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ON SOME NEW THEOREMS ON CERTAIN ANALYTIC AND MEROMORPHIC CLASSES OF NEVANLINNA TYPE ON THE COMPLEX PLANE

ROMI SHAMOYAN[1] AND OLIVERA MIHIĆ[2]

Abstract. We introduce and study certain new scales of analytic and meromorphic functions in the unit disc and solve some problems in these scales. We provide complete descriptions of zero sets, then we present some new parametric representations for these classes. Some of our results were known only previously for so called standard weights.

1. Introduction

Assuming that $\mathbf{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ is the unit disk of the finite complex plane $\mathbb{C}$, $\mathbf{T}$ is the boundary of $\mathbf{D}$, $\mathbf{T} = \{z \in \mathbb{C} : \ |z| = 1\}$ and $H(\mathbf{D})$ is the space of all functions holomorphic in $\mathbf{D}$ we introduce the classes of functions

$$N^{\infty}_{\alpha}(\mathbf{D}) = \{f \in H(\mathbf{D}) : T(r, f) \leq C_f (1 - r)^{-\alpha}, \ 0 \leq r < 1, \ \alpha \geq 0\},$$

where $T(r, f)$ is classical and well known Nevanlinna characteristic defined by $T(r, f) = \frac{1}{2\pi} \int_{\mathbf{T}} \log^+ |f(r\xi| d\xi$, where $a^+ = \max\{0, a\}, \ a \in \mathbb{R}$, (see for example [3]–[7]).

It is obvious that if $\alpha = 0$ then $N^{\infty}_0 = N$, where N is a classical Nevanlinna class. The following statement holds by Nevanlinna's classical result on the parametric representation of N (see [3]–[7]).

The N class coincides with the set of functions representable in the form

$$f(z) = C_\lambda z^\lambda B(z, \{z_k\}) \exp\left(\int_{-\pi}^{\pi} \frac{d\mu(\theta)}{1 - ze^{-i\theta}}\right), \ z \in \mathbf{D},$$

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where C_λ is a complex number, λ is a nonnegative integer, $B(z,\{z_k\})$ is the classical Blaschke product with zeros $\{z_k\}_{k=1}^{\infty}\subset \mathbf{D}$ enumerated according to their multiplicities and satisfying the Blaschke density condition $\sum_{k=1}^{\infty}(1-|z_k|)<\infty$, and $\mu(\theta)$ is any function of bounded variation on $[-\pi,\pi]$, (see [3]).

We denote by $B_\alpha^{p,q}(\mathbf{T})$, $0<p<\infty,\ 0<q\le\infty,\ \alpha>0$, the classical Besov space on the unit circle $\mathbf{T}$, (see [2]). Also, by $m_2(\xi)$ we denote standard normalized Lebesques area measure.

Everywhere below by $n_f(t)=n(t)$ we denote the quantity of zeros of an analytic function f in the unit disk $|z|\le t<1$ and by $Z(X)$ the zero set of an analytic class $X,\ X\subset H(\mathbf{D})$. By let $\{z_k\}_{k=1}^{\infty}$ be a sequence of numbers from $\mathbf{D}$ below we mean that $\{z_k\}_{k=1}^{\infty}$ is an arbitrary sequence from the unit disk enumerated by its growth $(|z_k|\le|z_{k+1}|\le\ldots)$ according to its multiplicity. Also, by n_k we denote $n(1-2^{-k})$, i.e. $n_k=n(1-2^{-k}),\ k=1,2,\ldots$

In all our assertions below we assume in advance that our functions are not identically zero or infinity.

Theorem A. (see [13]) Let $\alpha>0$ and $\beta>\alpha-1$, then the N_α^∞ class coincides with the set of functions representable in the form

$$f(z)=C_\lambda z^\lambda \Pi_\beta(z,\{z_k\})\exp\left(\int_{-\pi}^{\pi}\frac{\psi(e^{i\theta})d\theta}{(1-ze^{-i\theta})^{\beta+2}}\right),\ z\in\mathbf{D}, \tag{1.1}$$

where C_λ is a complex number, λ is a nonnegative integer, $\Pi_\beta(z,\{z_k\})$ is the Weierstrass - type product

$$\Pi_\beta(z,\{z_k\})=\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)\exp\left(\frac{-(\beta+1)}{\pi}\int_{\mathbf{D}}\frac{(1-|\xi|^2)^\beta\ln\left|1-\frac{\xi}{z_k}\right|}{(1-\overline{\xi}z)^{\beta+2}}dm_2(\xi)\right),$$

which converges absolutely and uniformly inside $\mathbf{D}$, where it present an analytic function with zeros $\{z_k\}_{k=1}^{\infty}$, $\{z_k\}_{k=1}^{\infty}\subset\mathbf{D}$ is a finite or infinite sequence with condition

$$n(\tau)\le\frac{c}{(1-\tau)^{\alpha+1}},$$

where $c>0$ is a positive constant and $\psi(e^{i\theta})$ is a real function of $B_{\beta-\alpha+1}^{1,\infty}(\mathbf{T})$.

In the theorem below, we also give a result which was established in [11] and in a sense similar to Theorem A.

Let $S_\alpha^p(\mathbf{D})$ be the class defined by

$$S_\alpha^p(\mathbf{D})=\{f\in H(\mathbf{D}):\|f\|_{S_\alpha^p}^p=\int_0^1(1-\tau)^\alpha T^p(\tau,f)d\tau<\infty,\ 0<p<\infty,\ \alpha>-1\}.$$

Theorem B. (see [11]) For $p\in(0,\infty)$, $\beta>\frac{\alpha+1}{p}$, $f\in S_\alpha^p(\mathbf{D})$ if and only if $f(z)$ admits representation $f(z)=C_\lambda z^\lambda\Pi_\beta(z,\{z_k\})\exp\left(\int_{-\pi}^{\pi}\frac{\psi(e^{i\theta})d\theta}{(1-ze^{-i\theta})^{\beta+1}}\right)$, $z\in\mathbf{D}$, where C_λ is a complex number, λ is a nonnegative integer, $\{z_k\}_{k=1}^{\infty}\subset\mathbf{D}$ is a sequence for which $\int_0^1(1-\tau)^{\alpha+p}[n(\tau)]^p d\tau<\infty$ and $\psi\in B_s^{1,p}(\mathbf{T})$, where $s=\beta-\frac{\alpha+1}{p}$.

One can easily see that Theorem A gives parametric representation of the spaces $N^{\infty}_{\alpha}(\mathbf{D})$ while Theorem B gives the parametric representation of $S^p_{\alpha}(\mathbf{D})$ analytic area Nevanlinna type spaces in the unit disk via certain infinite products in the unit disk. One of the goals of this paper is to obtain such parametric representation of the larger spaces

$$N^{\infty,p}_{\alpha,\beta}(\mathbf{D}) = \left\{ f \in H(\mathbf{D}) : \sup_{0 \le R < 1} (1-R)^{\beta} \int_0^R \left(\int_{\mathbf{T}} \ln^+ |f(|z|\xi)| d\xi \right)^p (1-|z|)^{\alpha} d|z| < \infty \right\}$$

where $0 < p < \infty$, $\alpha > -1$ and $\beta \ge 0$, and

$$N^p_{\alpha,\beta}(\mathbf{D}) = \{ f \in H(\mathbf{D}) : \int_0^1 \left(\int_{|z| \le R} (\ln^+ |f(z)|)(1-|z|)^{\alpha} dm_2(z) \right)^p (1-R)^{\beta} dR < \infty \},$$

where it is assumed that $\beta > -1$, $\alpha > -1$ and $0 < p < \infty$.

These analytic area Nevanlinna type classes were introduced recently in [14]. Note that various properties of $N^{\infty,p}_{\alpha,0}$ spaces are studied in [3] for $p = 1$ and in [11] for all p.

Thus it is natural to consider the problem on extension of these important results to all $N^{\infty,p}_{\alpha,\beta}$ classes. The zero set description problem can be stated in the following simple form: Assuming that X is a fixed subspace of $H(\mathbf{D})$ find a class Y of sequences such that the zero set of any function f, $f \in X$ is a sequence of Y and for any sequence $\{z_k\} \in Y$ there is a function f, $f \in X$ such that $f(z_k) = 0$, $k = 1, \ldots$

Note that for many classical analytic classes such as the spaces A^p_{α} this problem is still open (see [8]). On the other hand the complete descriptions of the zero sets of $\mathcal{N}^{\infty,p}_{\alpha,0}$ and N^{∞}_{α} are known (see [3]). One of the intentions of this paper is to solve this problem for mentioned new Nevanlinna type analytic classes in the unit disc and to establish the parametric representations of these classes, where the found description is used. We mention that several new results of this type are presented in [11], [14] for some classical Nevanlinna-Djrbashian analytic classes in the unit disc. So it is natural to consider this problem for $N^p_{\alpha,\beta}$ and $N^{\infty,p}_{\alpha,\beta}$.

