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A NOTE ON THE ISOTROPICAL GEOMETRY OF PRODUCTION SURFACES

SIMONA DECU[1] AND LEOPOLD VERSTRAELEN[2]

In a series of recent articles [1–4] Bang–Yen Chen a. o. obtained several nice *characterisations of economical properties of homogeneous production functions* $f : \mathbb{R}^n \to \mathbb{R}$, $(n \geq 2)$, *in terms of Euclidean geometrical properties of the corresponding production hypersurfaces* $M^n = \text{graph } f$ *in* $\mathbb{R}^{n+1}$; (for the definitions of the basic Euclidean and Riemannian geometrical notions as well as for the basic economical ones mentioned in the present text we refer to these articles and the references therein for their precise meanings and backgrounds). In particular, and, for the sake of simplicity of formulation, hereafter *restricting to the case of production functions* f *depending on* $n = 2$ *variables only and to their corresponding production surfaces* $M^2 = \text{graph } f$ *in* $\mathbb{R}^3$, as such we recall the following general new result of the following kind.

Theorem A. [1] *A homogeneous production function* $f : \mathbb{R}^2 \to \mathbb{R}$ *has constant return to scale if and only if the corresponding production surface has vanishing Gauss curvature, i.e. is a flat surface in the* 3*–dimensional Euclidean space* $\mathbb{E}^3$.

This result contains as special cases the also recently obtained following results of G. E. Vilcu and A. D. Vilcu.

Corollary B. [5, 6] *The two–factor Cobb–Douglas production function and the two–factor ACMS production function have constant returns to scale if and only if their corresponding production surfaces are flat surfaces in* $\mathbb{E}^3$.

For functions $f : \mathbb{R}^2 \to \mathbb{R}$ of two variables, say x and y, the corresponding graphs are the surfaces M^2 in $\mathbb{R}^3$ given by the equation, say $z = f(x, y)$, in Cartesian co–ordinates (x, y, z) in $\mathbb{R}^3$. *The Gauss curvature* K *of the Riemannian surfaces* (M^2, g), whereby g are *the Riemannian metrics induced on such surfaces* $M^2 : z = f(x, y)$ *in* $\mathbb{R}^3$ *from*

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the metric $ds^2 = dx^2 + dy^2 + dz^2$ *of the ambient Euclidean space* $\mathbb{E}^3 = (\mathbb{R}^3, ds^2)$ is given by $K = (f_{xx}f_{yy} - f_{xy}^2)/(1 + f_x^2 + f_y^2)^2$. Thus, *the vanishing of this Gauss curvature* K *is equivalent to the vanishing of the determinant* $\det H_f$ *of the Hessian matrix* H_f *of the function* $f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto z = f(x, y)$ *concerned.*

The point that we would like to make in the present paper essentially is the following; (cfr. [7]). In the modeling of several problems, like here describing a production, say f, in terms of some variables, say x and y, (or $x_1, \ldots, x_n$ for that matter), *qualitatively the values of the variables* x and y, (or the values of the variables $x_1, \ldots, x_n$), *and the corresponding values of* f, say $z = f(x, y)$, (or $z = f(x_1, \ldots, x_n)$), *are basically of different natures*. Therefore, it might not be considered to be altogether so unwise to assign *different measures* to the variables x and y, (or to the variables $x_1, \ldots, x_n$), on the one hand, and to the function values $z = f(x, y)$, (or $z = f(x_1, \ldots, x_n)$), on the other hand. And, amongst the possibilities of doing so in a more or less reasonable way, here *we will consider the surfaces* $M^2 : z = f(x, y)$ *in* $\mathbb{R}^3$, (or the hypersurfaces $M^n : z = f(x_1, \ldots, x_n)$ in $\mathbb{R}^{n+1}$), *as being situated in the one–fold isotropical space(s)* $I^{(3)} = (\mathbb{R}^3, ds_*^2 = dx^2 + dy^2)$, (or $I^{(n+1)} = (\mathbb{R}^{n+1}, ds_*^2 = dx_1^2 + \cdots + dx_n^2)$). In a kind of flexible analogy, this is similar to the standard geometrical description of the luminosity surfaces in the theory of vision [8, 9], whereby -though less outspoken than in the present case- the arguments x and y themselves too already are not exactly of the same quality either, but it does not harm so much to overlook these latter differences, at least in a first approach to a formal description of the realities under study. Also, this approach is well in accordance with such isotropical spaces after all naturally occurring as subspaces in the pseudo–Euclidean spaces which essentially base on our kind's visual–physical experiences in "our" space–time worlds; (cfr. [10]). The classical textbook reference for the geometry of surfaces $M^2 : z = f(x, y)$ in isotropical spaces $I^{(3)}$ is H. Sachs' *"Isotrope Geometrie des Raumes"* [11].

