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F-GEOMETRIC MEAN LABELING OF SOME CHAIN GRAPHS AND THORN GRAPHS

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Abstract. A function f is called a F-Geometric mean labeling of a graph $G(V,E)$ if $f : V(G) \to \{1,2,3,\ldots,q+1\}$ is injective and the induced function $f^* : E(G) \to \{1,2,3,\ldots,q\}$ defined as $f^*(uv) = \left\lfloor \sqrt{f(u)f(v)} \right\rfloor$, for all $uv \in E(G)$, is bijective. A graph that admits a F-Geometric mean labelling is called a F-Geometric mean graph. In this paper, we have discussed the F-Geometric mean labeling of some chain graphs and thorn graphs.

1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V,E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [3]. For a detailed survey on graph labeling, we refer [2].

Path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. $G \odot K_1$ is the graph obtained from G by attaching a new pendant vertex to each vertex of G. A star graph S_m is the complete bipartite graph $K_{1,m}$. $G \odot S_2$ is the graph obtained from G by attaching two pendant vertices at each vertex of G. If $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \ldots, v_{m+1}^{(i)}$ and $u_1, u_2, u_3 \ldots, u_n$ be the vertex of the star graph S_m and the path P_n, then the graph $(P_n; S_m)$ is obtained from n copies of S_m and the path P_n by joining u_i with the central vertex $v_1^{(i)}$ of the i^{th} copy of S_m by means of an edge, for $1 \le i \le n$. The $H-$ graph is obtained from two paths $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ of equal length by joining an edge $u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}$ when n is odd and $u_{\frac{n+2}{2}} v_{\frac{n}{2}}$ when n is even. Let G_1 and G_2 be any two graphs with p_1 and p_2 vertices, respectively. Then the cartesian product $G_1 \times G_2$ has $p_1 p_2$ vertices which are $\{(u,v)/u \in G_1, v \in G_2\}$. The edges are defined as follows: (u_1,v_1) and (u_2,v_2) are adjacent in $G_1 \times G_2$ if either

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$u_1 = u_2$ and v_1 and v_2 are adjacent in G_2 or u_1 and u_2 are adjacent in G_1 and $v_1 = v_2$. The Ladder graph L_n is obtained from $P_n \times P_2$.

The study of graceful graphs and graceful labeling methods first introduced by Rosa [5]. The concept of mean labeling was first introduced by S. Somasundaram and R. Ponraj [6] and it was developed in [4] and [7]. In [10], R. Vasuki et al. discussed the mean labeling of cyclic snake and armed crown. In [8], S. Somasundaram et al. defined the geometric mean labeling as follows.

A graph $G = (V, E)$ with p vertices and q edges is said to be a geometric mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, \ldots, q+1$ in such way that when each edge $e = uv$ is labeled with $f(uv) = \left\lfloor \sqrt{f(u)f(v)} \right\rfloor$ or $\left\lceil \sqrt{f(u)f(v)} \right\rceil$ then the edge labels are distinct.

In the above definition, the readers will get some confusion in finding the edge labels which edge is assigned by flooring function and which edge is assigned by ceiling function.

In [9], they have given the geometric mean labeling of the graph $C_5 \cup C_7$ as in the Figure 1.

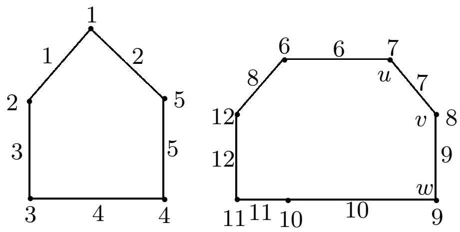

Figure 1. A Geometric mean labeling of $C_5 \cup C_7$.

From the above figure, for the edge uv, they have used flooring function $\left\lfloor \sqrt{f(u)f(v)} \right\rfloor$ and for the edge vw, they have used ceiling function $\left\lceil \sqrt{f(u)f(v)} \right\rceil$ for fulfilling their requirement. To avoid the confusion of assigning the edge labels in their definition, we just consider the flooring function $\left\lfloor \sqrt{f(u)f(v)} \right\rfloor$ for our discussion. Based on our definition, the F-Geometric mean labeling of the same graph $C_5 \cup C_7$ is given in Figure 2.

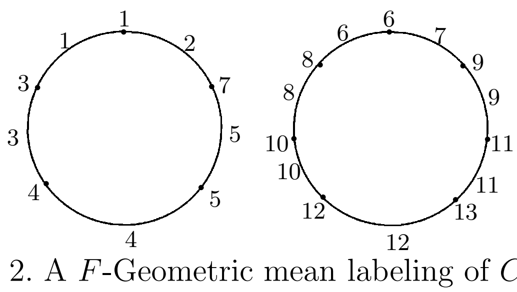

Figure 2. A F-Geometric mean labeling of $C_5 \cup C_7$

In [1], A. Durai Baskar et al. introduced Geometric mean graph.

A function f is called a F-Geometric mean labeling of a graph $G(V, E)$ if $f : V(G) \to \{1, 2, 3, \ldots, q+1\}$ is injective and the induced function $f^* : E(G) \to$

$\{1, 2, 3, \ldots, q\}$ defined as

$$f^*(uv) = \left\lfloor \sqrt{f(u)f(v)} \right\rfloor, \text{ for all } uv \in E(G),$$

is bijective. A graph that admits a F-Geometric mean labeling is called a F-Geometric mean graph.

The graph shown in Figure 3 is a F-Geometric mean graph.

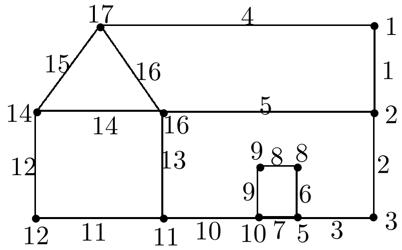

Figure 3. A F-Geometric mean graph

In this paper, we have discussed the F-Geometric mean labeling of some chain graphs and thorn graphs.

2. Main Results

The graph $G^*(p_1, p_2, \ldots, p_n)$ is obtained from n cycles of length $p_1, p_2, \ldots, p_n$ by identifying consecutive cycles at a vertex as follows. If the j^{th} cycle is of odd length, then its $\left(\frac{p_j+3}{2}\right)^{th}$ vertex is identified with the first vertex of $(j+1)^{th}$ cycle and if the j^{th} cycle is of even length, then its $\left(\frac{p_j+2}{2}\right)^{th}$ vertex is identified with the first vertex of $(j+1)^{th}$ cycle.

Theorem 2.1. *$G^*(p_1, p_2, \ldots, p_n)$ is a F-Geometric mean graph for any p_j, for $1 \leq j \leq n$.*

Proof. Let $\{v_i^{(j)}; 1 \leq j \leq n, 1 \leq i \leq p_j\}$ be the vertices of the n number of cycles.

For $1 \leq j \leq n-1$, the j^{th} and $(j+1)^{th}$ cycles are identified by a vertex $v^{(j)}_{\frac{p_j+3}{2}}$ and $v_1^{(j+1)}$ while p_j is odd and $v^{(j)}_{\frac{p_j+2}{2}}$ and $v_1^{(j+1)}$ while p_j is even.