Note that zero sets of the classes $N^{\infty,p}_{\alpha,\beta}$ are described in [11] for $\beta = 0$. Besides note that the above mentioned problems on zero sets description and parametric representation have various applications and are important in function theory, (see [4][5][6][12]).

It is not difficult to verify that all the above mentioned analytic classes are topological vector spaces with complete invariant metrics.

Some results of this paper concerning zero sets without proofs were given in a paper [14]. In this paper we add complete proofs to mentioned results announced in [14] and add new results for spaces of analytic and meromorphic functions. Moreover a new important feature here is that some our assertions will be provided in more general form than in mentioned paper that is for general weights for certain S class of functions. This kind of extensions are well-known and can be found recently in many papers of various authors, for example in [11].

Throughout the paper C, sometimes with indexes, stands for various positive constants which can be different even in a chain of inequalities and are independent of the discussed functions or variables.

The notation $A \asymp B$ means that there is a positive constant C, such that $\frac{B}{C} \leq A \leq CB$. We will write for two expressions $A \lesssim B$ if there is a positive constant C such that $A < CB$.

2. Preliminaries

In this separate section we collect various assertions and facts that will be used in sequel and some known propositions from theory of meromorphic functions that we will need later in proofs or for comparasion with our results.

Proposition A. (see [3])Let $\{z_k\}_{k=1}^{\infty}$ be a sequence in the unit disk, $\{z_k\}_{k=1}^{\infty} \subset \mathbf{D}$, satisfying condition $\sum_{k=1}^{\infty}(1-|z_k|)^{t+2} < \infty,\ t > -1$. Then for such a t the infinite product

(2.1)

$$\Pi_t(z, \{z_k\}) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) \exp\left(\frac{-(t+1)}{\pi} \int_{\mathbf{D}} \frac{(1-|\xi|^2)^t \ln\left|1 - \frac{\xi}{z_k}\right|}{(1-\overline{\xi}z)^{t+2}} dm_2(\xi)\right),\ z \in \mathbf{D},$$

converges absolutely and uniformly inside $\mathbf{D}$ where it presents an analytic function with zeros $\{z_k\}_{k=1}^{\infty}$.

The following known corollary shows that the infinite product we introduced above has a simple form for nonnegative integers.

Corollary 2.1. (*see* [3]) *Let $\{z_k\}_{k=1}^{\infty}$ be a sequence in the unit disk, $\{z_k\}_{k=1}^{\infty} \subset \mathbf{D}$, satisfying condition $\sum_{k=1}^{\infty}(1-|z_k|)^{q+2} < \infty,\ q \in \mathbb{Z}_+$. Then the infinite product*

$$\Pi_q(z, \{z_k\}) = \prod_{k=1}^{\infty} \overline{z}_k \left(\frac{z_k - z}{1 - \overline{z}_k z}\right) \exp \sum_{j=1}^{q+1} \frac{1}{j} \left(\frac{1-|z_k|^2}{1-\overline{z}_k z}\right)^j,\ z \in \mathbf{D},$$

converges absolutely and uniformly inside $\mathbf{D}$ *where it presents an analytic function with zeros* $\{z_k\}_{k=1}^{\infty}$.

It is easy to see that the factors of the infinite product from corollary arise in a simple way from the well-known Blaschke factors similarly as Weierstrass products (see [5] [7]).

Proposition B. (see [3]) Let $\{z_k\}_{k=1}^{\infty}$ be a sequence in the unit disk, $\{z_k\}_{k=1}^{\infty} \subset \mathbf{D}$, and $\sum_{k=1}^{\infty}(1-|z_k|)^{t+2} < \infty,\ t > -1$, then the following estimate holds for $\Pi_t(z, \{z_k\})$ product

$$\ln^+ |\Pi_t(z, \{z_k\})| \leq C_t \sum_{k=1}^{\infty} \frac{(1-|z_k|^2)^{t+2}}{|1 - z\overline{z}_k|^{t+2}},\ z \in \mathbf{D}$$

where $C_t > 0$ is a constant depending solely on t.

In the following proposition we introduce another infinite product which will be mentioned by us.

Proposition C. (see [4][5][7][12]) Let $\alpha > -1$. Let $\{z_k\}_{k=1}^{\infty}$ be a sequence of numbers from $\mathbf{D}$ and $\sum_{k=1}^{\infty}(1-|z_k|)^{\alpha+1} < \infty$. Then the infinite product $B_\alpha(z,\{z_k\})$ converges absolutely and uniformly inside $\mathbf{D}$ if $\sum_{k=1}^{\infty}(1-|z_k|)^{\alpha+1} < \infty$, where

$$B_\alpha(z,\{z_k\}) = \prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)\exp\left(-W_\alpha(z,z_k)\right) \text{ and}$$

$$W_\alpha(z,\xi) = \sum_{k=1}^{\infty}\frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)\Gamma(k+1)}\left((\overline{\xi}z)^k\int_{|\xi|}^{1}\frac{(1-x)^\alpha dx}{x^{k+1}} - \left(\frac{z}{\xi}\right)^k\int_0^{|\xi|}(1-x)^\alpha x^{k-1}dx\right),$$

$z,\xi\in\mathbf{D}$. The $B_\alpha(z,\{z_k\})$ product presents an analytic function in $\mathbf{D}$ with zeros only on $\{z_k\}_{k=1}^{\infty}$.

Remark 2.1. An interesting generalization of this product can be found in [3].

Now we will add to this section some facts from the theory of meromorphic functions that will be needed for our exposition (see [4][5][7][12]).

Let $f(z)$ be meromorphic function in $\mathbf{D}$ and let $f(z)=\sum_{k=m}^{\infty}C_k z^k$, be it is Loran expansion near $z=0$. Let $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ be sequences of poles and zeros of $f(z)$. We assume also $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ are counted by their growth according to their multiplicity.

The following formula of Poisson - Jensen is well known (see [4][5][7][12]).

$$\ln|f(z)| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln|f(\rho e^{i\theta})|\frac{(\rho^2-r^2)}{(\rho^2-2r\rho\cos(\theta-\varphi)+r^2)}d\theta$$

$$+\Sigma_{0<|a_\nu|<\rho}\ln\left|\frac{\rho(z-a_\nu)}{\rho^2-\overline{a}_\nu z}\right| - \Sigma_{0<|b_\nu|<\rho}\ln\left|\frac{\rho(z-b_\nu)}{\rho^2-\overline{b}_\nu z}\right| + m\ln\left(\frac{|z|}{\rho}\right),\ z\in\mathbf{D},$$

where m is a multiplicity of zero or pole of f in $z=0$, $|z|=r<\rho<1$.

Putting $z=0$ we get classical Jensen's formula that will be used in this paper (see, for example [4][5][7][12] and the references there). Moreover the formula we mentioned can be written in the following symmetric form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln^+|f(\rho e^{i\theta})d\theta + \Sigma_{0<|b_\nu|<\rho}\ln\left(\frac{\rho}{|b_\nu|}\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln^+\frac{1}{|f(\rho e^{i\theta})|}d\theta + \Sigma_{0<|a_\nu|<\rho}\ln\left(\frac{\rho}{|a_\nu|}\right) + \ln|C_m|,$$

(see [4][5][7][12]).

