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MULTIFUNCTIONAL ANALYTIC SPACES ON PRODUCTS OF BOUNDED STRICTLY PSEUDOCONVEX DOMAINS AND EMBEDDING THEOREMS

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Abstract. We provide new estimates for new multifunctional analytic spaces in products of pseudoconvex domains. We also obtain new sharp embedding theorems for mixed-norm analytic spaces in pseudoconvex domains.

1. Introduction

The theory of one functional analytic spaces on pseudoconvex domains is well-developed by various authors during last decades (see [4–6] and various references there). One of the goals of this paper among other things is to define for the first time in literature multifunctional analytic spaces in strictly pseudoconvex domains and to establish some basic properties of these spaces. We believe this new interesting objects can serve as a base for further generalizations and investigations in this active research area. Multifunctional spaces we mentioned above are closely connected also with so-called analytic function spaces on products of strictly pseudoconvex domains $D\times\cdots\times D$. Various such connections in analytic and harmonic function spaces were found and mentioned in [3, 7, 8]. We note basic properties of last spaces on product domains are closely connected on the other hand with so-called Trace operator [7, 8].We will add some new results related with Trace map for certain spaces of analytic functions on products of pseudoconvex domains. Next in second main part of paper we will turn to study of certain embedding theorems for some new mixed norm analytic classes in strictly pseudoconvex domains in $\mathbb{C}^n$.We note that in this paper we extend some theorems from [3] and [30] where they can be seen in context of unit ball. Proving estimates and embedding theorems in pseudoconvex domains we heavily use

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the technique which was developed recently in [1, 2]. In our embedding theorems and inequalities for analytic function spaces in pseudoconvex domains with smooth boundary the so-called Carleson- type measures constantly appear. We add some historical remarks on this important topic now. Carleson measures were introduced by Carleson [17] in his solution of the corona problem in the unit disk of the complex plane, and, since then, have become an important tool in analysis, and an interesting object of study *per se*. Let A be a Banach space of holomorphic functions on a domain $D \subset \mathbb{C}^n$; given $p \geq 1$, a finite positive Borel measure μ on D is a *Carleson measure* of A (for p) if there is a continuous inclusion $A \hookrightarrow L^p(\mu)$, that is there exists a constant $C > 0$ such that for every $f \in A$

$$\int_D |f|^p d\mu \leq C\|f\|_A^p,$$

we shall furthermore say that μ is a *vanishing Carleson measure* of A if the inclusion $A \hookrightarrow L^p(\mu)$ is compact.

Carleson studied this property [17] taking as Banach space A the Hardy spaces in unit disk Δ $H^p(\Delta)$, and proved that a finite positive Borel measure μ is a Carleson measure of $H^p(\Delta)$ for p if and only if there exists a constant $C > 0$ such that $\mu(S_{\theta_0,h}) \leq Ch$ for all sets

$$S_{\theta_0,h} = \{re^{i\theta} \in \Delta| \;\; 1-h \leq r < 1, \; |\theta - \theta_0| < h\}$$

(see also [19, 24]); in particular the set of Carleson measures of $H^p(\Delta)$ does not depend on p.

In 1975, Hastings [20] (see also Oleinik and Pavlov [24] and Oleinik [23]) proved a similar characterization for the Carleson measures of the Bergman spaces $A^p(\Delta)$, still expressed in terms of the sets $S_{\theta_0,h}$. Later Cima and Wogen [18] characterized Carleson measures for Bergman spaces in the unit ball $B^n \subset \mathbb{C}^n$, and Cima and Mercer [5] characterized Carleson measures of Bergman spaces in strongly pseudoconvex domains, showing in particular that the set of Carleson measures of $A^p(D)$ is independent of $p \geq 1$.

Cima and Mercer's characterization of Carleson measures of Bergman spaces is expressed using interesting generalizations of the sets $S_{\theta_0,h}$; for our aims, it will be more useful a different characterization, expressed via the intrinsic Kobayashi geometry of the domain. Given $z_0 \in D$ and $0 < r < 1$, let $B_D(z_0, r)$ denote the ball of center z_0 and radius $\frac{1}{2}\log\frac{1+r}{1-r}$ for the Kobayashi distance k_D of D (that is, of radius r with respect to the pseudohyperbolic distance $\rho = \tanh(k_D)$; see Section 2 for the necessary definitions). Then it is possible to prove (see Luecking [22] for $D = \Delta$, Duren and Weir [16] and Kaptanoğlu [21] for $D = B^n$, and [1, 2] for D strongly pseudoconvex) that a finite positive measure μ is a Carleson measure of $A^p(D)$ for p if and only if for some (and hence all) $0 < r < 1$ there is a constant $C_r > 0$ such that

$$\mu(B_D(z_0, r)) \leq C_r \nu(B_D(z_0, r))$$

for all $z_0 \in D$. (The proof of this equivalence in [2] relied on Cima and Mercer's characterization [5]).