We define $f : V[G^*(p_1, p_2, \ldots, p_n] \to \left\{1, 2, 3, \ldots, \sum\limits_{j=1}^{n} p_j + 1\right\}$ as follows:

$$f(v_i^{(1)}) = \begin{cases} 2i - 1 & 1 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \\ 2p_1 - 2(i-2) & \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \leq i \leq p_1 \end{cases} \quad \text{and}$$

for $2 \le j \le n$,

$$f(v_i^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1} p_k + 2i - 1 & 2 \le i \le \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \\ \sum\limits_{k=1}^{j-1} p_k + 2(i-1) & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \text{ and } p_j \text{ is odd} \\ \sum\limits_{k=1}^{j-1} p_k + 2(i-2) & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \text{ and } p_j \text{ is even} \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j - 2(i-2) & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 3 \le i \le p_j \end{cases}$$

The induced edge labeling is as follows:

$$f^*(v_i^{(1)} v_{i+1}^{(1)}) = \begin{cases} 2i - 1 & 1 \le i \le \left\lfloor \frac{p_1}{2} \right\rfloor \\ 2i - 1 & i = \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \text{ and } p_1 \text{ is odd} \\ 2p_1 - 2(i-1) & i = \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \text{ and } p_1 \text{ is even} \\ 2p_1 - 2(i-1) & \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \le i \le p_1 - 1, \end{cases}$$

$$f^*(v_{p_1}^{(1)} v_1^{(1)}) = 2,$$

for $2 \le j \le n$,

$$f^*(v_i^{(j)} v_{i+1}^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1} p_k + 2i - 1 & 1 \le i \le \left\lfloor \frac{p_j}{2} \right\rfloor \\ \sum\limits_{k=1}^{j-1} p_k + 2i - 1 & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \text{ and } p_j \text{ is odd} \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j - 2(i-1) & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \text{ and } p_j \text{ is even} \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j - 2(i-1) & \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \le i \le p_j - 1 \end{cases}$$

and

$$f^*(v_{p_j}^{(j)} v_1^{(j)}) = \sum_{k=1}^{j-1} p_k + 2.$$

Hence, f is a F-Geometric mean labeling of $G^*(p_1, p_2, \ldots, p_n)$. Thus the graph $G^*(p_1, p_2, \ldots, p_n)$ is a F-Geometric mean graph. □

A F-Geometric mean labeling of $G^*(10, 9, 12, 4, 5)$ is as shown in Figure 4.

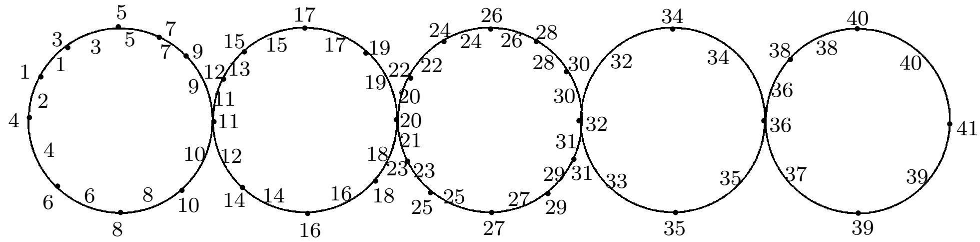

Figure 4. A F-Geometric mean labeling of $G^*(10, 9, 12, 4, 5)$

The graph $G'(p_1, p_2, \ldots, p_n)$ is obtained from n cycles of length $p_1, p_2, \ldots, p_n$ by identifying consecutive cycles at an edge as follows: The $\left(\frac{p_j+3}{2}\right)^{th}$ edge of j^{th} cycle is identified with the first edge of $(j+1)^{th}$ cycle when j is odd and the $\left(\frac{p_j+1}{2}\right)^{th}$ edge of j^{th} cycle is identified with the first edge of $(j+1)^{th}$ cycle when j is even.

Theorem 2.2. *$G'(p_1, p_2, \ldots, p_n)$ is a F-Geometric mean graph if all p_j's are odd or all p_j's are even, for $1 \le j \le n$.*

Proof. Let $\{v_i^{(j)}; 1 \le j \le n, 1 \le i \le p_j\}$ be the vertices of the n number of cycles.

Case (i) p_j is odd, for $1 \le j \le n$.

For $1 \le j \le n-1$, the j^{th} and $(j+1)^{th}$ cycles are identified by the edges $v^{(j)}_{\frac{p_j+1}{2}} v^{(j)}_{\frac{p_j+3}{2}}$ and $v_1^{(j+1)} v_{p_{j+1}}^{(j+1)}$ while j is odd and $v^{(j)}_{\frac{p_j-1}{2}} v^{(j)}_{\frac{p_j+1}{2}}$ and $v_1^{(j+1)} v_{p_{j+1}}^{(j+1)}$ while j is even.

We define $f : V[G'(p_1, p_2, \ldots, p_n)] \to \left\{1, 2, 3, \ldots, \sum\limits_{j=1}^{n} p_j - n + 2\right\}$ as follows:

$$f(v_i^{(1)}) = \begin{cases} 1 & i = 1 \\ 2i & 2 \le i \le \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \\ 2p_1 + 3 - 2i & \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \le i \le p_1 \end{cases}$$

and for $2 \le j \le n$,

$$f(v_i^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1} p_k - j + 2i + 2 & 2 \le i \le \left\lfloor \frac{p_j}{2} \right\rfloor \text{ and } j \text{ is even} \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j + 3 - j - 2i & \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \le i \le p_j - 1 \text{ and } j \text{ even} \\ \sum\limits_{k=1}^{j-1} p_k - j + 2i + 1 & 2 \le i \le \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \text{ and } j \text{ is odd} \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j + 4 - j - 2i & \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \le i \le p_j - 1 \text{ and } j \text{ odd.} \end{cases}$$

The induced edge labeling is as follows:

$$f^*(v_i^{(1)}v_{i+1}^{(1)}) = \begin{cases} 2i & 1 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\ 2p_1 + 1 - 2i & \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \leq i \leq p_1 - 1, \end{cases}$$
$$f^*(v_{p_1}^{(1)}v_1^{(1)}) = 1$$

and for $2 \leq j \leq n$,

$$f^*(v_i^{(j)}v_{i+1}^{(j)}) = \begin{cases} \sum_{k=1}^{j-1} p_k - j + 2i + 2 & 1 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor \text{ and } j \text{ is even} \\ \sum_{k=1}^{j-1} p_k + 2p_j + 1 - j - 2i & \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \leq i \leq p_j - 1 \text{ and } j \text{ even} \\ \sum_{k=1}^{j-1} p_k - j + 2i + 1 & 1 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor \text{ and } j \text{ is odd} \\ \sum_{k=1}^{j-1} p_k + 2p_j + 2 - j - 2i & \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \leq i \leq p_j - 1 \text{ and } j \text{ odd.} \end{cases}$$

Case (ii) p_j is even, for $1 \leq j \leq n$.

For $1 \leq j \leq n-1$, the j^{th} and $(j+1)^{th}$ cycles are identified by the edges $v_{\frac{p_j}{2}}^{(j)} v_{\frac{p_j+2}{2}}^{(j)}$ and $v_1^{(j+1)} v_{p_{j+1}}^{(j+1)}$.

We define $f : V[G'(p_1, p_2, \ldots, p_n)] \to \left\{1, 2, 3, \ldots, \sum_{j=1}^{n} p_j - n + 2\right\}$ as follows:

$$f(v_i^{(1)}) = \begin{cases} 1 & i = 1 \\ 2i & 2 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\ 2p_1 + 3 - 2i & \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \leq i \leq p_1 \end{cases}$$

and for $2 \leq j \leq n$,

$$f(v_i^{(j)}) = \begin{cases} \sum_{k=1}^{j-1} p_k - j + 2i + 1 & 2 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor \\ \sum_{k=1}^{j-1} p_k + 2p_j + 4 - j - 2i & \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \leq i \leq p_j - 1. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(v_i^{(1)}v_{i+1}^{(1)}) = \begin{cases} 2i & 1 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\ 2p_1 + 1 - 2i & \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \leq i \leq p_1 - 1, \end{cases}$$
$$f^*(v_{p_1}^{(1)}v_1^{(1)}) = 1$$

and for $2 \le j \le n$,

$$f^*(v_i^{(j)} v_{i+1}^{(j)}) = \begin{cases} \sum_{k=1}^{j-1} p_j - j + 2i + 1 & 1 \le i \le \lfloor \frac{p_j}{2} \rfloor \\ \sum_{k=1}^{j-1} p_k + 2p_j + 2 - j - 2i & \lfloor \frac{p_j}{2} \rfloor + 1 \le i \le p_j - 1. \end{cases}$$

Hence, f is a F-Geometric mean labeling of $G'(p_1, p_2, \ldots, p_n)$. Thus the graph $G'(p_1, p_2, \ldots, p_n)$ is a F-Geometric mean graph. □

A F-Geometric mean labeling of $G'(7, 5, 9, 13)$ and $G'(4, 8, 10, 6)$ is as shown in Figure 5.