We will need the Nevanlinna characteristic of meromorphic f function which can be expressed in the following form

$$\widetilde{T}_m(r,f) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln^+|f(\rho e^{i\theta})|d\theta + \Sigma_{|b_\nu|<r}\ln\left(\frac{r}{|b_\nu|}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln^+|f(\rho e^{i\theta})d\theta + N(r),$$

$N(r) = N(r,f) = \int_0^r \frac{\widetilde{n}(t,f)-\widetilde{n}(0,f)}{t}dt$, where $\widetilde{n}(r,f) = \{card\ b_k :\ |b_k| < r\}$, $\{b_k\}$ is a set of poles of a meromorphic f function in the unit disk. $\widetilde{T}_m(r,f)$ is growing, (see [4][5][7][12]), on $(0,1)$. And we define spaces of meromorphic functions with bounded characteristic, so that $\widetilde{T}_m(1,f) < \infty$, where $\widetilde{T}_m(1,f) = \lim_{r\to 1-0} \widetilde{T}_m(r,f) < \infty$. They are coinciding with the class of all meromorphic functions such that

$$f(z) = Cz^\lambda \frac{B(z,\{a_\nu\})}{B(z,\{b_\nu\})} \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\psi(\theta)\right),\ z \in \mathbf{D},$$

where λ is an integer number, ψ is a measure of bounded variation and $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ are sequences in $\mathbf{D}$ so that $\sum_{k=1}^\infty (1-|a_k|) < \infty$ and $\sum_{k=1}^\infty (1-|b_k|) < \infty$. In the second part of this paper we will show that such type results are valid for some larger scales of meromorphic functions in the unit disk $\mathbf{D}$. We mention that for some classes of mentioned type of meromorphic functions such results are known. We give such result below.

Theorem C. (see [3][4][5][12]) Let f be meromorphic function in $\mathbf{D}$ and

$$\int_0^1 (1-r)^\alpha \widetilde{T}_m(r,f)dr < \infty.$$

Let also $f(z) = C_\lambda z^\lambda + \ldots,\ C_\lambda \neq 0$, be it is Loran expansion near $z = 0$, then

$$f(z) = K_\alpha \overline{C}_\lambda z^\lambda \frac{\Pi_\alpha(z,\{a_k\})}{\Pi_\alpha(z,\{b_k\})} \exp\left(\frac{2(\alpha+1)}{\pi}\int_0^1\int_{-\pi}^{\pi} (1-\rho^2)^\alpha \frac{\ln|f(\rho e^{i\theta})|\rho d\rho d\theta}{(1-z\rho e^{-i\theta})^{\alpha+2}}\right),$$

where $K_\alpha = \exp\left(\lambda(\alpha+1)\int_0^1 (1-\rho)^\alpha \ln\frac{1}{\rho}d\rho\right)$, C_λ is a complex number, λ is a non-negative integer, $z \in \mathbf{D}$, $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ are sequences of zeros and poles of $f(z)$, $\sum_{k=1}^\infty (1-|a_k|)^{\alpha+2} < \infty$ and $\sum_{k=1}^\infty (1-|b_k|)^{\alpha+2} < \infty$.

Remark 2.2. Interesting parametric representations for various classes of meromorphic functions can be found in [3].

We mention now another result on parametric representations of certain classes of meromorphic functions. In [4] [5][12] the following space of all meromorphic in the unit disk functions were introduced. Let $f \in M(\mathbf{D}),\ \alpha > -1,\ r \in (0,1)$, then we put $m_\alpha(r,f) = \frac{r^{-\alpha}}{2\pi}\int_{-\pi}^{\pi}\left(\int_0^r (r-t)^\alpha \ln|f(te^{i\varphi})|dt\right)^+ d\varphi$, and let

$$T_\alpha(r,f) = m_\alpha(r,f) + \frac{r^{-\alpha-1}}{\Gamma(\alpha+2)}\int_0^r \frac{(r-t)^{\alpha+1}}{t}(\widetilde{n}(t)-n(0))dt + \frac{n(0)}{\Gamma(\alpha+2)}\ln r,$$

where $\widetilde{n}(t)$ is a number of poles in $\mathbf{D}_t = \{z \in \mathbb{C}| :\ |z| < t\}$. Finally we define $MN_\alpha := \{f \in M(\mathbf{D}) : \sup_{0<r<1} T_\alpha(r,f)dr < \infty\}$.

Theorem D. (see [4][5][12]) The MN_α class coincides with the class of all meromorphic functions in $\mathbf{D}$ such that

$$f(z) = C_\lambda z^\lambda \frac{B_\alpha(z,\{a_k\})}{B_\alpha(z,\{b_k\})} \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{2}{(1-e^{-i\theta}z)^{\alpha+2}} - 1\right)d\psi(\theta)\right),\ z \in \mathbf{D},$$

where C_λ is a complex number, λ is a positive integer, $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ are arbitrary sequences in $\mathbf{D}$ so that $\sum_{k=1}^\infty(1-|a_k|)^{\alpha+2}<\infty$ and $\sum_{k=1}^\infty(1-|b_k|)^{\alpha+2}<\infty$, ψ is an arbitrary real function of bounded variation.

Less general results on zero sets can be seen in [1] and [15].

3. Main results

Here is the plan of this main section. First we describe zero sets of analytic classes $N^p_{\alpha,\beta}$ and $N^\infty_{\alpha,\beta}$ in the unit disc. Then we using these assertions provide complete parametric representations of corresponding analytic and meromorphic spaces. Note our results can be considered as complete analoques of results for other analytic and meromorphic classes in the unit disc that we provided in our previous section.

We add first some additional preliminaries concerning weights from S class and some their nice properties (see [11] and references there).

Let us denote by S the class of all slowly varying functions, i.e. the class of all positive measurable functions $\omega(t)$ on $(0,1]$ such that there are constants $m=m_\omega,\ M=M_\omega$ and $=q_\omega$ satisfying: $0<m,q<1$ and

$$m\le\frac{\omega(\lambda r)}{\omega(r)}\le M,\quad 0<r<1,\quad q\le\lambda\le 1,$$

see [9] for detailed study of such functions. The constants m,M,q are the structural constants of the slowly varying function ω. We note that functions $\omega(r)=r^\alpha,\alpha\in\mathbb{R}$ are in class S. In fact, for any $\omega\in S$ there is an $\beta\ge 0$ depending on the structural constants of ω such that $\omega(r)\ge Cr^\beta$, $0<r\le 1$. For more details on these weights see [10]. The following property (B) of these functions from these classes will be used by us below in proof of so-called "dyadic lemma" for these functions for proof of this property (B) we refer the reader to [10] where it is given as lemma. The property says that in $\omega\in S$ then there are two bounded measurable functions ϵ and ϵ_1 so that the following equality holds

$$\omega(x)=exp\tau,$$

where

$$\tau=\epsilon_1+\int_x^1\frac{\epsilon(u)}{u}du,$$

where $x\in(0,1)$ moreover this ϵ function satisfies the following condition - it is in closed interval between two numbers $-\alpha_\omega$ and β_ω, where

$$\alpha_\omega=\frac{\ln m_\omega}{\ln q_\omega}$$

and β_ω is the same as α_ω but with M_ω instead of m_ω and q_ω^{-1} instead of q_ω. All these facts can be found in [11].

The following lemma will be called "dyadic estimate" or "dyadic lemma" on $\omega(r)$ weights from S class below when it will be needed for general versions of our propositions (see for this lemma from [11]). Let $r_k=1-\frac{1}{2^k},\ k\in\mathbb{N}$ or $k=0$. Then the following estimate holds for all $\omega \in S$ so that $\beta_\omega < 1$

$$C_1\omega\left(\frac{1}{2^k}\right)2^{-k} \leq \int_{r_k}^{r_{k+1}} \omega(1-t)dt \leq C_2\omega\left(\frac{1}{2^k}\right)2^{-k}.$$

This assertion (see [11]) which follows directly from integration by parts for $\beta_\omega \in (0,1)$ and $\alpha_\omega > 0$ and $-\alpha_\omega \leq \epsilon(u) \leq \beta_\omega$ shows that in those "proofs with standard $(1-r)^\alpha$ weights" where the base of it is a dyadic decomposition of unit interval these $(1-r)^\alpha$ weights can be replaced by general ω weights. We will use this remark called (A) or "dyadic estimate" below at the end of paper.

As it was said property (B) is used for the proof of property (A) [11] and this property (A) in his turn is a base of the following estimate property (C) which was also proved in [11] and will lead us to direct extensions of our assertions below on zero sets. The property (C) is the following.

Let $\omega \in S$ and p is any positive finite real number, $\tau > -1$, then if one of the following expressions is finite then the other one is also finite ([11])

$$\int_0^1 \omega(1-r)n(r)^p(1-r)^\tau dr$$

$$\sum_0^\infty n_k^p \omega\left(\frac{1}{2^k}\right)2^{k(-\tau-1)}$$

for all ω from S with $\beta_\omega < 1$.