Thus we will have a new geometrical characterization of Carleson measures of Bergman spaces, and it turns out that this *geometrical* characterization is very important for the study of the various properties of Toeplitz operators; but first it is necessary to widen the class of Carleson measures under consideration. Given $\theta > 0$, we say that a finite positive Borel measure μ is a (geometric)*θ-Carleson measure* if for some (and hence all) $0 < r < 1$ there is a constant $C_r > 0$ such that

$$\mu(B_D(z_0, r)) \leq c_r \nu(B_D(z_0, r))^\theta$$

for all $z_0 \in D$; and we shall say that μ is a (geometric) *vanishing θ-Carleson measure* if for some (and hence all) $0 < r < 1$ the quotient $\frac{\mu(B_D(z_0,r))}{\nu(B_D(z_0,r))^\theta}$ tends to 0 as $z_0 \to \partial D$. Note a 1-Carleson measures are usual Carleson measures of $A^p(D)$, and we know [1, 2] that θ-Carleson measures are exactly the Carleson measures of suitably weighted Bergman spaces. Note also that when $D = B^n$ a q-Carleson measure in the sense of [21, 34] is a $(1 + \frac{q}{n+1})$-Carleson measure in our sense.

In this paper we are however more interested in Carleson type measures for some new Bergman-type mixed norm spaces.

Throughout this paper constants are denoted by C and C_i, $\quad i = 1, \dots,$ they are positive and may not be the some at each occurrence.

2. Preliminaries on geometry of strongly pseudoconvex domains

In this section we provide a chain of facts,properties and estimates on the geometry of strongly convex domains which we will use heavily in all our proofs below. Practically all of them are taken from recent interesting papers of Abate and coauthors [1, 2]. In particular, we following these papers provide several results on the boundary behavior of Kobayashi balls, and we formulate a vital submean property for nonnegative plurisubharmonic functions in Kobayashi balls.

We now recall first the standard definition and the main properties of the Kobayashi distance which can be seen in various books and papers; we refer for example to [12, 13] and [14] for details . Let k_Δ denote the Poincare distance on the unit disk $\Delta \subset \mathbb{C}^n$. If X is a complex manifold, the Lempert function δ_X:$X \times X \to \mathbb{R}^+$ of X is defined by

$$\delta_X(z, w) = \inf\{k_\Delta(\zeta, \eta) | \text{there exists a holomorphic } \phi : \Delta \to X \text{ with } \phi(\zeta) = z \text{ and } \phi(\eta) = w\}$$

for all $z, w \in X$. The Kobayashi pseudodistance $k_X : X \times X \to \mathbb{R}^+$ of X is the smallest pseudodistance on X bounded below by δX. We say that X is (Kobayashi) hyperbolic if k_X is a true distance — and in that case it is known that the metric topology induced by k_X coincides with the manifold topology of X (see, e.g., Proposition 2.3.10 in [12]). For instance, all bounded domains are hyperbolic (see, e.g., Theorem 2.3.14 in [12]). The following properties are well known in literature. The Kobayashi (pseudo)distance is contracted by holomorphic maps: if $f : X \to Y$ is a holomorphic map then for all $z, w \in X$

$$k_Y(f(z), f(w)) \le k_X(z, w).$$

Next the Kobayashi distance is invariant under biholomorphisms, and decreases under inclusions: if $D_1 \subset D_2 \subset\subset \mathbb{C}^n$ are two bounded domains we have $k_{D_2}(z, w) \le k_{D_1}(z, w)$ for all $z, w \in D_1$. Further the Kobayashi distance of the unit disk coincides with the Poincare distance. Also, the Kobayashi distance of the unit ball $B^n \subset \mathbb{C}^n$ coincides with the well known in many applications so-called Bergman distance (see, e.g., Corollary 2.3.6 in [12], see also [30, 34]).

If X is a hyperbolic manifold, $z_0 \in X$ and $r \in (0; 1)$ we shall denote by $B_X(z_0, r)$ the Kobayashi ball of center z_0 and radius $\frac{1}{2} \log \frac{1+r}{1-r}$

$$B_X(z_0, r) = \{z \in X | \quad \tanh\ k_X(z_0, z) < r\}.$$

We can see that $\rho_X = \tanh\ k_X$ is still a distance on X, because tanh is a strictly convex function on $\mathbb{R}^+$. In particular, ρ_{B^n} is the pseudohyperbolic distance of B^n.