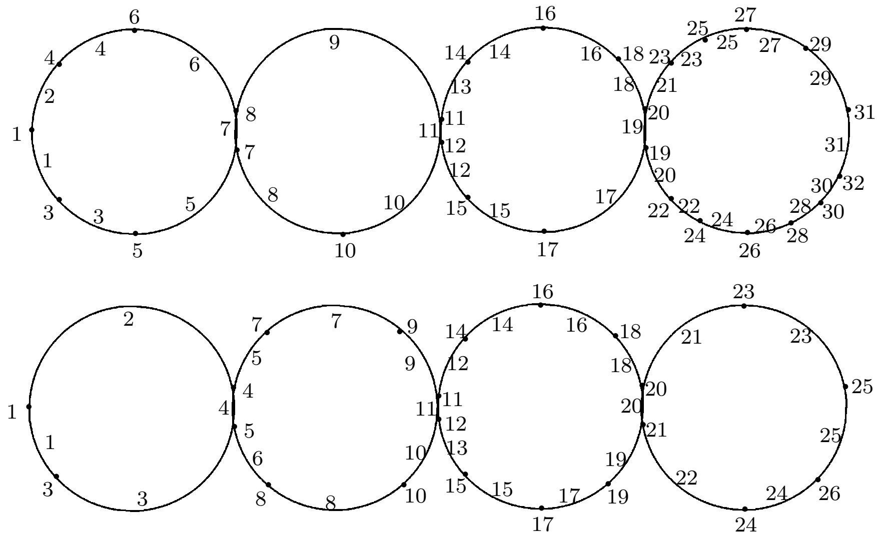

Figure 5. F-Geometric mean labeling of $G'(7, 5, 9, 13)$ and $G'(4, 8, 10, 6)$

The graph $\widehat{G}(p_1, m_1, p_2, m_2, \ldots, m_{n-1}, p_n)$ is obtained from n cycles of length $p_1, p_2, \ldots, p_n$ and $(n-1)$ paths on $m_1, m_2, \ldots, m_{n-1}$ vertices respectively by identifying a cycle and a path at a vertex alternatively as follows: If the j^{th} cycles is of odd length, then its $\left(\frac{p_j+3}{2}\right)^{th}$ vertex is identified with a pendant vertex of j^{th} path and if the j^{th} cycle is of even length, then its $\left(\frac{p_j+2}{2}\right)^{th}$ vertex is identified with a pendant vertex of j^{th} path while the other pendant vertex of the j^{th} path is identified with the first vertex of the $(j+1)^{th}$ cycle.

Theorem 2.3. *$\widehat{G}(p_1, m_1, p_2, m_2, \ldots, m_{n-1} p_n)$ is a F-Geometric mean graph for any p_j's and m_j's.*

Proof. Let $\{v_i^{(j)}; 1 \le j \le n, 1 \le i \le p_j\}$ and $\{u_i^{(j)}; 1 \le j \le n-1, 1 \le i \le m_j\}$ be the n number of cycles and $(n-1)$ number of paths respectively.

For $1 \leq j \leq n-1$, the j^{th} cycle and j^{th} path are identified by a vertex $v^{(j)}_{\frac{p_j+2}{2}}$ and $u^{(j)}_1$ while p_j is even and $v^{(j)}_{\frac{p_j+3}{2}}$ and $u^{(j)}_1$ while p_j is odd. And the j^{th} path and $(j+1)^{th}$ cycle are identified by a vertex $u^{(j)}_{m_j}$ and $v^{(j+1)}_1$.

We define $f : V[\widehat{G}(p_1, m_1, p_2, m_2, \ldots, m_{n-1}, p_n)] \to \left\{1, 2, 3, \ldots, \sum\limits_{j=1}^{n-1}(p_j + m_j) + p_n - n + 2\right\}$ as follows:

$$f(v^{(1)}_i) = \begin{cases} 2i-1 & 1 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \\ 2p_1 + 4 - 2i & \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \leq i \leq p_1, \end{cases}$$

$$f(u^{(1)}_i) = p_1 + i, \text{ for } 2 \leq i \leq m_1,$$

for $2 \leq j \leq n$,

$$f(v^{(j)}_i) = \begin{cases} \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2i - j & 2 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \\ \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2i - j - 1 & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \text{ and } p_j \text{ is odd} \\ \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2i - j - 3 & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \text{ and } p_j \text{ is even} \\ \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2p_j - 2i - j + 5 & \left\lfloor \frac{p_j}{2} \right\rfloor + 3 \leq i \leq p_j \end{cases}$$

and for $3 \leq j \leq n$,

$$f(u^{(j-1)}_i) = \sum_{k=1}^{j-2}(p_k + m_k) + p_{j-1} + i + 2 - j, \text{ for } 2 \leq i \leq m_{j-1}$$

The induced edge labeling is as follows:

$$f^*(v^{(1)}_i v^{(1)}_{i+1}) = \begin{cases} 2i-1 & 1 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\ 2i-1 & i = \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \text{ and } p_1 \text{ is odd} \\ 2p_1 - 2i + 2 & i = \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \text{ and } p_1 \text{ is even} \\ 2p_1 - 2i + 2 & \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \leq i \leq p_1 - 1, \end{cases}$$

$$f^*(v^{(1)}_{p_1} v^{(1)}_1) = 2,$$

$$f^*(u^{(1)}_i u^{(1)}_{i+1}) = p_1 + i, \text{ for } 1 \leq i \leq m_1 - 1,$$

for $2 \le j \le n$,

$$f^*(v_i^{(j)} v_{i+1}^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2i - j & 1 \le i \le \left\lfloor \frac{p_j}{2} \right\rfloor \\ \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2i - j & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \text{ and} \\ & p_j \text{ is odd} \\ \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2p_j - 2i - j + 3 & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \text{ and} \\ & p_j \text{ is even} \\ \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2p_j - 2i - j + 3 & \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \le i \le p_j - 1, \end{cases}$$

$$f^*(v_{p_j}^{(j)} v_1^{(j)}) = \sum_{k=1}^{j-1}(p_k + m_k) - j + 3$$

and for $3 \le j \le n$,

$$f^*(u_i^{(j-1)} u_{i+1}^{(j-1)}) = \sum_{k=1}^{j-2}(p_k + mk) + p_{j-1} + i + 2 - j, \text{ for } 1 \le i \le m_{j-1} - 1.$$

Hence, f is a F-Geometric mean labeling of $\widehat{G}(p_1, m_1, p_2, m_2 \ldots, m_{n-1}, p_n)$. Thus the graph $\widehat{G}(p_1, m_1, p_2, m_2 \ldots, m_{n-1}, p_n)$ is a F-Geometric mean graph. $\square$

A F-Geometric mean labeling of $\widehat{G}(8, 4, 5, 6, 10)$ is as shown in Figure 6.

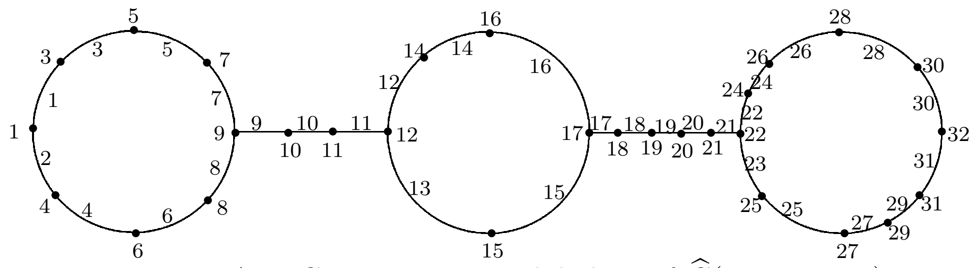

Figure 6. A F-Geometric mean labeling of $\widehat{G}(8, 4, 5, 6, 10)$

Theorem 2.4. *$C_n \odot K_1$ is a F-Geometric mean graph, for $n \ge 3$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n and let u_i be the pendant vertices attached at each v_i, for $1 \le i \le n$. Consider the graph $C_n \odot K_1$, for $n \ge 4$.
Case (i) $\lfloor \sqrt{2n+1} \rfloor$ is odd.