It is seems now natural to replace α or β in the names of spaces we introduced above by ω or ω_1 like this N^p_{ω,ω_1} for example if ω and ω_1 are in S class. This type of procedure for analytic area Nevanlinna type classes in unit disk is not new we refer the reader to, for example, [11] and references there. So here $(1-r)^\alpha$ weight for example is now $\omega(1-r)$.

Theorem 3.1. *Let $0 < p < \infty$, $\alpha > -1$ and let $\beta > -1$. Then*

$$\sum_{k=1}^\infty \frac{n_k^p}{2^{k(2p+1+\alpha p+\beta)}} < \infty \tag{3.1}$$

if and only if $\{z_k\} \in Z\left(N^p_{\alpha,\beta}\right)$. If (3.1) is true, then $\Pi_t(z,\{z_k\}) \in N^p_{\alpha,\beta}$ for $t > \max\left[\alpha+\beta/p+\max(1,1/p), \alpha+1\right]$. Moreover more general assertion is true, let $f \in N^p_{\alpha,\beta}$, where $\omega \in S, \beta_\omega < 1$, and p and α as above then for zero set of this function the following condition holds

$$\sum_0^\infty n_k^p \omega\left(\frac{1}{2^k}\right)2^{k(p+1)+k(\alpha+1)p} < \infty.$$

Proof. We start the proof with the following vital short remark which concerns the second part of theorem (the case of general weights). To show that general part we have to repeat all arguments we see below after this remark, but to replace $(1-r)^\beta$ there by $\omega(1-r)$ step by step. This will lead us to a integral expression at final step which with direct application of (C) property mentioned by us above will finally immediately give us the assertion we need to prove. This remark will allow us to concentrate now fully to the proof of that part of theorem where the general weights are not discussed.

Let $f \in N^p_{\alpha,\beta}(\mathbf{D})$. Then, without loss of generality it can be assumed that $f(0) = 1$, $f(z_k) = 0$ $(k = 1, 2, \ldots)$. Hence, by Jensen's inequality (see [3])

$$\begin{aligned} I &= \int_0^1 \left[\int_0^R (1-\tau)^\alpha d\tau \int_0^\tau \frac{n(u)}{u} du\right]^p (1-R)^\beta dR \\ &\le C_2 \int_0^1 \left[\int_{|z|<R} \log^+ |f(z)|(1-|z|)^\alpha dm_2(z)\right]^p (1-R)^\beta dR. \end{aligned}$$

Further, it is obvious that

$$\int_0^\tau \frac{n(u)}{u} du \ge \int_{\tau-\frac{R-\tau}{2}}^\tau \frac{n(u)}{u} du \ge C_2 n\left(\frac{3\tau - R}{2}\right)\frac{R-\tau}{2},$$

and

$$\|f\|^p_{N^p_{\alpha,\beta}} \ge C_2 \int_{C_1}^1 \left[\int_C^R (R-\tau)^{\alpha+1} n\left(\frac{3\tau-R}{2}\right) d\tau\right]^p (1-R)^\beta dR$$

for any numbers $R < 3\tau$, $C_1 > C$, $C < R$, $C_1 < R < 1$, $C, C_1 > 0$, $\alpha > 0$. Besides, one can see that the following implications are true

$$\frac{3\tau - R}{2} = \rho \quad \Rightarrow \quad \tau = \frac{2\rho + R}{3} \quad \Rightarrow \quad R - \tau = \frac{2(R-\rho)}{3}.$$

Hence,

$$\int_C^R (R-\tau)^{\alpha+1} n\left(\frac{3\tau-R}{2}\right) d\tau \ge C_2 \int_{(3C-R)/2}^R n(\rho)(R-\rho)^{\alpha+1} d\rho.$$

Suppose $C = (4R-1)/3$. Then $(3C-R)/2 = R - (1-R)/2$ and

$$\begin{aligned} \|f\|^p_{N^p_{\alpha,\beta}} &\ge C_2 \int_{C_1}^1 \left[\int_{R-\frac{1-R}{2}}^R n(\rho)(R-\rho)^{\alpha+1} d\rho\right]^p (1-R)^\beta dR \\ &\ge C_2 \int_{C_1}^1 \left[n\left(\frac{3R-1}{2}\right)\right]^p (1-R)^{(\alpha+1)p+\beta+p} dR \\ &\ge C_2 \int_{C_1^*}^1 [n(\rho)]^p (1-\rho)^{(\alpha+1)p+\beta+p} d\rho \asymp \sum_{k=1}^\infty \frac{n_k^p}{2^{k(p+1)} 2^{k(\alpha+1)p+k\beta}} \end{aligned}$$

since

$$\int_0^1 f(\rho)d\rho = \sum_{k=1}^\infty \int_{\tau_k}^{\tau_{k+1}} f(\tau)d\tau \tag{3.2}$$

for any $f \in L^1(0,1)$ and $\tau_k = 1 - \frac{1}{2^k}$ $(k = 0, 1, 2, \ldots)$ and

$$n(s_1) \le n(s_2) \text{ when } 0 \le s_1 \le s_2 < 1.$$

For $\alpha \in (-1, 0]$, a similar argument leads to the estimate

$$\|f\|^p_{N^p_{\alpha,\beta}} \geq C_2 \int_{C_1}^1 \left[\int_{R-\frac{1-R}{2}}^R n(\rho)(R-\rho) \left(\frac{3-2\rho-R}{3} \right)^\alpha d\rho \right]^p (1-R)^\beta dR$$
$$\geq C_2 \int_{C_1}^1 \left[n\left(\frac{3R-1}{2} \right) \right]^p (1-R)^{(\alpha+1)p+\beta+p} dR.$$

Then, we continue as in the above case $\alpha > 0$ and come to the desired statement.

For proving the converse statement, fix a number t so that Proposition A and Proposition B are applicable. We assume such t exists. Further, observe that $|\log|f||$ and $\log^+|f|$ both belong to $N^p_{\alpha,\beta}(\mathbf{D})$ if just one of them is of $N^p_{\alpha,\beta}(\mathbf{D})$. Hence, for $z = \rho e^{i\varphi}$, $\tau = t+2$ we get

$$\int_{-\pi}^{\pi} \big| \log |\Pi_t(z, \{z_k\})| \big| d\varphi \leq C \sum_{k=1}^{\infty} (1-|z_k|)^{t+2} \int_{-\pi}^{\pi} \frac{d\varphi}{|1 - z_k e^{i\varphi}|^\tau}.$$

Hence, for great enough values of t

$$\int_0^R T(\Pi_t, \rho)(1-\rho)^\alpha d\rho \leq C \int_0^R \sum_{k=1}^{\infty} \frac{(1-|z_k|)^{t+2}}{(1-z_k\rho)^{t+1}} (1-\rho)^\alpha d\rho$$
$$\leq C \int_0^R (1-\rho)^\alpha \int_0^1 \frac{(1-s)^{t+2}}{(1-s\rho)^{t+1}} dn(s) d\rho = J(R, f).$$

It is easy to show

$$\int_0^1 \frac{(1-s)^{t+2}}{(1-s\rho)^{t+1}} dn(s) \leq C \int_0^1 \frac{(1-s)^{t+1}}{(1-s\rho)^{t+1}} n(s) ds,$$

and hence

$$\int_0^1 \frac{(1-s)^{t+1}}{(1-s\rho)^{t+1}} n(s) ds \leq C \sum_{k=1}^{\infty} \frac{n_k}{2^{k(t+2)}(1-\tau_k\rho)^{t+1}}, \ \tau_k = 1 - \frac{1}{2^k}.$$

Consequently, for $p \leq 1$

$$J \leq C \sum_{k=1}^{\infty} \int_0^R \frac{(1-\rho)^\alpha d\rho}{(1-\tau_k\rho)^{t+1}} \frac{n_k}{2^{k(t+2)}} \leq C \sum_{k=1}^{\infty} \frac{n_k}{2^{k(t+2)}(1-\tau_k R)^{(t+1)-(\alpha+1)}},$$

and by the inequality $[\sum_{k=1}^{\infty} a_k]^p \leq \sum_{k=1}^{\infty} a_k^p$ $(p \leq 1)$ we get

$$\int_0^1 J^p(f, R)(1-R)^\beta dR \leq C \sum_{k=1}^{\infty} \frac{n_k^p}{2^{k(2p+1+\alpha p+\beta)}}.$$