The Kobayashi distance of bounded strongly pseudoconvex domains with smooth boundary has several important properties. First of all, it is complete (see, e.g., Corollary 2.3.53 in [12]), and hence closed Kobayashi balls are compact. It is vital that we can describe the boundary behavior of the Kobayashi distance: if $D \subset\subset \mathbb{C}^n$ is a strongly pseudoconvex bounded domain and $z_0 \in D$, there exist $c_0, C_0 > 0$ such that for every $z \in D$

$$c_0 - \frac{1}{2} \log d(z, \partial D) \le k_D(z_0, z) \le C_0 - \frac{1}{2} \log d(z, \partial D),$$

where $d(\cdot, \partial D)$ denotes the Euclidean distance from the boundary of D (see Theorems 2.3.51 and 2.3.52 in [12]). We provide some facts on Kobayashi balls of B^n; for proofs see Section 2.2.2 in [12], Section 2.2.7 in [15] and [16]. The ball $B_{B^n}(z_0, r)$ is given by

$$B_{B^n}(z_0, r) = \{z \in B^n | \quad \frac{(1 - \|z_0\|^2)(1 - \|z\|^2)}{|1 - \langle z, z_0 \rangle|^2} > 1 - r^2\}.$$

Geometrically, it is an ellipsoid of (Euclidean) center

$$c = \frac{1 - r^2}{1 - r^2 \|z_0\|^2} z_0,$$

its intersection with the complex line C_{z_0} is an Euclidean disk of radius

$$r \frac{1 - \|z_0\|^2}{1 - r^2 \|z_0\|^2},$$

and its intersection with the affine subspace through z_0 orthogonal to z_0 is an Euclidean ball of the larger radius

$$r \sqrt{\frac{1 - \|z_0\|^2}{1 - r^2 \|z_0\|^2}}.$$

Let ν denote the Lebesque volume measure of R^{2n}, normalized so that $\nu(B^n) = 1$. Then the volume of a Kobayashi ball $B_{B^n}(z_0, r)$ is given by (see [16])

$$\nu(B_{B^n}(z_0, r)) = r^{2n}\Big(\frac{1 - \|z_0\|^2}{1 - r^2\|z_0\|^2}\Big)^{n+1}.$$

A similar estimate is valid for the volume of Kobayashi balls in strongly pseudoconvex bounded domains:

Lemma 2.1. [1, 2] *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex bounded domain. Then there exist $c_1 > 0$ and, for each $r \in (0; 1)$, a $C_{1,r} > 0$ depending on r such that*

$$c_1 r^{2n} d(z_0, \partial D)^{n+1} \le \nu(B_D(z_0, r)) \le C_{1,r} d(z_0, \partial D)^{n+1}$$

for every $z_0 \in D$ and $r \in (0, 1)$.

Let $d\nu_t(z) = (\delta(z))^t d\nu(z)$, $t > -1$. Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$. We shall use the following notations:

- $\delta : D \to \mathbb{R}^+$ will denote the Euclidean distance from the boundary, that is $\delta(z) = d(z, \partial D)$;
- given two non-negative functions $f, g : D \to \mathbb{R}^+$ we shall write $f \preceq g$ to say that there is $C > 0$ such that $f(z) \le Cg(z)$ for all $z \in D$. The constant C is independent of $z \in D$, but it might depend on other parameters (r, θ, etc.);
- given two strictly positive functions $f, g : D \to \mathbb{R}^+$ we shall write $f \approx g$ if $f \preceq g$ and $g \preceq f$, that is if there is $C > 0$ such that $C^{-1}g(z) \le f(z) \le Cg(z)$ for all $z \in D$;
- ν will be the Lebesque measure;
- $H(D)$ will denote the space of holomorphic functions on D, endowed with the topology of uniform convergence on compact subsets;
- given $1 \le p \le +\infty$, the *Bergman space* $A^p(D)$ is the Banach space $L^p(D) \cap H(D)$, endowed with the L^p-norm;
- more generally, given $\beta \in \mathbb{R}$ we introduce the *weighted Bergman space*

  $$A^p(D, \beta) = L^P(\delta^\beta \nu) \cap H(D)$$

  endowed with the norm

  $$\|f\|_{p,\beta} = \Big[\int_D |f(\zeta)|^p \delta^\beta(\zeta) d\nu(\zeta)\Big]^{\frac{1}{p}}$$

  if $1 \le p < \infty$, and with the norm

  $$\|f\|_{\infty,\beta} = \|f\delta^\beta\|_\infty$$

  if $p = \infty$;
- $K : D \times D \to \mathbb{C}$ will be the Bergman kernel of D; The K_t is a kernel of type t, see [35]. Note $K = K_{n+1}$ (see [1, 35]);

- for each $z_0 \in D$ we shall denote by $k_{z_0} : D \to \mathbb{C}$ the normalized Bergman kernel defined by
$$k_{z_0}(z) = \frac{K(z,z_0)}{\sqrt{K(z_0,z_0)}} = \frac{K(z,z_0)}{\|K(\cdot,z_0)\|_2};$$
- given $r \in (0,1)$ and $z0 \in D$, we shall denote by $B_D(z_0,r)$ the Kobayashi ball of center z_0 and radius $\frac{1}{2}\log\frac{1+r}{1-r}$.

See, e.g., [12–14, 25] for definitions, basic properties and applications to geometric function theory of the Kobayashi distance; and [26–29] for definitions and basic properties of the Bergman kernel. Let us now recall a number of results proved in [2]. The first two give information about the shape of Kobayashi balls:

Lemma 2.2. [2, Lemma 2.1] *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, and $r \in (0,1)$. Then*
$$\nu(B_D(\cdot,r)) \approx \delta^{n+1},$$
(where the constant depends on r).