We define $f : V[C_n \odot K_1] \to \{1, 2, 3, \ldots, 2n+1\}$ as follows:

$$f(v_i) = \begin{cases} 1 & i = 1 \\ 2i & 2 \le i \le \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor + 1 \\ 2i+1 & \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor + 2 \le i \le n \end{cases} \quad \text{and}$$

$$f(u_i) = \begin{cases} 2 & i = 1 \\ 2i-1 & 2 \le i \le \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor \\ 2i+1 & i = \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor + 1 \\ 2i & \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor + 2 \le i \le n. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(v_i v_{i+1}) = \begin{cases} 2i & 1 \le i \le \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor \\ 2i+1 & \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor + 1 \le i \le n-1, \end{cases}$$

$$f^*(v_1 v_n) = \left\lfloor \sqrt{2n+1} \right\rfloor \quad \text{and}$$

$$f^*(u_i v_i) = \begin{cases} 2i-1 & 1 \le i \le \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor \\ 2i & \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor + 1 \le i \le n. \end{cases}$$

Case (ii) $\left\lfloor \sqrt{2n+1} \right\rfloor$ is even.

We define $f : V[C_n \odot K_1] \to \{1, 2, 3, \ldots, 2n+1\}$ as follows:

$$f(v_i) = \begin{cases} 1 & i = 1 \\ 2i & 2 \le i \le \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor \\ 2i+1 & \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor + 1 \le i \le i \le n \end{cases} \quad \text{and}$$

$$f(u_i) = \begin{cases} 2 & i = 1 \\ 2i-1 & 2 \le i \le \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor \\ 2i & \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor + 1 \le i \le n. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(v_i v_{i+1}) = \begin{cases} 2i & 1 \le i \le \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor - 1 \\ 2i+1 & \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor \le i \le n-1, \end{cases}$$

$$f^*(v_1 v_n) = \left\lfloor \sqrt{2n+1} \right\rfloor \quad \text{and}$$

$$f^*(u_i v_i) = \begin{cases} 2i-1 & 1 \le i \le \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor \\ 2i & \left\lfloor \frac{\sqrt{2n+1}}{2} \right\rfloor + 1 \le i \le n. \end{cases}$$

Hence, the graph $C_n \odot K_1$, for $n \geq 4$ admits F-Geometric mean labeling. $\square$

For $n = 3$, a F-Geometric mean labeling of $C_3 \odot K_1$ is as shown in Figure 7.

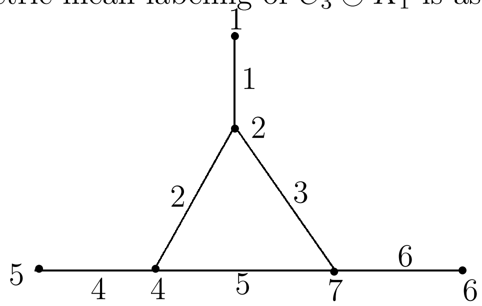

Figure 7. A F - Geometric mean labeling of $C_3 \odot K_1$

A F-Geometric mean labeling of $C_{12} \odot K_1$ is as shown in Figure 8.

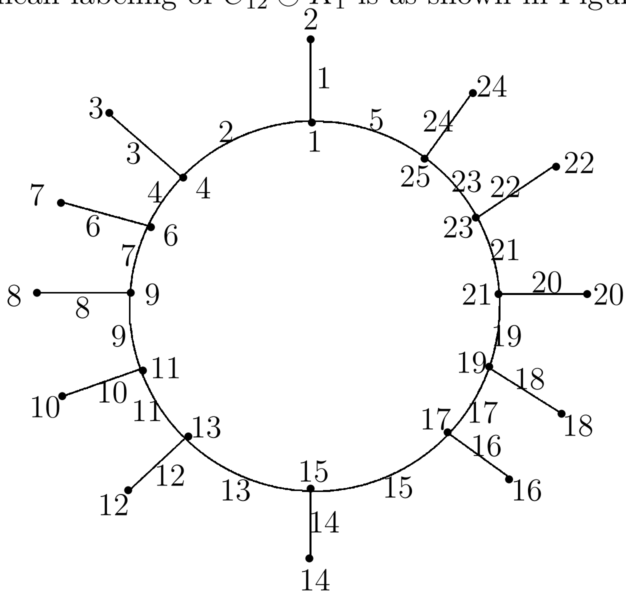

Figure 8. A F-Geometric mean labeling of $C_{12} \odot K_1$

Theorem 2.5. *$C_n \odot S_2$ is a F-Geometric mean graph for $n \geq 3$.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle C_n. Let $v_1^{(i)}$ be the pendant vertices at each vertex u_i, for $1 \leq i \leq n$. Therefore,

$$V[C_n \odot S_2] = V(C_n) \cup \{v_1^{(i)}, v_2^{(i)}; 1 \leq i \leq n\}$$

and

$$E[C_n \odot S_2] = E(C_n) \cup \{u_i v_1^{(i)}, u_i v_2^{(i)}; 1 \leq i \leq n\}.$$

Case (i) $\lfloor\sqrt{6n}\rfloor$ is a multiple of 3.

We define $f : V[C_n \odot S_2] \to \{1, 2, 3, \ldots, 3n+1\}$ as follows:

$$f(u_i) = \begin{cases} 3i-1 & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \\ 3i+1 & i = \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \\ 3i & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 2 \le i \le n, \end{cases}$$

$$f(v_1^{(i)}) = \begin{cases} 3i-2 & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \\ 3i-1 & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n \end{cases}$$

and

$$f(v_2^{(i)}) = \begin{cases} 3i & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \\ 3i+1 & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 2 \le i \le n. \end{cases}$$

The induced edge labeling is as follows

$$f^*(u_iu_{i+1}) = \begin{cases} 3i & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor - 1 \\ 3i+1 & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \le i \le n-1, \end{cases}$$

$$f^*(u_nu_1) = \left\lfloor \sqrt{6n} \right\rfloor,$$

$$f^*(u_iv_1^{(i)}) = \begin{cases} 3i-2 & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \\ 3i-1 & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n \end{cases}$$

and

$$f^*(u_iv_2^{(i)}) = \begin{cases} 3i-1 & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \\ 3i & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n. \end{cases}$$

Case (ii) $\lfloor\sqrt{6n}\rfloor$ is not a multiple of 3.

We define $f : V[C_n \odot S_2] \to \{1, 2, 3, \ldots, 3n+1\}$ as follows:

$$f(u_i) = \begin{cases} 3i-1 & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \\ 3i & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n, \end{cases}$$

$$f(v_1^{(i)}) = \begin{cases} 3i-2 & 1 \le i \le \left\lfloor \frac{\sqrt{6n}+1}{3} \right\rfloor \\ 3i-1 & \left\lfloor \frac{\sqrt{6n}+1}{3} \right\rfloor + 1 \le i \le n \end{cases}$$

and

$$f(v_2^{(i)}) = \begin{cases} 3i & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \\ 3i+1 & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n. \end{cases}$$

The induced edge labeling is as follows

$$f^*(u_i u_{i+1}) = \begin{cases} 3i & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \\ 3i+1 & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n-1, \end{cases}$$

$$f^*(u_n u_1) = \left\lfloor \sqrt{6n} \right\rfloor,$$

$$f^*(u_i v_1^{(i)}) = \begin{cases} 3i-2 & 1 \le i \le \left\lfloor \frac{\sqrt{6n}+1}{3} \right\rfloor \\ 3i-1 & \left\lfloor \frac{\sqrt{6n}+1}{3} \right\rfloor + 1 \le i \le n \end{cases}$$

and

$$f^*(u_i v_2^{(i)}) = \begin{cases} 3i-1 & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \\ 3i & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n. \end{cases}$$

Hence, f is a F-Geometric mean labeling of $C_n \odot S_2$. Thus the graph $C_n \odot S_2$ is a F-Geometric mean graph, for $n \ge 3$. □

A F-Geometric mean labeling of $C_6 \odot S_2$ is as shown in Figure 9.