The proof of $p \le 1$ case is complete. If $p > 1$, then the following estimates are true:

$$\int_0^1 J^p(f,R)(1-R)^\beta dR \le C \int_0^1 \left[\sum_{k=1}^\infty \int_0^R \frac{(1-\rho)^\alpha d\rho}{(1-\tau_k\rho)^{t+1}} \frac{n_k}{2^{k(t+2)}}\right]^p (1-R)^\beta dR$$
$$\le C \int_0^1 \left[\sum_{k=1}^\infty \frac{n_k}{2^{k(t+2)}} \frac{1}{(1-\tau_k R)^{t+1-(\alpha+1)}}\right]^p (1-R)^\beta dR.$$

Or, which is the same,

$$\begin{aligned} M &= \int_0^1 J^p(f,R)(1-R)^\beta dR \\ &\lesssim \int_0^1 (1-R)^\beta \left[\int_0^R (1-\rho)^\alpha \int_0^1 \frac{(1-s)^{t+2}}{(1-s\rho)^{t+1}} dn(s) d\rho\right]^p dR \\ &\lesssim \int_0^1 (1-R)^\beta \left[\int_0^R (1-\rho)^\alpha \int_0^1 \frac{(1-s)^{t+1}}{(1-s\rho)^{t+1}} n(s) d(s) d\rho\right]^p dR \\ &= \int_0^1 (1-R)^\beta \left[\int_0^R (1-\rho)^\alpha \sum_{k=1}^\infty \int_{1-2^{-k}}^{1-2^{-(k+1)}} \frac{(1-s)^{t+1}}{(1-s\rho)^{t+1}} n(s) d(s) d\rho\right]^p dR \\ &\lesssim \int_0^1 (1-R)^\beta \left[\int_0^R \sum_{k=1}^\infty n_k \frac{2^{-k(t+2)}(1-\rho)^\alpha d\rho}{(1-\rho\tau_k)^{t+1}}\right]^p dR \\ &\lesssim \int_0^1 (1-R)^\beta \left[\sum_{k=1}^\infty n_k 2^{-k(t+2)} \frac{1}{(1-R\tau_k)^{t-\alpha)}}\right]^p dR, \quad t > \alpha. \end{aligned}$$

Hence,

$$\begin{aligned} M &\lesssim \int_0^1 (1-R)^\beta \left[\int_0^1 \frac{n(\rho)(1-\rho)^{t+1}}{(1-\rho R)^{t-\alpha}} d\rho\right]^p dR \\ &= \int_0^1 (1-R)^{\beta/p} \left(\int_0^R + \int_R^1\right) \psi(R) dR = I_1 + I_2 \end{aligned}$$

for any function $\psi \ge 0$ such that $\|\psi\|_{L^q} = 1$ for $1/p + 1/q = 1$. Using the Hardy and Hölder inequalities, one can be convinced that

$$\begin{aligned} I_1 &= \int_0^1 n(\rho)(1-\rho)^{t+1} \int_0^\rho \frac{(1-R)^{\beta/p}\psi(R)}{(1-\rho R)^{t-\alpha}} dR d\rho \\ &\lesssim \int_0^1 n(\rho)(1-\rho)^{t+1+\frac{\beta}{p}+\alpha-t+1} \int_0^\rho \frac{\psi(R)}{(1-R)} dR d\rho, \end{aligned}$$

i.e.

$$I_1 \le \int_0^1 \frac{\psi(\tau)}{1-\tau} \int_\tau^1 n(\rho)(1-\rho)^{\frac{\beta}{p}+\alpha+2} d\rho d\tau$$
$$\le \left(\int_0^1 \psi^q(\tau)d\tau\right)^{\frac{1}{q}} \cdot \left(\int_0^1 \left(\int_0^{1-\tau} n(1-t)t^{\frac{\beta}{p}+\alpha+2}dt\right)^p d\tau\right)^{\frac{1}{p}},$$

and hence

$$I_1 \lesssim \int_0^1 n(\rho)^p (1-\rho)^{p+\beta+\alpha p+p} d\rho.$$

For $\beta < 0$ above we used $(1-R)^{\frac{\beta}{p}} < (1-\rho)^{\frac{\beta}{p}}$, $R \le \rho < 1$ for $\beta \ge 0$, $(1-R)^{\frac{\beta}{p}} < (1-\rho R)^{\frac{\beta}{p}}$, $\rho, R \in (0,1)$, for $t > \max\big\{(\alpha+\beta/p) + \max\{1, 1/p\},\ (\alpha+1)\big\}$. Besides for $\beta \ge 0$ again by Hölder and Hardy inequalities we will have

$$I_2 = \int_0^1 n(\rho)(1-\rho)^{t+1} \int_\rho^1 \frac{(1-R)^{\beta/p}\psi(R)dR}{(1-\rho R)^{t-\alpha}}$$
$$\lesssim \int_0^1 n(\rho)(1-\rho)^{1+\frac{\beta}{p}+\alpha} \int_0^{1-\rho} \psi(1-u)dud\rho$$
$$\lesssim \left[\int_0^1 n(\rho)^p(1-\rho)^{p+\beta+\alpha p+p}d\rho\right]^{1/p} \left[\int_0^1 \left(\frac{1}{1-\rho}\int_0^{1-\rho}\psi(1-u)du\right)^q\right]^{1/q}$$
$$\lesssim B \cdot C\|\psi\|_{L^q}, \quad q > 1,$$

where

$$B = \left[\int_0^1 n(\rho)^p(1-\rho)^{2p+\beta+\alpha p}d\rho\right]^{1/p} \asymp \left[\sum_{k=1}^\infty \frac{n_k^p}{2^{k(p+1)}2^{k(\alpha+1)p+k\beta}}\right]^{1/p}$$

for $t > \max\big\{(\alpha+\beta/p) + \max\{1, 1/p\},\ (\alpha+1)\big\}$.

The estimate of I_2 in case of $\beta < 0$ needs small modification of mentioned arguments and we omit details.

Now we shall show that for great enough numbers t Proposition A and Proposition B are applicable. To this end, we prove that if $t > \max\big\{(\alpha+\beta/p) + \max\{1, 1/p\},$ $(\alpha+1)\big\}$, then $\sum_{k=1}^\infty (1-|z_k|)^{t+2} < \infty$. Hence, the condition

$$\sum_{k=1}^\infty \frac{n_k^p}{2^{k(\beta+\alpha p+2p+1)}} < \infty$$

will imply the convergence of the product $\Pi_t(z, \{z_k\})$.

Indeed, the obvious inequality

$$\int_0^1 n^p(\tau)(1-\tau)^{\beta+\alpha p+2p}d\tau < +\infty$$

implies that

$$\int_{\tau_1}^1 n^p(\tau)(1-\tau)^{\beta+\alpha p+2p}d\tau \to 0 \quad \text{as} \quad \tau_1 \to 1.$$

Hence, for $\beta+\alpha p+2p>-1$

$$n^p(\tau)(1-\tau)^{\beta+\alpha p+2p+1}\to 0 \quad \text{as} \quad \tau\to 1,$$

and therefore $n(\tau)\le C(1-\tau)^{-(\beta+\alpha p+2p+1)/p}$ $(0<\tau<1)$. Consequently,

$$\begin{aligned}\sum_{k=1}^{\infty}(1-|z_k|)^{t+2} &\le C\sum_{k=1}^{\infty}\sum_{z_k\in B_k}(1-|z_k|)^{t+2}n_k\\ &\le C\sum_{k=1}^{\infty}\sum_{z_k\in B_k}(1-|z_k|)^{t+2-(\beta+\alpha p+2p+1)/p}\\ &\le C\sum_{k=1}^{\infty}\frac{1}{2^{k[t-(\beta+1)/p-\alpha]}}<+\infty.\end{aligned}$$

Theorem 3.1 is proved. □

Our next theorem can be formulated also partially for these regularly varying weights, namely denote by $N^{\infty,p}_{\alpha,\beta}$, $\omega\in S$, $\alpha\ge 0$ the space of all functions $f\in H(\mathbf{D})$ satisfying the condition

$$\sup_{0<R<1}\omega(1-R)\int_0^R T(r,f)^p(1-r)^\alpha dr<\infty.$$

Theorem 3.2. *Let* $0<p<\infty$, $\alpha\ge 0$ *and* $\omega\in S$, $\beta>0$. *Then*

$$n(\tau)\le c(1-\tau)^{-\frac{\alpha+p+1}{p}}\omega^{\frac{-1}{p}}(1-\tau),\quad \tau\in(0,1) \tag{3.3}$$

only if $\{z_k\}\in Z\left(N^{\infty,p}_{\alpha,\omega}\right)$. *If* $\omega(r)=r^\beta$ *and* (3.3) *is true then* $\Pi_t(z,\{z_k\})\in N^{\infty,p}_{\alpha,\beta}$ *for sufficiently large* t. *Taking* $\omega(r)=r^\beta$ *we obtain the following sharp result for zero sets of this class. Namely*

$$n(\tau)\le c(1-\tau)^{-\frac{\alpha+\beta+p+1}{p}},\quad \tau\in(0,1)$$

if and only if $\{z_k\}\in Z\left(N^{\infty,p}_{\alpha,\beta}\right)$.

