_{-M} A_{M-\beta}^1$ into $H^{s'}$. Moreover, it is easy to verify that M_c^* acts as a multiplier operator from $D_{-M} A_{M-\beta}^1$ to $H^{s'}$ generated by the same double indexed sequence c. Next, using definition of the space $D_{-M} A_{M-\beta}^1$ we see that the double indexed sequence

$$c' = \left\{ \frac{\Gamma(k + n/2)\Gamma(M + 1)}{\Gamma(k + n/2 + M + 1)} c_j^k, k \geq 0, 1 \leq j \leq d_k \right\}$$

acts as a multiplier from $A_{M-\beta}^1$ to $H^{s'}$. Set $g' = g_{c'}$, then we have $g = \Lambda_{M+1} g'$. Since $A_{M-\beta}^1 = B_{M-\beta+1}^{1,1}$ we have $c' \in M_H(B_{M-\beta+1}^{1,1}, H^{s'})$ and Theorem 1 from [3] gives, with $m > M - \beta$

$$\sup_{0 \le \rho < 1} \sup_{y' \in \mathbb{S}} (1 - \rho)^{-(M-\beta+1)+m+1} \left(\int_{\mathbb{S}} |\Lambda_{m+1}(g' * P_{x'})(\rho y')|^{s'} dx' \right)^{1/s'} < \infty.$$

But this implies

$$\sup_{0 \le \rho < 1} \sup_{y' \in \mathbb{S}} (1 - \rho)^{\beta+m+1} \left(\int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^{s'} dx' \right)^{1/s'}$$

$$= C \sup_{0 \le \rho < 1} \sup_{y' \in \mathbb{S}} (1 - \rho)^{\beta+m+1} \left(\int_{\mathbb{S}} |\Lambda_{m+M+2}(g' * P_{x'})(\rho y')|^{s'} dx' \right)^{1/s'}$$

$$\le C \sup_{0 \le \rho < 1} \sup_{y' \in \mathbb{S}} (1 - \rho)^{\beta+m-M} \left(\int_{\mathbb{S}} |\Lambda_{m+1}(g' * P_{x'})(\rho y')|^{s'} dx' \right)^{1/s'} < \infty.$$

Since $\beta > 0$, the last estimate implies (4.4).

Now we assume c satisfies (4.4) and choose $f \in H^s$. Set $h = M_c f$. Then, using (4.2), we have

$$|h(r^2 x')| \le \int_{\mathbb{S}} |(g * P_{x'})(ry')f(ry')| dy' \le N_{s'}(g)(1 - r)^{-\beta} M_s(f, r),$$

which means that $(1 - r)^\beta M_\infty(h, r) \le C \|f\|_{H^s}$.   □

For the case $0 < s \le 1$ we refer reader to [3].

The next theorem deals with $B_\alpha^{p,q}$ spaces on the unit ball. For another results on multipliers from $B_\alpha^{p,q}$ spaces we refer to [3].

Theorem 4.2. *Let* $1 < p < \infty$, $1 \le q < \infty$, $\alpha, \beta > 0$ *and let* q' *be the exponent conjugate to* q. *Assume* $m > \max(\alpha - \beta - 1, \beta - 1)$. *Then* $c \in M_H(B_\alpha^{p,q}, A_\beta^\infty)$ *if and only if the function* $g = g_c$ *satisfies the following condition*

$$(4.5) \quad M_{q'}(g) = \sup_{0 \le \rho < 1} \sup_{y' \in \mathbb{S}} (1 - \rho)^{m+1+\beta-\alpha} \left(\int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^{q'} dx' \right)^{1/q'} < \infty.$$

Proof. For necessity of this condition we use a duality result $(B_\alpha^{p,q})^* \cong B_\alpha^{p',q'}$ from [20], see [3] for another application of this duality to multiplier problems. Since the arguments are analogous to those in the proof of Theorem 4.1 we omit details.

Now we prove sufficiency of the condition (4.5). Assume $f \in B_\alpha^{\infty,q}$ and assume c satisfies (4.5). Set $h = M_c f$. Using (4.2) we obtain

$$|\Lambda_{m+1} h(r^2 x')| \leq \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(ry') f(ry')| dy'$$
$$\leq M_{q'}(g)(1-r)^{\alpha-\beta-m-1} M_q(f,r),$$

which gives

$$(1-r)^{m+1+\beta} M_\infty(\Lambda_{m+1}h, r) \leq C(1-r)^\alpha M_q(f,r) \leq C\|f\|_{B_\alpha^{\infty,q}}.$$

Hence $(1-r)^\beta M_\infty(h,r) \leq C\|f\|_{B_\alpha^{\infty,q}}$, see [7] Chapter 7. Therefore $M_c : B_\alpha^{\infty,q} \to A_\beta^\infty$. Since we have embedding $B_\alpha^{p,q} \hookrightarrow B_\alpha^{\infty,q}$ for all $0 < p \leq \infty$, see [7], the proof is completed. $\qquad\square$

Finally we state two results on multipliers in spaces where definitions of norms involve derivatives, proofs will appear elsewhere. Let $\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ denote the gradient of a smooth function f on $\Omega \subset \mathbb{R}^n$. We set

$$DA_\alpha^p = \{f \in h(\mathbb{B}) : \|f\|_{DA_\alpha^p} = |f(0)| + \|\nabla f\|_{A_\alpha^p} < \infty\}, \qquad \alpha > 0, 0 < p < \infty,$$

$$DB_\alpha^{p,q} = \{f \in h(\mathbb{B}) : \|f\|_{DB_\alpha^{p,q}} = |f(0)| + \|\nabla f\|_{B_\alpha^{p,q}} < \infty\}, \qquad \alpha > 0, 0 < p, q < \infty.$$

Theorem 4.3. *Let $1 < s < \infty$, $\alpha, \beta > 0$, $0 < p \leq 1$ and let s' be the exponent conjugate to s. Then $c \in M_H(DB_\alpha^{p,1}, H_\beta^s)$ if and only if the function $g = g_c$ satisfies the following condition*

$$(4.6) \qquad L_{s'}(g) = \sup_{0 \leq \rho < 1} \sup_{y' \in \mathbb{S}} (1-\rho)^{m+2+\beta-\alpha} \left(\int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^{s'} dx' \right)^{1/s'} < \infty.$$

Since $DA_\alpha^p = DB_{\frac{\alpha+1}{p}}^{p,p}$, see [7], taking $p = 1$ we obtain the following corollary.

Corollary 4.1. *Let $1 < s < \infty$, $\alpha, \beta > 0$ and let s' be the exponent conjugate to s. Then $c \in M_H(DA_\alpha^1, H_\beta^s)$ if and only if the function $g = g_c$ satisfies the following condition*

$$(4.7) \qquad K_{s'}(g) = \sup_{0 \leq \rho < 1} \sup_{y' \in \mathbb{S}} (1-\rho)^{m+1+\beta-\alpha} \left(\int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^{s'} dx' \right)^{1/s'} < \infty.$$

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