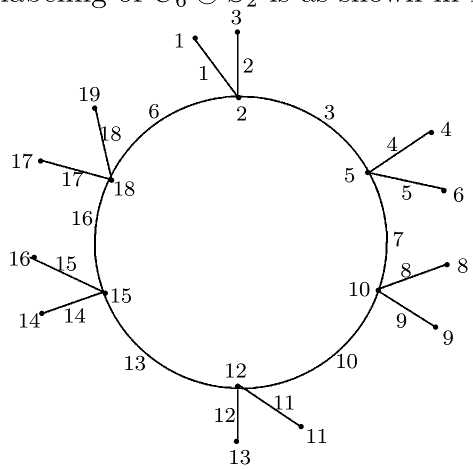

Figure 9. A F-Geometric mean labeling of $C_6 \odot S_2$

Theorem 2.6. *$(P_n; S_m)$ is a F-Geometric mean graph, for $m \le 2$ and $n \ge 1$.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path P_n and Let $v_1^{(i)}, v_2^{(i)}, \ldots, v_{m+1}^{(i)}$ be the vertices of the star graph S_m such that $v_1^{(i)}$ is the central vertex of S_m, for $1 \le i \le n$.

Case (i) $m = 1$

We define $f : V[(P_n; S_m)] \to \{1, 2, 3, \ldots, 3n\}$ as follows:

$$f(u_i) = \begin{cases} 3i & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 3i - 2 & 1 \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

$$f(v_1^{(i)}) = 3i - 1, for \quad 1 \leq i \leq n$$

and

$$f(v_2^{(i)}) = \begin{cases} 3i - 2 & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 3i & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labeling is as follows

$$f^*(u_i u_{i+1}) = 3i, \ for \qquad 1 \leq i \leq n - 1,$$

$$f^*(u_i v_1^{(i)}) = \begin{cases} 3i - 1 & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 3i - 2 & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

and

$$f^*(v_1^{(i)} v_2^{(i)}) = \begin{cases} 3i - 2 & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 3i - 1 & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

Case (ii) $m = 2$

We define $f : V[(P_n; S_m)] \to \{1, 2, 3, \ldots, 4n\}$ as follows:

$$f(u_i) = \begin{cases} 4i & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 4i - 2 & 1 \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

$$f(v_1^{(i)}) = 4i - 1, for \quad 1 \leq i \leq n,$$

$$f(v_2^{(i)}) = 4i - 3, for \quad 1 \leq i \leq n$$

and

$$f(v_3^{(i)}) = \begin{cases} 4i - 2 & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 4i & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labeling is as follows:

$$f^*(u_i u_{i+1}) = 4i, \ for \qquad 1 \leq i \leq n - 1,$$

$$f^*(u_i v_1^{(i)}) = \begin{cases} 4i - 1 & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 4i - 2 & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(v_1^{(i)} v_2^{(i)}) = 4i - 3, \quad \text{for } 1 \leq i \leq n$$

and

$$f^*(v_1^{(i)} v_3^{(i)}) = \begin{cases} 4i - 2 & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 4i - 1 & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

Hence, f is a F-Geometric mean labeling of $(P_n; S_m.)$ Thus the graph $(P_n; S_m)$ is a F-Geometric mean graph, for $m \leq 2$ and $n \geq 1$. □

A F-Geometric mean labeling of $(P_7; S_1)$ and $(P_8; S_2)$ is as shown in Figure 10.

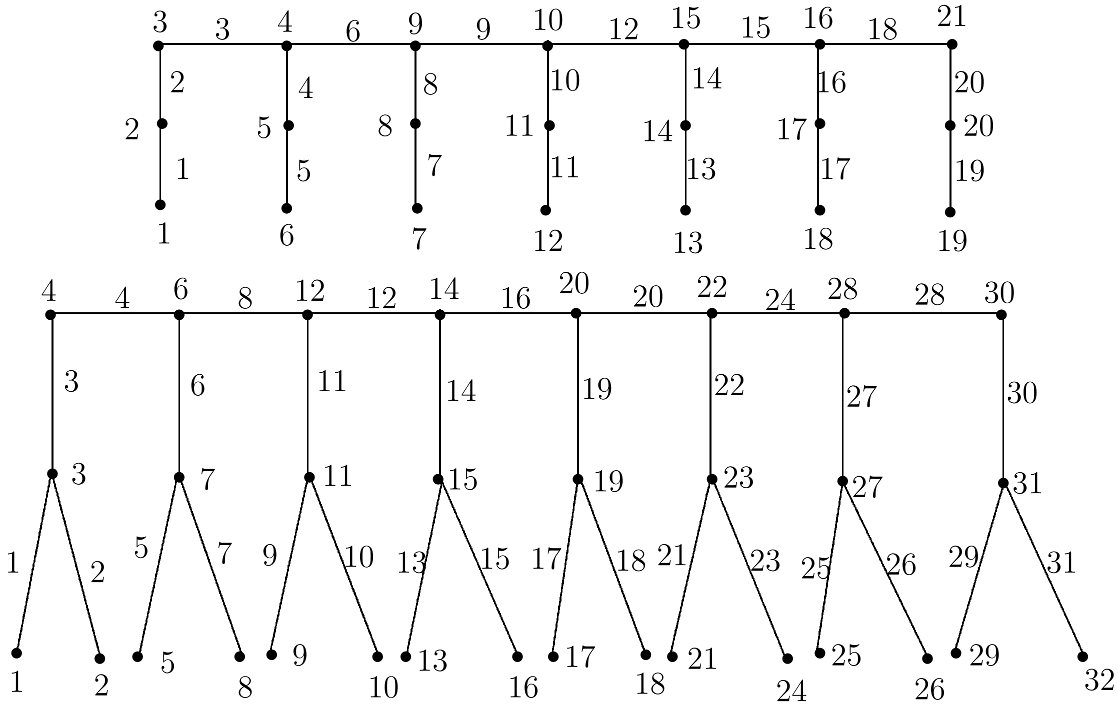

Figure 10. A F-Geometric mean labeling of $(P_7; S_1)$ and $(P_8; S_2)$

Theorem 2.7. *For a $H-$graph G, $G \odot K_1$ is a F-Geometric mean graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of G. Therefore

$$V(G \odot K_1) = V(G) \cup \{u_i', v_i'; 1 \leq i \leq n\}$$

and

$$E(G \odot K_1) = E(G) \cup \{u_i u_i', v_i v_i'; 1 \leq i \leq n\}.$$

Case (i) $n \equiv 0(mod\ 4)$.