Proof. Without loss of generality we assume $f(0)=1$, $f(z_k)=0$ $(k=1,2,\ldots)$. Then by Jensen's inequality used more accurately than in [3]

$$\begin{aligned}J&=\sup_{C_1<R<1}\left(\int_{R/3}^R\left[\int_{C^*}^\tau\frac{n(u)}{u}du\right]^p(1-\tau)^\alpha d\tau\right)\omega(1-R)\\ &\le C\sup_{C_1<R<1}\int_0^R\left[\int_{\mathrm{T}}\log^+|f(\tau\xi)|d\xi\right]^p(1-\tau)^\alpha d\tau\,\omega(1-R)\\ &\le C,\end{aligned}$$

where $C_1 > 0$ and $C^* = \tau - \frac{R-\tau}{2}$. Estimating the left-hand side of the above inequality from below, we get

$$\begin{aligned} J &\geq \sup_{C_1<R<1} \left(\int_{R/3}^{R} \left[n\left(\frac{3\tau - R}{2}\right)\right]^p \left(\frac{R-\tau}{2}\right)^{p+\alpha} d\tau \right) \omega(1-R) \\ &\geq \sup_{C_1<R<1} \left(\int_{R-\frac{1-R}{2}}^{R} [n(\rho)]^p (R-\rho)^{\alpha+p} d\rho \right) \omega(1-R) \\ &\geq \sup_{C_1<R<1} \left(\left[n\left(\frac{3R-1}{2}\right)\right]^p (1-R)^{1+p+\alpha} \omega(1-R) \right) \\ &\geq C[n(\rho)]^p (1-\rho)^{1+p+\alpha} \omega(1-\rho). \end{aligned}$$

Hence, combining the estimate from below with the estimate from above we obtain $n(\rho) \leq C(1-\rho)^{-(\alpha+1+p)/p} \omega^{\frac{-1}{p}}(1-\rho)$ for any $\rho \in (0,1)$.

To prove the converse statement for partial case $\omega(r) = r^\beta$, we use the latter inequality for $n(\rho)$ and Propositions above that give estimates for $\omega(r) = r^\beta$ and for $\Pi_t(z, \{z_k\})$. Let $t > \frac{\alpha+\beta+1}{p} - 1$. We have, as in proof of Theorem 3.1

$$\begin{aligned} \left\| \Pi_t(\cdot, \{z_k\}) \right\|^p_{N^{\infty,p}_{\alpha,\beta}} &\leq C \sup_{C_1<R<1} \omega(1-R) \int_0^R (1-\rho)^\alpha \left[\int_0^1 \frac{(1-\nu^2)^{t+1}}{(1-\rho\nu)^{t+1}} n(\nu) d\nu \right]^p d\rho \\ &\leq C \sup_{C_1<R<1} \omega(1-R) \int_0^R (1-\rho)^\alpha \left[\int_0^1 \frac{(1-s)^{t+1-\frac{\alpha+p+1}{p}} \omega^{\frac{-1}{p}}(1-r)}{(1-\rho s)^{t+1}} ds \right]^p d\rho \\ &\leq C \sup_{C_1<R<1} \omega(1-R) \int_0^R \frac{\omega^{-1}(1-\rho)}{1-\rho} d\rho \leq C. \end{aligned}$$

Now, setting $\widetilde{\beta} = t - \frac{\alpha+\beta+1}{p} > -1$ and using partial integration we have

$$\sum_{|z_k|<R} (1-|z_k|)^{t+2} = \int_0^R (1-s)^{t+2} dn(s) \leq C \int_0^R (1-s)^{\widetilde{\beta}} ds < +\infty.$$

Therefore applications of Proposition A and B is justified. The proof is complete. □

Proofs of Theorem 3.1 and Theorem 3.2 are based on classical arguments, (see [3]), but with more accurate and delicate attention to estimates in them.

¿From these theorems we using standard argument that already were done in [3][11][13]. We get immediately the following parametric representations complete analogues of theorems we provided in previous sections.

Theorem 3.3. *If $0 < p < \infty$, $\alpha > -1$ and $\beta > -1$, then the class $N^p_{\alpha,\beta}$ coincides with the set of functions representable for $z \in \mathbf{D}$ as*

$$f(z) = c_\lambda z^\lambda \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) \exp\left\{ \frac{t+1}{\pi} \int_0^1 \int_{-\pi}^{\pi} \frac{(1-\rho^2) \ln \left| 1 - \frac{\rho e^{i\varphi}}{z_k} \right|}{(1-\rho e^{-i\varphi} z)^{t+2}} \rho d\rho d\varphi \right\} \exp\{h(z)\},$$

where $t > \max\left\{\alpha + \frac{\beta}{p} + \max(1, 1/p), \alpha + 1\right\}$, $c_\lambda \in \mathbb{C}$, $\lambda \geq 0$,

$$\sum_{k=1}^{\infty} \frac{n_k^p}{2^{k(2p+1+\alpha p+\beta)}} < \infty$$

and $h \in H(\mathbf{D})$ *is a function satisfying the condition*

$$\int_0^1 \left[\int_0^R \left(\int_{-\pi}^{\pi} |h(\tau e^{i\varphi}) d\varphi\right) (1-\tau)^\alpha d\tau\right]^p (1-R)^\beta dR < \infty.$$

If in particular $\widetilde{q} = \alpha$, $\widetilde{q} \in \mathbb{Z}_+$ *then*

$$f(z) = \overline{C}_\lambda z^\lambda \prod_{k=1}^{\infty} \left(1 - \frac{1-|z_k|^2}{1-\overline{z}_k z}\right) \exp\left(\sum_{j=1}^{\widetilde{q}+1} \frac{1}{j}\left(\frac{1-|z_k|^2}{1-\overline{z}_k z}\right)^j\right)$$
$$\times \exp\left(\frac{2(\widetilde{q}+1)}{\pi} \int_0^1 \int_{-\pi}^{\pi} (1-\rho^2)^{\widetilde{q}} \frac{\ln|f(\rho e^{i\theta})| \rho d\rho d\theta}{(1-z\rho e^{-i\theta})^{\widetilde{q}+2}}\right),$$

where C_λ *is a complex number,* $C_\lambda \neq 0$, λ *is a nonnegative integer and* $z \in \mathbf{D}$.

Proof. Let us first show the first part of theorem. Let $f \in N^p_{\alpha,\beta}(\mathbf{D})$. Note, if $f,\ g \in N^p_{\alpha,\beta}(\mathbf{D})$, $Z(f) \supset Z(g)$ then $\frac{f}{g} \in N^p_{\alpha,\beta}(\mathbf{D})$. Note also for mentioned t, $\Pi_t(z, \{z_k\}) \in N^p_{\alpha,\beta}(\mathbf{D})$. Hence $\psi(z) = \frac{f(z)}{C_\lambda z^\lambda \Pi_t(z,\{z_k\})} \in N^p_{\alpha,\beta}(\mathbf{D})$. It remains to use the following two equalities

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln|\psi(re^{i\varphi})| d\varphi = \ln|\psi(0)|$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\ln|\psi(re^{i\varphi})|\right| d\varphi = \frac{1}{\pi} \int_{-\pi}^{\pi} \ln^+|\psi(re^{i\varphi})| d\varphi - \ln|\psi(0)|,$$

hence

$$\int_0^1 (1-r)^\beta \left(\int_0^r \int_{-\pi}^{\pi} (1-R)^\alpha \left|\ln|\psi(Re^{i\varphi})|\right| dR d\varphi\right)^p dr < \infty.$$

It remains to put $h(z) = \ln(\psi(z))$, $z \in \mathbf{D}$, where we choose the main branch of logarithm. The reverse follows from Theorem 3.1 and the fact that

$$\ln^+|\Pi_\beta(z, \{b_k\}) \cdot \exp h(z)| \leq \ln^+|\Pi_\beta(z, \{b_k\})| + \ln^+|\exp h(z)|.$$