Lemma 2.3. [2, Lemma 2.2] *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then there is $C > 0$ such that*
$$\frac{C}{1-r}\delta(z_0) \geq \delta(z) \geq \frac{1-r}{C}\delta(z_0)$$
for all $r \in (0,1), z_0 \in D$ and $z \in B_D(z_0,r)$.

Definition 2.1. Let $D \subset\subset \mathbb{C}^n$ be a bounded domain, and $r > 0$. An r-lattice in D is a sequence $\{a_k\} \subset D$ such that $D = \bigcup\limits_k B_D(a_k,r)$ and there exists $m > 0$ such that any point in D belongs to at most m balls of the form $B_D(a_k,R)$, where $R = \frac{1}{2}(1+r)$.

The existence of r-lattices in bounded strongly pseudoconvex domains is ensured by the following

Lemma 2.4. [2, Lemma 2.5] *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then for every $r \in (0,1)$ there exists an r-lattice in D, that is there exist $m \in N$ and a sequence $\{a_k\} \subset D$ of points such that $D = \bigcup\limits_{k=0}^{\infty} B_D(a_k,r)$ and no point of D belongs to more than m of the balls $B_D(a_k,R)$, where $R = \frac{1}{2}(1+r)$, $\nu_\alpha(B_D(a_k,R)) = (\delta^\alpha(a_k))\nu(B_D(a_k,R)), \alpha > -1$.*

We will call r-lattice sometimes the family $B_D(a_k,r)$. Dealing with K kernel we always assume $K(z,a_k) \asymp K(a_k,a_k)$ for any $z \in B_D(a_k,r)$, $r \in (0;1)$ (see [1, 2]). We shall use a submean estimate for nonnegative plurisubharmonic functions on Kobayashi balls.

Lemma 2.5. [2, Corollary 2.8] *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Given $r \in (0,1)$, set $R = \frac{1}{2}(1+r) \in (0,1)$. Then there exists a $C_r > 0$ depending on r such that for all $z_0 \in D$ and for all $z \in B_D(z_0, r)$*

$$\chi(z) \leq \frac{C_r}{\nu(B_D(z_0,r))} \int\limits_{B_D(z_0,R)} \chi d\nu$$

for every nonnegative plurisubharmonic function $\chi : D \to \mathbb{R}^+$.

We will use this lemma for $\chi = |f(z)|^q, f \in H(D), q \in (0;\infty)$. Obviously using properties of $\{B_D(a_k, R)\}$ Kobayashi balls we have the following estimates for Bergman space $A^p_\alpha(D)$

$$\|f\|^p_{A^p_\alpha} = \int\limits_D |f(w)|^p \delta^\alpha(w) d\nu(w) \asymp \sum_{k=1}^{\infty} [\max_{z \in B_D(a_k,R)} |f(z)|^p] \nu_\alpha B_D(a_k, R)$$
$$\asymp \sum_{k=1}^{\infty} \int\limits_{B_D(a_k,R)} |f(z)|^p \delta^\alpha(z) d\nu(z), \quad 0 < p < \infty, \quad \alpha > -1.$$

Let now

$$A(p,q,\alpha) = \left\{ f \in H(D) : \sum_{k=1}^{\infty} \Big(\int\limits_{B_D(a_k,R)} |f(z)|^p \delta^\alpha(z) d\nu(z) \Big)^{\frac{q}{p}} < \infty \right\},$$

where $0 < p, q < \infty,\ \alpha > -1$. These are Banach spaces if $\min(p,q) \geq 1$.

These $A(p,q,\alpha)$ spaces (or their multifunctional generalizations) can be considered as natural extensions of classical Bergman spaces in strictly D pseudoconvex domains with smooth boundary for which $\{B_D(a_k, R)\}$ family exists related to r-lattice $\{(a_k)\}$ (see [1, 2]). It is natural to consider the problem of extension of classical results on $A^p_\alpha(D)$ spaces to these $A(p,q,\alpha)$ spaces. Some our results are motivated with this problem.

We now collect a few facts on the (possibly weighted) L^p-norms of the Bergman kernel and the normalized Bergman kernel. The first result is classical (see, e.g., [1, 2]).

Lemma 2.6. [1] *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then*

$$\|K(\cdot, z_0)\|_2 = \sqrt{K(z_0,z_0)} \approx \delta^{-\frac{n+1}{2}}(z_0) \quad \text{and} \quad \|k_{z_0}\|_2 \equiv 1$$

for all $z_0 \in D$.

The next result is the main result of this section, and contains the weighted L^p-estimates we shall need.