We define $f : V(G \odot K_1) \rightarrow \{1, 2, 3, \ldots, 4n\}$ as follows:

$$f(u_i) = \begin{cases} 2i-1 & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 2i & 1 \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

$$f(u_i') = \begin{cases} 2i & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 2i-1 & 1 \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

$$f(v_i) = \begin{cases} 2n-3+4i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } i \text{ is odd} \\ 2n-1+4i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ and } i \text{ is even,} \end{cases}$$

$$f(v_{n+1-i}) = \begin{cases} 2n-2+4i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } i \text{ is odd} \\ 2n+4i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ and } i \text{ is even,} \end{cases}$$

and

$$f(v_i') = \begin{cases} f(v_i)+2 & 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1 \text{ and } i \text{ is odd} \\ f(v_i)-2 & \lfloor \frac{n}{2} \rfloor + 1 \le i \le n \text{ and } i \text{ is odd} \\ f(v_i)-2 & 1 \le i \le \lfloor \frac{n}{2} \rfloor \text{ and } i \text{ is even} \\ f(v_i)+2 & \lfloor \frac{n}{2} \rfloor + 2 \le i \le n \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labeling is as follows:

$$f^*(u_iu_{i+1}) = 2i,\ for\ 1 \le i \le n-1,$$
$$f^*(u_iu_i') = 2i-1,\ for\ 1 \le i \le n,$$
$$f^*(v_iv_{i+1}) = 2n-1+4i,\ for\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor,$$
$$f^*(v_{n+1-i}v_{n-i}) = 2n+4i,\ for\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1,$$
$$f^*(v_iv_i') = \begin{cases} f(v_i) & 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1 \text{ and } i \text{ is odd} \\ f(v_i)-2 & \lfloor \frac{n}{2} \rfloor + 1 \le i \le n \text{ and } i \text{ is odd} \\ f(v_i)-2 & 1 \le i \le \lfloor \frac{n}{2} \rfloor \text{ and } i \text{ is even} \\ f(v_i) & \lfloor \frac{n}{2} \rfloor + 2 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

and

$$f^*(u_{i+1}v_i) = 2n,\ for\ i = \left\lfloor \frac{n}{2} \right\rfloor.$$

Case (ii) $n \equiv 1 (mod\ 4)$.

We define $f : V(G \odot K_1) \to \{1, 2, 3, \ldots, 4n\}$ as follows:

$$f(u_i) = 2i,\ for\ 1 \le i \le n, f(u_i') = 2i-1,\ for\ 1 \le i \le n,$$
$$f(v_i) = \begin{cases} 2n-3+4i & 1 \le i \le \lfloor \frac{n}{2} \rfloor + 1 \text{ and } i \text{ is odd} \\ 2n-1+4i & 1 \le i \le \lfloor \frac{n}{2} \rfloor \text{ and } i \text{ is even,} \end{cases}$$
$$f(v_{n+1}-i) = \begin{cases} 2n-2+4i & 1 \le i \le \lfloor \frac{n}{2} \rfloor \text{ and } i \text{ is odd} \\ 2n+4i & 1 \le i \le \lfloor \frac{n}{2} \rfloor \text{ and } i \text{ is even} \end{cases}$$

and

$$f(v_i') = \begin{cases} f(v_i)+2 & 1 \le i \le \lfloor \frac{n}{2} \rfloor \text{ and } i \text{ is odd} \\ f(v_i)+1 & i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } i \text{ is odd} \\ f(v_i)+2 & \lfloor \frac{n}{2} \rfloor + 3 \le i \le n \text{ and } i \text{ is odd} \\ f(v_i)-2 & 1 \le i \le n \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labeling is as follows:

$$f^*(u_iu_{i+1}) = 2i,\ for\ 1 \le i \le n-1,$$
$$f^*(u_iu_i') = 2i-1,\ for\ 1 \le i \le n,$$
$$f^*(v_iv_{i+1}) = 2n-1+4i,\ for\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor,$$
$$f^*(v_{n+1-i}v_{n-i}) = 2n+4i,\ for\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor,$$

$$f^*(v_iv_i') = \begin{cases} f(v_i) & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ f(v_i) - 2 & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

and

$$f^*(u_iv_i) = 2n, \ for \ i = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Case (iii) $n \equiv 2(mod\ 4)$.

We define $f : V(G \odot K_1) \to \{1, 2, 3, \ldots, 4n\}$ as follows:

$$f(u_i) = \begin{cases} 2i & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 2i - 1 & 1 \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

$$f(u_i') = \begin{cases} 2i - 1 & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 2i & 1 \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

$$f(v_i) = \begin{cases} 2n - 1 + 4i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ and } i \text{ is odd} \\ 2n - 3 + 4i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } i \text{ is even,} \end{cases}$$

$$f(v_{n+1-i}) = \begin{cases} 2n + 4i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ and } i \text{ is odd} \\ 2n - 2 + 4i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } i \text{ is even,} \end{cases}$$

and

$$f(v_i') = \begin{cases} f(v_i) - 2 & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ and } i \text{ is odd} \\ f(v_i) + 2 & \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n \text{ and } i \text{ is odd} \\ f(v_i) + 2 & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } i \text{ is even} \\ f(v_i) - 2 & \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

The induced edge labeling is as follows:

$$f^*(u_iu_{i+1}) = 2i, \ for \ 1 \leq i \leq n - 1,$$

$$f^*(u_iu_i') = 2i - 1, \ for \ 1 \leq i \leq n,$$

$$f^*(v_iv_{i+1}) = 2n - 1 + 4i, \ for \ 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor,$$

$$f^*(v_{n+1-i}v_{n-i}) = 2n + 4i, \ for \ 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1,$$

$$f^*(v_iv_i') = \begin{cases} f(v_i) - 2 & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ and } i \text{ is odd} \\ f(v_i) & \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n \text{ and } i \text{ is odd} \\ f(v_i) & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } i \text{ is even} \\ f(v_i) - 2 & \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

and

$$f^*(u_{i+1}v_i) = 2n, \ for \ i = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Case (iv) $n \equiv 3(mod\ 4)$.

We define $f : V(G \odot K_1) \to \{1, 2, 3, \dots, 4n\}$ as follows:

$$f(u_i) = 2i,\ for\ 1 \le i \le n, f(u_i') = 2i - 1,\ for\ 1 \le i \le n,$$

$$f(v_i) = \begin{cases} 2n - 1 + 4i & 1 \le i \le \lfloor \frac{n}{2} \rfloor \text{ and } i \text{ is odd} \\ 2n - 3 + 4i & 1 \le i \le \lfloor \frac{n}{2} \rfloor + 1 \text{ and } i \text{ is even,} \end{cases}$$

$$f(v_{n+1-i}) = \begin{cases} 2n + 4i & 1 \le i \le \lfloor \frac{n}{2} \rfloor \text{ and } i \text{ is odd} \\ 2n - 2 + 4i & 1 \le i \le \lfloor \frac{n}{2} \rfloor \text{ and } i \text{ is even,} \end{cases}$$

and

$$f(v_i') = \begin{cases} f(v_i) - 2 & 1 \le i \le \lfloor \frac{n}{2} \rfloor \text{ and } i \text{ is odd} \\ f(v_i) + 1 & i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } i \text{ is odd} \\ f(v_i) - 2 & \lfloor \frac{n}{2} \rfloor + 3 \le i \le n \text{ and } i \text{ is odd} \\ f(v_i) + 2 & 1 \le i \le n \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labeling is as follows:

$$f^*(u_iu_{i+1}) = 2i,\ for\ 1 \le i \le n - 1,$$
$$f^*(u_iu_i') = 2i - 1,\ for\ 1 \le i \le n,$$
$$f^*(v_iv_{i+1}) = 2n - 1 + 4i,\ for\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor,$$
$$f^*(v_{n+1-i}v_{n-i}) = 2n + 4i,\ for\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor,$$
$$f^*(v_iv_i') = \begin{cases} f(v_i) - 2 & 1 \le i \le n \text{ and } i \text{ is odd} \\ f(v_i) + 2 & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

and

$$f^*(u_iv_i) = 2n,\ for\ i = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Hence, f is a F-Geometric mean labeling of $G \odot K_1$. Thus the graph $G \odot K_1$ is a F-Geometric mean graph. □

A F-Geometric mean labeling of $G \odot K_1$ is as shown in Figure 11.