The proof of second part of Theorem 3.3 follows directly from Corollary 2.1. Theorem 3.3 is proved. □

Theorem 3.4. *If* $0 < p < \infty$, $\alpha \geq 0$ *and* $\beta > 0$, *then the class* $N^{\infty,p}_{\alpha,\beta}$ *coincides with the set of functions representable for* $z \in \mathbf{D}$ *as*

$$f(z) = c_\lambda z^\lambda \prod_{k=1}^{\infty}\left(1 - \frac{z}{z_k}\right) \exp\left\{\frac{t+1}{\pi} \int_0^1 \int_{-\pi}^{\pi} \frac{(1-\rho^2)\ln\left|1 - \frac{\rho e^{i\varphi}}{z_k}\right|}{(1-\rho e^{-i\varphi} z)^{t+2}} \rho d\rho d\varphi\right\} \exp\{h(z)\},$$

where

$$n(\tau) \leq c(1-\tau)^{-\frac{\alpha+\beta+p+1}{p}}, \quad \tau \in (0,1),$$

c_λ is a complex number, $\lambda \geq 0$ and $h \in H(\mathbf{D})$ is a function satisfying the condition

$$\sup_{0<R<1} \int_0^R \left(\int_{\mathbf{T}} |h(\tau\xi) d\xi \right)^p (1-\tau)^\alpha d\tau (1-R)^\beta dR < \infty.$$

If in particular $\widetilde{q} = \alpha,\ \widetilde{q} \in \mathbb{Z}_+$ then

$$f(z) = \overline{C}_\lambda z^\lambda \prod_{k=1}^\infty \left(1 - \frac{1-|z_k|^2}{1-\overline{z}_k z}\right) \exp\left(\sum_{j=1}^{\widetilde{q}+1} \frac{1}{j} \left(\frac{1-|z_k|^2}{1-\overline{z}_k z} \right)^j \right)$$
$$\times \exp\left(\frac{2(\widetilde{q}+1)}{\pi} \int_0^1 \int_{-\pi}^{\pi} (1-\rho^2)^{\widetilde{q}} \frac{\ln |f(\rho e^{i\theta})| \rho d\rho d\theta}{(1-z\rho e^{-i\theta})^{\widetilde{q}+2}} \right),$$

where C_λ is a complex number, $C_\lambda \neq 0$, λ is a nonnegative integer and $z \in \mathbf{D}$.

The proof of Theorem 3.4 is based on same arguments and we do not present that proof here.

Remark 3.1. It is not difficult to extend the statements and the proof of Theorem 3.1 to the weights we used in Theorem 3.2.

Remark 3.2. It is clear that obtain a parametric representations of classes we study in this paper via $B_t(z, \{z_k\})$ all we have to do is to show, for example, that if $f \in X,\ X = N^p_{\alpha,\beta}(\mathbf{D})$ or $X = N^{\infty,p}_{\alpha,\beta}(\mathbf{D})$ then $f \in S^1_\tau(\mathbf{D})$ for some big enough $\tau > 0$ then apply theorems we just formulated above. To do that partially we formulate the following propositions.

To obtain parametric representations of $N^p_{\alpha,\beta}(\mathbf{D})$ and $N^{\infty,p}_{\alpha,\beta}(\mathbf{D})$ classes via $B_\alpha(z, \{z_k\})$ infinite Blaschke type products we can use some embeddings and known parametric representations for analytic classes or area Nevanlinna type with quazinorms $\int_0^1 \left(\int_{\mathbf{T}} \log^+ |f(z)| d\xi \right)^p (1-|z|)^\alpha dm_2(z) < \infty$, for certain $0 < p < \infty,\ \alpha > -1$, that were obtained.

First we formulate a result that will be used by us.

Theorem E. (see [13]) Let $0 < p < \infty,\ \alpha > 0$. Then $MS^p_\alpha(\mathbf{D})$ coinciding with the class of f functions such that

$$f(z) = e^{i\alpha + mK_\beta} z^m \frac{B_\beta(z, \{a_k\})}{B_\beta(z, \{b_k\})} \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{2}{(1-e^{-i\varphi}z)^{\beta+1}} - 1 \right) \psi(e^{i\varphi}) d\varphi \right), z \in \mathbf{D},$$

$\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$, $(0 < |a_k| \leq |a_{k+1}|,\ 0 < |b_k| \leq |b_{k+1}|,\ k = 1, 2, \ldots)$, are arbitrary sequences of points from $\mathbf{D}$, so that

$$\int_0^1 n^p(r, f)(1-r)^{\alpha+p} dr < \infty \quad \text{and} \quad \int_0^1 n^p(r, \frac{1}{f})(1-r)^{\alpha+p} dr < \infty,$$

where $\beta \in \left(\frac{\alpha+1}{p},\ \frac{\alpha+1}{p} + 2 \right)$, $\psi \in B^s_{1,p}(\mathbf{T}),\ s = \beta - \frac{\alpha+1}{p},\ K_\beta = \beta \sum_{k=1}^\infty \frac{1}{k(k+\beta)}$ and $\psi(e^{i\theta}) = \lim_{r\to 1-0} \frac{1}{\Gamma(\beta)} \int_0^r (r-t)^{\beta-1} \ln |f(te^{i\varphi})| dt$.

The following proposition for $\omega(r) = r^\alpha$ will be proved and used below, we decided however to formulate it in general form for all functions from S_1 class it is S class with $\beta_\omega < 1$ and the proof of this follows directly from "dyadic lemma" for ω weights or remark (A) which was mentioned and formulated above by us.

Proposition 3.1. *Let $\omega \in S_1,\ f \in H(\mathbf{D})$.*

1) *Let $\alpha > -1,\ p \le 1,\ \gamma > 0,\ \alpha > \gamma - 1$. Then*

$$\int_{\mathbf{D}} \log^+ |f(z)|(1-|z|)^\alpha \omega(1-|z|)dm_2(z)$$
$$\le C \left(\int_0^1 \omega(1-|z|)(1-|z|)^{(\alpha+1)p-\gamma p-1} \left(\sup_{0<\tau\le|z|} T(\tau,f)(1-\tau)^\gamma \right)^p d|z| \right)^{\frac{1}{p}};$$

2) *Let $\beta > -1,\ \gamma \ge 0,\ 0 < q < \infty$. Then*

$$\left(\int_0^1 \omega(1-|z|)(1-\tau)^{\beta+(\gamma+1)q} \left(\int_{\mathbf{T}} \log^+ |f(\tau\xi)|dm(\xi) \right)^q d\tau \right)^{\frac{1}{q}}$$
$$\le C \left(\int_0^1 \omega(1-|z|)(1-\tau)^\beta \left(\int_{|z|<\tau} \log^+ |f(z)|(1-|z|)^\gamma dm_2(z) \right)^q d\tau \right)^{\frac{1}{q}}.$$

Proof. Let $\tau_n = 1 - \frac{1}{2^n},\ n \in \mathbb{N},\ p \le 1,\ \widetilde{f}(z) = \log^+ |f(z)|$. Then we have

$$\left(\int_{\mathbf{D}} \widetilde{f}(z)(1-|z|)^\alpha dm_2(z) \right)^p \lesssim \sum_{k=1}^\infty 2^{-kp(\alpha+2)} \left(M_1(\tau_k, \widetilde{f}) \right)^p$$
$$\lesssim \sum_{k=1}^\infty 2^{-kp(\alpha+1)} \sup_{0<\rho\le\tau_k} \left(M_1(\rho,\widetilde{f})(1-\rho)^\gamma \right)^p 2^{k\gamma p}$$
$$\lesssim \sum_{k=1}^\infty \int_{1-2^{-k-2}}^{1-2^{-k-3}} (1-|z|)^{(\alpha+1)p-\gamma p-1} \sup_{0<\rho\le|z|} \left(M_1(\rho,\widetilde{f})(1-\rho)^\gamma \right)^p d|z|$$
$$\le C \int_0^1 (1-|z|)^{(\alpha+1)p-\gamma p-1} \left(\sup_{0<\rho\le|z|} T(\tau,f)(1-\tau)^\gamma \right)^p d|z|.$$