Theorem 2.1. [1] *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, and let $z_0 \in D$ and $1 \le p < \infty$. Then*

$$\int_D |K(\zeta, z_0)|^p \delta^\beta(\zeta) d\nu(\zeta) \preceq \begin{cases} \delta^{\beta-(n+1)(p-1)}(z_0), & for\ -1 < \beta < (n+1)(p-1); \\ |\log \delta(z_0)|, & for\ \beta = (n+1)(p-1); \\ 1, & for\ \beta > (n+1)(p-1). \end{cases}$$

In particular:

(i) $\|K(\cdot, z_0)\|_{p,\beta} \preceq \delta^{\frac{\beta}{p}-\frac{n+1}{q}}(z_0)$ *and* $\|k_{z_0}\|_{p,\beta} \preceq \delta^{\frac{n+1}{2}+\frac{\beta}{p}-\frac{n+1}{q}}(z_0)$ *when* $-1 < \beta < (n+1)(p-1)$*, where $q > 1$ is the conjugate exponent of p (and $\frac{n+1}{q} = 0$ when $p = 1$);*

(ii) $\|K(\cdot, z_0)\|_{p,\beta} \preceq 1$ *and* $\|k_{z_0}\|_{p,\beta} \preceq \delta^{\frac{n+1}{2}}(z_0)$ *when* $\beta > (n+1)(p-1)$*;*

(iii) $\|K(\cdot, z_0)\|_{p,(n+1)(p-1)} \preceq \delta^{-\varepsilon}(z_0)$ *and* $\|k_{z_0}\|_{p,(n+1)(p-1)} \preceq \delta^{\frac{n+1}{2}-\varepsilon}(z_0)$ *for all $\varepsilon > 0$.*

Furthermore,

(iv) $\|K(\cdot, z_0)\|_{\infty,\beta} \approx \delta^{\beta-(n+1)}(z_0)$ *and* $\|k_{z_0}\|_{\infty,\beta} \approx \delta^{\beta-\frac{n+1}{2}}(z_0)$ *for all* $0 \le \beta < n+1$*; and* $\|K(\cdot, z_0)\|_{\infty,\beta} \approx 1$ *and* $\|k_{z_0}\|_{\infty,\beta} \approx \delta^{\frac{n+1}{2}}(z_0)$ *for all* $\beta \ge n+1$

Note again results from section 3 in context of unit ball can be seen in [3], while all results of section 4 can be seen in [30] in case of unit ball, all our proofs hence are sketchy since arguments are similar.

A complete analogue of this theorem is valid also for K_t kernel $t > 0$ (see [35]).

3. Multifunctional analytic spaces in pseudoconvex domains with smooth boundary

We will need for all proofs various properties of r-lattice of D (see [1, 2]) and various nice properties of Kobayashi balls from recent papers [1] and [2]. We listed all these properties in detail in previous section.

Theorem 3.1. *Let $\alpha > -1, F_i \in H(D \times \cdots \times D), i = 1, \ldots, m, D^t = D \times \cdots \times D, t \in \mathbb{N}$. Let $0 < p_i, q_i < \infty, i = 1, \ldots, m$ so $\sum_{i=1}^m \frac{p_i}{q_i} = 1$. Then we have*

$$\int_D |F_1(w, \ldots, w)|^{p_1} \ldots |F_m(w, \ldots, w)|^{p_m} \delta^\alpha(w) d\nu(w)$$

$$\le c \prod_{i=1}^m \left(\int_D \cdots \int_D |F_i(w_1, \ldots, w_t)|^{q_i} \prod_{j=1}^t \delta^{\beta_i}(w_j) d\nu(w_j) \right)^{\frac{p_i}{q_i}},$$

where $\beta_i = \frac{(n+1+\alpha)q_i}{tmp_i} - (n+1) > -1, i = 1, \ldots, m$.

If all $p_i = q_j = p$ above then we get the "limit case" of Theorem 3.1.

Theorem 3.2. *Let $F_i \in H(D \times \cdots \times D), i = 1, \ldots, m, D^t = D \times \cdots \times D, t \in \mathbb{N}$, $\alpha > -1$, $\alpha > tn - n - 1$, $\beta_i > -1$, $0 < p < \infty$. Then*

$$\int_D \prod_{i=1}^m |F_i(w, \ldots, w)|^p \delta^\alpha(w) d\nu(w) \le c \int_D \ldots \int_D \prod_{i=1}^m |F_i(w_1, \ldots, w_t)|^p \prod_{j=1}^t \delta^{\beta_j}(w_j) d\nu(w_j),$$

where

$$\beta_i = \frac{(n+1+\alpha)}{t} - (n+1), i = 1, \ldots, t.$$

Remark 3.1. Note for $t = 1, m = 1$ these estimates are obvious. For $m = 1, t > 1$ in case of (unit disk, polydisk) these estimates can be found in [8]. For case of unit ball $D = B^n \subset C^n$ these results can be found in [3].