Theorem 2.8. *For a $H-$graph G, $G \odot S_2$ is a F-Geometric mean graph.*

Proof. Let $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ be the vertices of G. Therefore,

$$V[G \odot S_2] = V(G) \cup \{u_i', u_i'', v_i', v_i''; 1 \le i \le n\}$$

and

$$E[G \odot S_2] = E(G) \cup \{u_iu_i', u_iu_i'', v_iv_i', v_iv_i''; 1 \le i \le n\}.$$

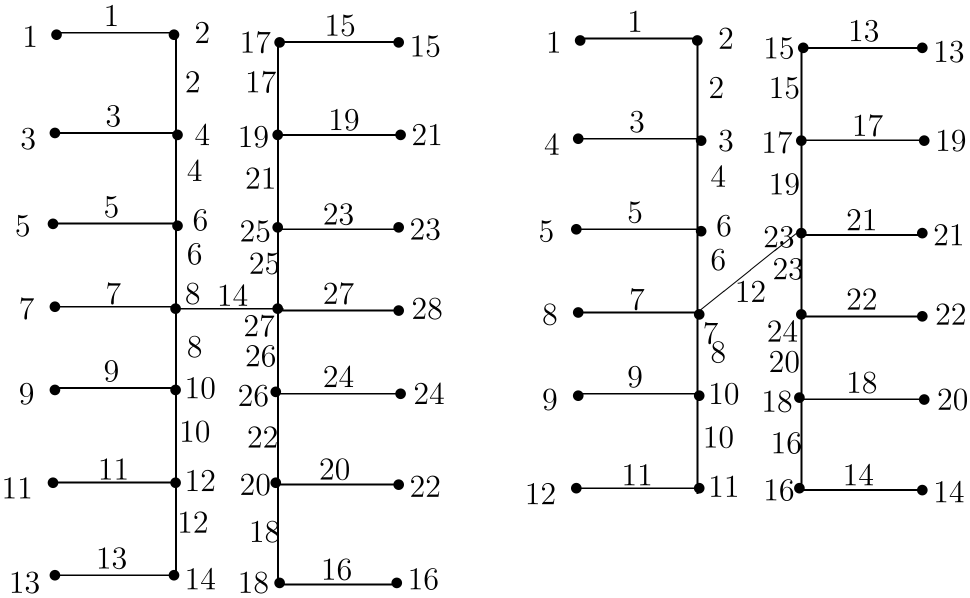

Figure 11. A F-Geometric mean labeling of H graph $G \odot K_1$

Case (i) n is odd.

We define $f : V(G \odot S_2) \to \{1, 2, 3, \ldots, 6n\}$ as follows:

$$f(u_i) = 3i - 1,\ for\ 1 \leq i \leq n,$$

$$f(u_i') = 3i - 2,\ for\ 1 \leq i \leq n,$$

$$f(u_i'') = 3i,\ for\ 1 \leq i \leq n,$$

$$f(v_i) = 3n - 2 + 6i,\ for\ 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor,$$

$$f(v_{n+1-i}) = 3n - 3 + 6i,\ for\ 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

$$f(v_i') = \begin{cases} f(v_i) - 2 & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ f(v_i) - 1 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ f(v_i) + 2 & \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n \end{cases}$$

and

$$f(v_i'') = \begin{cases} f(v_i) + 2 & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ f(v_i) - 2 & \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(u_i u_{i+1}) = 3i,\ for\ 1 \leq i \leq n - 1,$$

$$f^*(u_i u_i') = 3i - 2,\ for\ 1 \leq i \leq n,$$

$$f^*(u_i u_i'') = 3i - 1,\ for\ 1 \leq i \leq n,$$

$$f^*(u_i v_i) = 3n,\ for\ i = \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

$$f^*(v_i v_{i+1}) = 3n + 6i,\ for\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor,$$
$$f^*(v_{n+1-i} v_{n-i}) = 3n - 1 + 6i,\ for\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor,$$
$$f^*(v_i v_i') = \begin{cases} f(v_i) - 2 & 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ f(v_i) - 1 & i = \lfloor \frac{n}{2} \rfloor + 1 \\ f(v_i) & \lfloor \frac{n}{2} \rfloor + 2 \le i \le n \end{cases}$$

and

$$f^*(v_i v_i'') = \begin{cases} f(v_i) & 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ f(v_i) - 2 & \lfloor \frac{n}{2} \rfloor + 1 \le i \le n. \end{cases}$$

Case (ii) n is even.

We define $f : V(G \odot S_2) \to \{1, 2, 3, \ldots, 6n\}$ as follows:

$$f(u_i) = 3i - 1,\ for\ 1 \le i \le n,$$
$$f(u_i') = 3i - 2,\ for\ 1 \le i \le n,$$
$$f(u_i'') = 3i,\ for\ 1 \le i \le n - 1,$$
$$f(u_n'') = 3n + 1,$$
$$f(v_i) = 3n + 1 + 6i,\ for\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1,$$
$$f(v_{n+1-i}) = 3n + 6(i - 1),\ for\ 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1,$$
$$f(v_n) = 3n + 2,$$
$$f(v_i') = \begin{cases} f(v_i) - 2 & 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ f(v_i) + 2 & \lfloor \frac{n}{2} \rfloor + 1 \le i \le n - 1, \end{cases}$$
$$f(v_n') = f(v_n) + 1$$

and

$$f(v_i'') = \begin{cases} f(v_i) + 2 & 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1 \\ f(v_i) - 1 & i = \lfloor \frac{n}{2} \rfloor \\ f(v_i) - 2 & \lfloor \frac{n}{2} \rfloor + 1 \le i \le n. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(u_i u_{i+1}) = 3i,\ for\ 1 \le i \le n - 1,$$
$$f^*(u_i u_i') = 3i - 2,\ for\ 1 \le i \le n,$$
$$f^*(u_i u_i'') = 3i - 1,\ for\ 1 \le i \le n,$$
$$f^*(u_{i+1} v_i) = 3n + 1,\ for\ i = \left\lfloor \frac{n}{2} \right\rfloor,$$
$$f^*(v_i v_{i+1}) = 3n + 3 + 6i,\ for\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor,$$

$$f^*(v_{n+1-i}v_{n-i}) = 3n - 4 + 6i, \ \ for \ 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor,$$
$$f^*(v_nv_{n-1}) = 3n + 3,$$
$$f^*(v_iv_i') = \begin{cases} f(v_i) - 2 & 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ f(v_i) & \lfloor \frac{n}{2} \rfloor + 1 \le i \le n \end{cases}$$

and

$$f^*(v_iv_i'') = \begin{cases} f(v_i) & 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1 \\ f(v_i) - 1 & i = \lfloor \frac{n}{2} \rfloor \\ f(v_i) - 2 & \lfloor \frac{n}{2} \rfloor + 1 \le i \le n. \end{cases}$$

Hence, f is a F-Geometric mean labeling of $G \odot S_2$. Thus the graph $G \odot S_2$ is a F-Geometric mean graph. □

A F-Geometric mean labeling of $G \odot S_2$ is as shown in Figure 12.

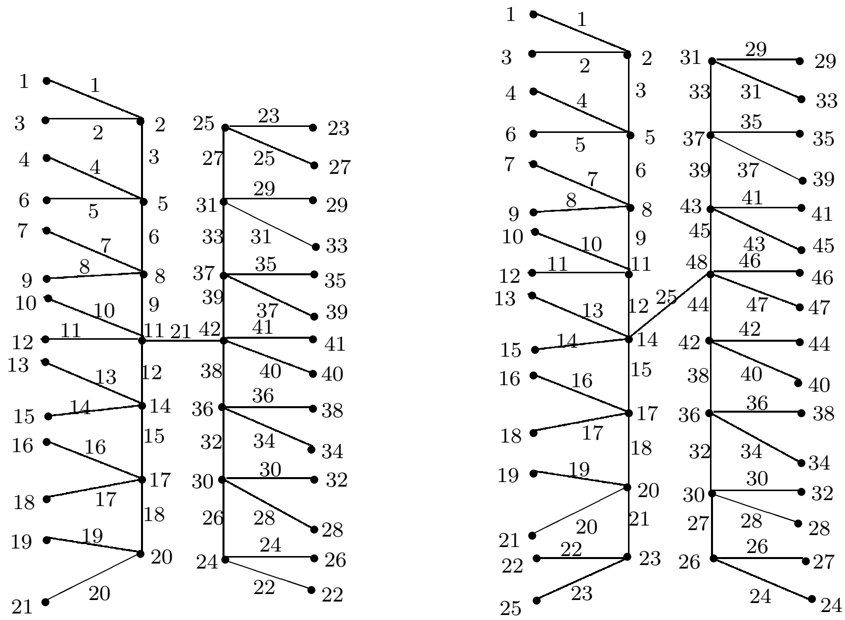

Figure 12. A F-Geometric mean labeling of H graph $G \odot S_2$

Theorem 2.9. *$L_n \odot K_1$ is a F-Geometric mean graph, for $n \ge 2$.*

Proof. Let $V(L_n) = \{u_i, v_i; 1 \le i \le n\}$ be the vertex set of the ladder L_n and $E(L_n) = \{u_iv_i; 1 \le i \le n\} \cup \{u_iu_{i+1}, v_iv_{i+1}; 1 \le i \le n-1\}$ be the edge set of the ladder L_n. Let w_i be the pendent vertex adjacent to u_i, and x_i be the pendent vertex adjacent to v_i, for $1 \le i \le n$.