Let us show the second estimate

$$\begin{aligned}
&\int_0^1 (1-\tau)^{\beta+(\gamma+1)q} \left(\int_{\mathbf{T}} \widetilde{f}(r\xi)dm(\xi)\right)^q d\tau \\
&\lesssim \sum_{k=1}^{\infty} 2^{-k(\beta+(\gamma+1)q+1)} \left(M_1(\tau_k, \widetilde{f})\right)^q \\
&\lesssim \sum_{k=1}^{\infty} \left(\int_{\tau_k<|z|<\tau_{k+1}} \widetilde{f}(z)(1-|z|)^\gamma dm_2(z)\right)^q 2^{-k(\beta+1)} \\
&\lesssim \sum_{k=1}^{\infty} \int_{\tau_{k+1}}^{\tau_{k+2}} (1-\tau)^\beta \left(\int_{|z|<\tau} \widetilde{f}(z)(1-|z|)^\gamma dm_2(z)\right)^q d\tau \\
&\lesssim \int_0^1 (1-\tau)^\beta \left(\int_{|z|<\tau} \widetilde{f}(z)(1-|z|)^\gamma dm_2(z)\right)^q d\tau.
\end{aligned}$$

□

Similar estimate can be proved for $N^{\infty,p}_{\alpha,\beta}(\mathbf{D})$. We state it as

Proposition 3.2. *Let* $f \in H(\mathbf{D}),\ 0<p<\infty,\ \alpha>1,\ -1<\beta<0.$ *Then*

$$\begin{aligned}
&\int_0^1 \left(\int_{\mathbf{T}} \log^+|f(z)|d\xi\right)^p (1-|z|)^\alpha d|z| \\
&\qquad \le C \sup_{R<1} \left(\int_0^R T^p(\tau,f)(1-\tau)^\beta d\tau\right)(1-R)^{\alpha-1}.
\end{aligned}$$

The proof of Proposition 3.2 follows directly from the fact that if $f \ge 0$ and $f(r_1) \le f(r_2)$ for $r_1 \ge r_2$ on $(0,\infty),\ q>1$, then

$$\frac{(q-1)^{q-1}}{q^q} \sup_x (x^q f(x)) \le \sup_x x^{q-1} \int_x^\infty f(t)dt,$$

which can be found in [9]. We omit the prove of last assertion.

Remark 3.3. Similar assertions are true for parallel meromorphic spaces.

Remark 3.4. The analogs of Theorems 3.1 and 3.2 on zero sets and parametric representations are true for the area Nevanlinna type classes in the upper half-plane $\mathbb{C}_+$, which are the analogs of the analytic classes we considered above (see [16]).

A classical result of meromorphic function theory says that every meromorphic function of bounded characteristic f, can be expressed as

$$f = \frac{f_1}{f_2},\ f_1, f_2 \in H^\infty(\mathbf{D}),$$

where H^∞ is a set of all bounded analytic functions, (see [7] and [12]). In short, a meromorphic function of certain class can be obtained as a factor of two functions from certain analytic class, a subspace of $H(\mathbf{D})$. We will obtain complete analogue of this result for meromorphic spaces we study.

First we define meromorphic classes

$$M^p_{\alpha,\beta}(\mathbf{D}) = \left\{ f \in M(\mathbf{D}) \ : \ \int_0^1 \left(\int_0^R \widetilde{T}_m(r,f)(1-r)^\alpha dr \right)^p (1-R)^\beta dR < \infty \right\},$$

$$M^{\infty;p}_{\alpha,\beta_1}(\mathbf{D}) = \left\{ f \in M(\mathbf{D}) \ : \ \sup_{0<R<1} \int_0^R \left(\widetilde{T}_m(r,f)(1-r)^\alpha dr \right)^p (1-R)^{\beta_1} < \infty \right\},$$

where $0 < p < \infty$, $\alpha > -1$, $\beta > -1$ and $\beta_1 > 0$.

Our next theorem shows us parametric representations for meromorphic spaces in the unit disk we defined above can be readily derived from parametric representation of corresponding analytic classes in the unit disk we obtained already in Theorems 3.3 and 3.4. We note again we do not consider in this paper parametric representations of analytic or meromorphic classes via Besov spaces on the unit circle we formulated in Section 2.

Theorem 3.5. *Let* $0 < p < \infty$, $\alpha > -1$ *and* $\beta > -1$.

1) *The* $M^p_{\alpha,\beta}(\mathbf{D})$ *class coincides with the space of all meromorphic function, so that* $f(z) = \frac{g(z)}{\Pi_t(z,\{b_k\})}$, $z \in \mathbf{D}$, $g \in N^p_{\alpha,\beta}(\mathbf{D})$ *and* $\{b_k\}_{k=1}^\infty$ *is a sequence from unit disk,*

$$\sum_{k=1}^\infty \frac{n_k^p}{2^{k(2p+1+\alpha p+\beta)}} < \infty \tag{3.4}$$

and $t > \max\left[\alpha + \beta/p + \max(1, 1/p), \alpha + 1\right]$.

2) *Two sequences of complex numbers from* $\mathbf{D}$ $\{a_k\}_{k=1}^\infty$ *and* $\{b_k\}_{k=1}^\infty$, $|a_k| \le |a_{k+1}|$ *and* $|b_k| \le |b_{k+1}|$, $k = 1, 2, \ldots$, *are zeros and poles of a function from* $M^p_{\alpha,\beta}(\mathbf{D})$ *if and only if for both of these sequences* (3.4) *holds.*

Remark 3.5. Combining results of Theorem 3.3 and Theorem 3.5 we immediately can get a parametric representations of $M^p_{\alpha,\beta}$ meromorphic spaces we consider in this paper as analogies of Theorems C, D formulated above.

Proof of Theorem 3.5. We start with the first part of the theorem. Let $f \in M^p_{\alpha,\beta}(\mathbf{D})$ and $\{b_k\}_{k=1}^\infty$ be the sequence of poles of f. Then

$$\int_0^1 \left(\int_0^R N(r)(1-r)^\alpha dr \right)^p (1-R)^\beta dR < \infty, \ \Pi_t(z,\{b_k\}) \in N^p_{\alpha,\beta}(\mathbf{D}).$$

Since

$$\int_0^1 \left(\int_0^R \widetilde{T}_m(r,f)(1-r)^\alpha dr \right)^p (1-R)^\beta dR < \infty.$$

Hence $g = f \cdot \Pi_t(z,\{b_k\}) \in N^p_{\alpha,\beta}(\mathbf{D})$. Hence we have what we need.

Let us show the reverse implication.

Let $g \in N^p_{\alpha,\beta}(\mathbf{D})$ and condition (3.4) holds for $\{b_k\}_{k=1}^{\infty}$ sequence from $\mathbf{D}$. Further, since for $f(z) = \frac{g(z)}{\Pi_t(z,\{b_k\})}$ we have

$$\ln^+ |f(z)| \le \ln^+ |g(z)| + \ln^+ \left| \frac{1}{\Pi_t(z,\{b_k\})} \right|,$$

all we have to show that $\frac{1}{\Pi_t(z,\{b_k\})}$ is also from $M^p_{\alpha,\beta}(\mathbf{D})$, (poles of f and $\frac{1}{\Pi_t(z,\{b_k\})}$ are the same). But it is true since $T(r,\Pi_t(z,\{b_k\})) = \widetilde{T}_m(r, \frac{1}{\Pi_t(z,\{b_k\})}) + \ln |\Pi_t(0,\{b_k\})|$, which follows directly from Jensen's equality mentioned above (see [7], [12]) and which says

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln^+ \left|\widetilde{f}(re^{i\varphi})\right| d\varphi + \int_0^r \frac{\widetilde{n}(t,\widetilde{f})}{t}dt$$
$$= \ln\left|\widetilde{f}(0)\right| + \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln^+ \frac{1}{\left|\widetilde{f}(re^{i\varphi})\right|}d\varphi + \int_0^r \frac{\widetilde{n}(t,\frac{1}{\widetilde{f}})}{t}dt,$$

$\widetilde{f}$ is meromorphic. The proof of the second part follows directly from previous assertions concerning about $M^p_{\alpha,\beta}(\mathbf{D})$ and will be omitted. Theorem 3.5 is proved. □

A very similar assertion of Theorem 3.5 with similar proof is true for $M^{\infty,p}_{\alpha,\beta_1}(\mathbf{D})$ spaces.

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[1]Bryansk University
Bryansk,
Russia
E-mail address: rshamoyan@yahoo.com

[2]Fakultet organizacionih nauka,
Beogradski univerzitet,
Serbia
E-mail address: oliveradj@fon.rs