Theorem 3.3. *Let $f_k(z_1, \ldots, z_t) \in H(D \times \cdots \times D)$, $0 < p_k, q_k < \infty$, $k = 1, \ldots, m$, so that $\sum_{j=1}^m \frac{p_j}{q_j} = 1$. Let $l_s > -1$, $\alpha_j^s > -1$, $s = 1, \ldots, t$, $j = 1, \ldots, m$ so that $\frac{q_j l_s}{m p_j} = n + 1 + \alpha_j^s$, $s = 1, \ldots, t$, $j = 1, \ldots, m$. Then we have for $\{a_k\}$ - r-lattice*

$$\sum_{k=1}^\infty \prod_{j=1}^m |f_j(a_k, \ldots, a_k)|^{p_j} \Big(\delta(a_k)\Big)^{\sum_{i=1}^t l_i}$$
$$\le c \prod_{j=1}^m \left(\int_D \cdots \int_D |f_j(z_1, \ldots, z_t)|^{q_j} d\nu_{\alpha_j^1}(z_1) \ldots d\nu_{\alpha_j^t}(z_t) \right)^{\frac{p_j}{q_j}}.$$

Remark 3.2. For case of unit ball $B \subset C^n$ these results can be found in [3]. For $m = 1$, $t = 1$ this result for unit ball can be found in [3] and [34]. For unit disk $D = \{z \in C : |z| < 1\}$ these estimates were found much earlier by various authors (see for example [33, 34] and references there).

Theorem 3.4. *Let μ be a positive Borel measure on D and let $\{a_k\}$ be a sequence an r-lattice from Kobayashi balls.*

(i) *Let $f_j \in H(D), j = 1, \ldots, m$, $0 < p_i, q_i < \infty$, $i = 1, \ldots, m+1$, $\sum_{i=1}^{m+1} \frac{p_i}{q_i} = 1$. If $\left(\sum_{k=1}^\infty \Big(\int_{B_D(a_k,R)} d\mu(z) \Big)^{\frac{q_{m+1}}{p_{m+1}}} \right)^{\frac{p_{m+1}}{q_{m+1}}} \le c$, then*

$$\int_D \prod_{i=1}^m |f_i(z)|^{p_i} \delta(z)^{m(n+1)} d\mu(z) \le c \left[\prod_{i=1}^m \left(\sum_{k=1}^\infty \Big(\int_{B_D(a_k,R)} |f_i(z)|^{p_i} d\nu(z) \Big)^{\frac{q_i}{p_i}} \right)^{\frac{p_i}{q_i}} \right].$$

(ii) *Let* $f_j \in H(D), j = 1, \dots, m,\ 0 \le p_i < q_i < \infty, i = 1, \dots, m+1,$ *so that* $\sum_{i=1}^{m+1} \frac{p_i}{q_i} = 1$. *If*

$$\left(\sum_{k=1}^{\infty}\Big(\int\limits_{B_D(a_k,R)} d\mu(z)\Big)^{\frac{q_{m+1}}{p_{m+1}}}\right)^{\frac{p_{m+1}}{q_{m+1}}} \le c < \infty,\ R \in (0,1); \tag{3.1}$$

then we have following estimate

$$\int\limits_D \prod_{i=1}^m |f_i(z)|^{p_i} (\delta(z))^{(n+1)\sum_{i=1}^m \frac{p_i}{q_i}} d\mu(z) \le c \prod_{i=1}^m \left(\int\limits_D |f_i(z)|^{q_i} d\nu(z)\right)^{\frac{p_i}{q_i}}.$$

Below based on preliminaries we provided complete proofs of our assertions will be given, some proofs are missed. We refer the reader for them to [3], where analogues for unit ball can be found. The main idea is the adaptation of r-lattice of D to r-lattice of unit ball and we leave this partially to readers.

Various results on product domains D^m can be seen in [9] and for other product domains in [10]. Hence our results can be seen as good completion of results from [9] and [10]. All these results can be seen also as direct extentions of estimates previously known in polydisk which is the simple case.

Again our proofs are paralleled to the unit ball case and we will omit some of them here. We (shortly speaking) should heavily use in all proofs certain nice properties of r-lattice in bounded strictly pseudoconvex domains which was introduced in [1]. In the case of unit ball we heavily used similar properties of an r-lattice, but for unit ball (see [3]), moreover our arguments are similar.

The proof of Theorem 3.1. Let $\{a_k\}$ be an R-lattice. Using properties we listed above we have

$$\begin{aligned} I &= \int\limits_D \prod_{i=1}^m |F_i(w,\dots,w)|^{p_i} \delta^\alpha(w) d\nu(w) \\ &\le c \sum_{k=1}^\infty \int\limits_{B_D(a_k,R)} \prod_{i=1}^m |F_i(w,\dots,w)|^{p_i} \delta^\alpha(w) d\nu(w) \\ &\le c \sum_{k=1}^\infty \sup_{z \in B_D(a_k,R)} \prod_{i=1}^m |F_i(z,\dots,z)|^{p_i} \nu_\alpha(B_D(a_k,R)) \end{aligned}$$

$$\leq c\sum_{k_1=1}^{\infty}\cdots\sum_{k_t=1}^{\infty}\Big[\sup_{\substack{z_1\in B_D(a_{k_1},R)\\ \cdots\cdots\cdots\\ z_t\in B_D(a_{k_t},R)}}|F_1|^{p_1}\ldots\sup_{\substack{z_1\in B_D(a_{k_1},R)\\ \cdots\cdots\cdots\\ z_t\in B_D(a_{k_t},R)}}|F_m|^{p_m}\ldots\Big]$$
$$\times\Big[\delta^{\frac{n+1+\alpha}{t}}(a_{k_1})\ldots\delta^{\frac{n+1+\alpha}{t}}(a_{k_t})\Big],$$

where $F_1=F_1(z_1,\ldots,z_t), F_m=F_m(z_1,\ldots,z_t)$, $\nu_\alpha(B_D(a_k,R))=(\delta^\alpha(a_k))\nu(B_D(a_k,R))$, $\alpha>-1$.