We define $f : V(L_n \odot K_1) \to \{1, 2, 3, \dots, 5n-1\}$ as follows:

$$
\begin{aligned}
f(u_1) &= 3,\\
f(v_1) &= 4,\\
f(x_1) &= 2,\\
f(u_i) &= 5i-3,\ \ for\ 2 \le i \le n,\\
f(v_i) &= 5i-2,\ \ for\ 2 \le i \le n,\\
f(w_i) &= 5i-4,\ \ for\ 1 \le i \le n
\end{aligned}
$$

and

$$f(x_i) = 5i-1,\ \ for\ 2 \le i \le n.$$

The induced edge labeling is as follows

$$
\begin{aligned}
f^*(u_1v_1) &= 3,\\
f^*(v_1x_1) &= 2,\\
f^*(u_iv_i) &= 5i-3,\ \ for\ 2 \le i \le n,\\
f^*(w_iu_i) &= 5i-4,\ \ for\ 1 \le i \le n,\\
f^*(v_ix_i) &= 5i-2,\ \ for\ 2 \le i \le n,\\
f^*(u_iu_{i+1}) &= 5i-1,\ \ for\ 1 \le i \le n-1,\\
f^*(v_iv_{i+1}) &= 5i,\ \ for\ 1 \le i \le n-1.
\end{aligned}
$$

Hence f is a F-Geometric mean labeling of $L_n \odot K_1$. Thus the graph $L_n \odot K_1$ is a F-Geometric mean graph, for $n \ge 2$. □

A F-Geometric mean labeling of $L_8 \odot K_1$ is as shown in Figure 13.

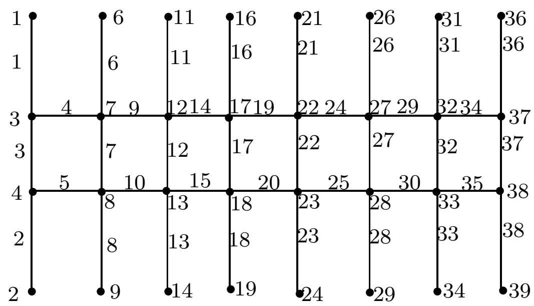

Figure 13. A F-Geometric mean labeling of $L_8 \odot K_1$

Theorem 2.10. *$L_n \odot S_2$ is a F-Geometric mean graph, for $n \ge 2$.*

Proof. Let $V(L_n) = \{u_i, v_i; 1 \le i \le n\}$ be the vertex set of the ladder L_n and $E(L_n) = \{u_iv_i; 1 \le i \le n\} \cup \{u_iu_{i+1}, v_iv_{i+1}; 1 \le i \le n-1\}$ be the edge set of the ladder L_n.

Let $w_1^{(i)}$ and $w_2^{(i)}$ be the pendant vertices at each vertex u_i, for $1 \leq i \leq n$ and $x_1^{(i)}$ and $x_2^{(i)}$ be the pendant vertices at each vertex v_i, for $1 \leq i \leq n$. Therefore

$$V(L_n \odot S_2) = V(L_n) \cup \{w_1^{(i)}, w_2^{(i)}, x_1^{(i)}, x_2^{(i)}; 1 \leq i \leq n\}$$

and

$$E(L_n \odot S_2) = E(L_n) \cup \{u_i w_1^{(i)}, u_i w_2^{(i)}, v_i x_1^{(i)}, v_i x_2^{(i)}; 1 \leq i \leq n\}.$$

We define $f : V(L_n \odot S_2) \to \{1, 2, 3, \ldots, 7n-1\}$ as follows :

$$f(u_i) = \begin{cases} 3 & i = 1 \\ 7i-2 & 2 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-5 & 3 \leq i \leq n \text{ and } i \text{ is odd,} \end{cases}$$

$$f(v_i) = \begin{cases} 5 & i = 1 \\ 7i-4 & 2 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-1 & 3 \leq i \leq n \text{ and } i \text{ is odd,} \end{cases}$$

$$f(w_1^{(i)}) = \begin{cases} 7i-3 & 1 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-6 & 1 \leq i \leq n \text{ and } i \text{ is odd,} \end{cases}$$

$$f(w_2^{(i)}) = \begin{cases} 2 & i = 1 \\ 7i-1 & 2 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-4 & 3 \leq i \leq n \text{ and } i \text{ is odd,} \end{cases}$$

$$f(x_1^{(i)}) = \begin{cases} 7i-6 & 1 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-3 & 1 \leq i \leq n \text{ and } i \text{ is odd,} \end{cases}$$

$$f(x_2^{(i)}) = \begin{cases} 6 & i = 1 \\ 7i-5 & 2 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-2 & 3 \leq i \leq n \text{ and } i \text{ is odd.} \end{cases}$$

The induced edge labeling is as follows:

$$f^*(u_i u_{i+1}) = 7i-1, \ for \ 1 \leq i \leq n-1,$$
$$f^*(u_i v_i) = 7i-4, \ for \ 1 \leq i \leq n,$$
$$f^*(v_i v_{i+1}) = 7i, \ for \ 1 \leq i \leq n-1,$$

$$f^*(u_i w_1^{(i)}) = \begin{cases} 7i-3 & 1 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-6 & 1 \leq i \leq n \text{ and } i \text{ is odd,} \end{cases}$$

$$f^*(u_i w_2^{(i)}) = \begin{cases} 7i-2 & 1 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-5 & 1 \leq i \leq n \text{ and } i \text{ is odd,} \end{cases}$$

$$f^*(v_i x_1^{(i)}) = \begin{cases} 7i-6 & 1 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-3 & 1 \leq i \leq n \text{ and } i \text{ is odd,} \end{cases}$$

and

$$f^*(v_i x_2^{(i)}) = \begin{cases} 7i-5 & 1 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-2 & 1 \leq i \leq n \text{ and } i \text{ is odd.} \end{cases}$$

Hence, f is a F-Geometric mean labeling of $L_n \odot S_2$. Thus the graph $L_n \odot S_2$ is a F-Geometric mean graph, for $n \geq 2$. □

A F-Geometric mean labeling of $L_7 \odot S_2$ is as shown in Figure 14.

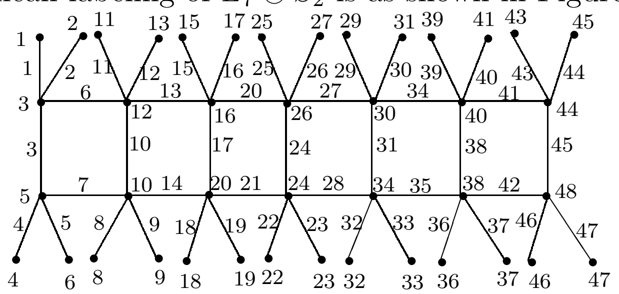

Figure 14. A F-Geometric mean labeling of $L_7 \odot S_2$

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