The metrics g_* *induced on such surfaces* $M^2 : z = f(x, y)$ *in* $\mathbb{R}^3$ *from the metric* $ds_*^2 = dx^2 + dy^2$ *of the one–fold isotropical space* $I^{(3)} = (\mathbb{R}^3, ds_*^2)$ *are always positive definite*, i.e. all surfaces (M^2, g_*) are properly Riemannian, and *all these Riemannian surfaces* (M^2, g_*) *are flat*, i.e. *the Gauss curvature* K, or still, *the intrinsic* or *absolute curvature* of all these surfaces (M^2, g_*) vanishes identically. Next, we will briefly comment on each of *the main extrinsic curvatures* of such surfaces $M^2 : z = f(x, y)$ in $I^{(3)} = (\mathbb{R}^3, ds_*^2)$. Firstly, *the relative curvature* $\tilde{K}$ of M^2 in $I^{(3)}$, which inspired by Euclidean surface theory is defined as $\tilde{K} = \tilde{k}_1.\tilde{k}_2$ whereby $\tilde{k}_1$ and $\tilde{k}_2$ are *the isotropical principal curvatures* of M^2 in $I^{(3)}$, is given by the determinant of the Hessian of $f : \tilde{K} = \det H_f = f_{xx}f_{yy} - f_{xy}^2$. Hence, in terms of the extrinsic geometry of surfaces in isotropical spaces, the previous results could be reformulated as follows.

Theorem 1.1. *A homogeneous production function* f *of two variables, (i.p. the two–factor Cobb–Douglas function and also the two–factor ACMS function), has constant return to scale if and only if the relative curvature of the corresponding production surface* M^2 *in* $I^{(3)}$ *vanishes identically.*

Secondly, *the isotropical mean curvature* $\tilde{H} = (\tilde{k}_1 + \tilde{k}_2)/2$ is determined by the Laplacian $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ of f: $\tilde{H} = (\Delta f)/2 = (f_{xx} + f_{yy})/2 = (\mathrm{tr} H_f)/2$, the *isotropical minimal surfaces* M^2 in $I^{(3)}$ thus being the graphs of *the harmonic functions* $f : \mathbb{R}^2 \to \mathbb{R}$. Finally, *the isotropical Casorati curvature* $\tilde{C} = (\tilde{k}_1^2 + \tilde{k}_2^2)/2 = 2\tilde{H} - \tilde{K}$ of $M^2 : z = f(x, y)$ in $I^{(3)}$ is given by $\tilde{C} = (f_{xx}^2 + f_{xy}^2 + f_{yx}^2 + f_{yy}^2)/2 = \frac{1}{2}\|H_f\|^2$, i.e. basically is given by the standard norm of the Hessian of f. Hence, any such graph surface M^2 in $I^{(3)}$ has *identically vanishing curvature* $\tilde{C}$ *if and only if* $f_{xx} = f_{xy} = f_{yx} = f_{yy} = 0$, i.e. if M^2 *is a plane in* $\mathbb{R}^3$. So, in particular the next proposition follows.

Proposition 1.1. *A production function $f : \mathbb{R}^2 \to \mathbb{R}$ is a perfect substitute if and only if the isotropical Casorati curvature of the corresponding production surface M^2 vanishes identically.*

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[1]Research Department,
SIAT, Romania
E-mail address: simona.decu@gmail.com

[2]Section of Geometry,
KU Leuven, Belgium
E-mail address: leopold.verstraelen@wis.kuleuven.be