Using Holder inequality for m-functions and again properties of r-lattice we listed above we have

$$I\leq c\prod_{s=1}^{m}\Bigg(\sum_{k_1,\ldots,k_t=1}^{\infty}\sup_{\substack{z_1\in B_D(a_{k_1},R)\\ \cdots\cdots\cdots\\ z_t\in B_D(a_{k_t},R)}}|F_s(z_1,\ldots,z_t)|^{q_s}\prod_{s=1}^{t}\delta^{\frac{(n+1+\alpha)q_s}{tmp_s}}(a_{k_s})\Bigg)^{\frac{p_s}{q_s}}$$

$$\leq c\prod_{s=1}^{m}\Bigg(\sum_{k_1,\ldots,k_t=1}^{\infty}\int\limits_{B_D(a_{k_1},\widetilde{R})}\cdots\int\limits_{B_D(a_{k_t},\widetilde{R})}|F_s(w_1,\ldots,w_t)|^{q_s}d\nu(w_1)\ldots d\nu(w_t)\prod_{j=1}^{t}\delta^{\beta_j}(a_{k_j})\Bigg)^{\frac{p_s}{q_s}}$$

$$\leq c\Bigg(\int\limits_D\cdots\int\limits_D|F_1(w_1,\ldots,w_t)|^{q_1}\prod_{s=1}^{t}(1-|w_s|)^{\beta_s}d\nu(w_1)\ldots d\nu(w_t)\Bigg)^{\frac{p_1}{q_1}}\times\cdots\times$$
$$\times\Bigg(\int\limits_D\cdots\int\limits_D|F_m(w_1,\ldots,w_t)|^{q_m}\prod_{j=1}^{t}(1-|w_j|)^{\beta_j}d\nu(w_1)\ldots d\nu(w_t)\Bigg)^{\frac{p_m}{q_m}},$$

where $\widetilde{R}=\frac{1+R}{2}, R\in(0,1)$. □

The proof of Theorem 3.2 can be obtained by small modification of Theorem 2.1 and we omit here details, refereing the reader also to unit ball case (see [3]).

The proof of Theorem 3.3 has similarities with the proof we provided above and with unit ball case (see [3]) from our paper and we again omit details. The base of the proof (as in unit ball case) is the following obvious basic inequality

$$\sum_{k=1}^{\infty}\sup_{z\in B_D(a_k,R)}\prod_{j=1}^{m}|f_j(z,\ldots,z)|^{p_j}(\delta^{\sum_{i=1}^{t}l_i}(z))$$
$$\leq c\Bigg[\sum_{k_1,\ldots,k_t=1}^{\infty}\sup_{\substack{z_1\in B_D(a_{k_1},R)\\ \cdots\cdots\cdots\\ z_t\in B_D(a_{k_t},R)}}|f_1(z_1,\ldots,z_t)|^{p_1}\Big(\prod_{k=1}^{t}\delta^{l_k^1}(z_k)\Big)\Bigg]\times$$

$$\times\cdots\times\Bigg[\sum_{k_1,\dots,k_t=1}^{\infty}\sup_{\substack{z_1\in B_D(a_{k_1},R)\\ \cdots\cdots\cdots\\ z_t\in B_D(a_{k_t},R)}}|f_m(z_1,\dots,z_t)|^{p_m}\Big(\prod_{k=1}^{t}\delta^{l_k^m}(z_k)\Big)\Bigg],$$

where $\sum_{j=1}^{t} l_k^j = l_k,\ m\in\mathbb{N}, t\in\mathbb{N}, R\in(0,1)$.

The proof of Theorem 3.4. We again follow arguments of unit ball case and properties of r-lattice from [1] which we listed above we omit the first part referring to our mentioned paper and concentrate only on proof of second part of theorem.

Note also the proof of first part has similarities with the proof of second part below. We have the following estimates. Suppose (3.1) holds then we have by Holder inequality

$$\begin{aligned}
J &= \int_D \prod_{i=1}^{m}|f_i(z)|^{p_i}\Big(\delta^{(n+1)\sum_{i=1}^{m}\frac{p_i}{q_i}}(z)\Big)d\mu(z)\\
&= \sum_{k=1}^{\infty}\int_{B_D(a_k,r)}\Big(\prod_{i=1}^{m}|f_i(z_i)|^{p_i}\Big)\delta^{\tau}(z)d\mu(z)\\
&\le c\sum_{k=1}^{\infty}\prod_{i=1}^{m}\sup_{z\in B_D(a_k,r)}|f_i(z)|^{p_i}\int_{B_D(a_k,r)}\delta^{(n+1)\sum_{i=1}^{m}\frac{p_i}{q_i}}(z)d\mu(z)\\
&\le c\Big(\sum_{k=1}^{\infty}\sup_{z\in B_D(a_k,r)}|f_1(z)|^{q_1}\delta^{n+1}(z)\Big)^{\frac{p_1}{q_1}}\times\cdots\times\Big(\sum_{k=1}^{\infty}\sup_{z\in B_D(a_k,r)}|f_m(z)|^{q_m}\delta^{n+1}(z)\Big)^{\frac{p_m}{q_m}}\\
&\quad\times\Big(\sum_{k=1}^{\infty}\Big[\int_{B_D(a_k,r)}d\mu(z)\Big]^{\frac{q_{m+1}}{p_{m+1}}}\Big)^{\frac{p_{m+1}}{q_{m+1}}}\\
&\le c\prod_{i=1}^{m}\Big(\int_D|f_i(z)|^{q_i}d\nu(z)\Big)^{\frac{p_i}{q_i}}\Big(\sum_{k=1}^{\infty}\Big[\int_{B_D(a_k,r)}d\mu(z)\Big]^{\frac{q_{m+1}}{p_{m+1}}}\Big)^{\frac{p_{m+1}}{q_{m+1}}}\\
&\le c\prod_{i=1}^{m}\Big(\int_D|f_i(z)|^{q_i}d\nu(z)\Big)^{\frac{p_i}{q_i}}.
\end{aligned}$$

The theorem is proved. □

Remark 3.3. We note that various (not sharp embedding theorems) can be obtained from the following simple observation also related with r-lattices for various embedding of type (we give only one function model)

$$\int_D |f(z)|^p d\mu(z)\le c\|f\|_Y^p,$$

where Y is a holomorphic subspace of $H(D)$. We have for $\{a_k\}$ - r-lattice

$$\int_D |f(z)|^p \delta^\alpha(z) d\mu(z) \le \sum_{k=1}^{\infty} \max_{z \in B_D(a_k,r)} |f(z)|^p \delta^\alpha(z) \mu(B_D(a_k,r)) \le c \int_D |f(z)|^p g_1(z) \delta^\alpha(z) d\nu(z),$$

where $g_1(z) = \sum_{k=1}^{\infty} \delta^{-(n+1)}(a_k)[\mu(B_D(a_k,r))][\chi_{B_D(a_k,r)}(z)],\ z \in D, 0 < p < \infty,\ \alpha > -1.$

Remark 3.4. Note also our estimates can be partially extended to some mixed norm spaces defined on product domains. For these spaces we refer the reader to [11].

Note for that an embedding theorem from [11] should be used at final step of proofs above.

We define these mixed norm spaces on product domains as spaces with quazinorms (see [11] for $m = 1$ case)

$$\sum_{|\alpha_1| \le k_1} \sum_{|\alpha_m| \le k_m} \int_0^{r_0^1} \dots \int_0^{r_0^m} \left(\int_{\partial D_r^1} \dots \int_{\partial D_r^m} |D^{\overrightarrow{\alpha}}_{z_1..z_m} f|^p d\sigma_r \right)^{\frac{q}{p}} r_1^{\delta \frac{q}{p} - 1} \dots r_m^{\delta \frac{q}{p} - 1} dr_1 \dots dr_m,$$

where $0 < p < \infty, 0 < q \le \infty, \delta > 0, D_r = \{z : \rho(z) < -r\}$; ∂D_r it is boundary, by $d\sigma_r$ we denote the normalized Lebesque measure on ∂D_r , where $D^{\overrightarrow{\alpha}} f$ is fractional derivative of f.

Note for $m = 1, k_m = 0$ and $p = q$ we get ordinary Bergman space $A^p_\beta(D)$ with quazinorm

$$\left(\int_D |f(z)|^p \delta^\beta(z) d\nu(z) \right)^{\frac{1}{p}};\ 0 < p < \infty, \beta > -1,$$

and for $m > 1, k_i = 0, i = 1, \dots, m, p = q$ we get the Bergman spaces on product domains with quazinorm

$$\left(\int_D \dots \int_D |f(z_1, \dots, z_m)|^p \delta^{\beta_1}(z_1) \dots \delta^{\beta_m}(z_m) d\nu(z_1) \dots d\nu(z_m) \right)^{\frac{1}{p}};$$

where $0 < p < \infty, \beta_j > -1, j = 1, \dots, m.$

4. On some new sharp embedding theorems for certain new mixed norm spaces in strictly pseudoconvex domain with smooth boundary

The theory of analytic spaces in bounded strictly pseudoconvex domains was developed rapidly during last decades (see [4–6, 11, 28, 31, 32]). Several Carleson-type sharp embedding theorems for such spaces are known today (see [1, 5] and references there). The goal of this paper is to add to this known list several new sharp assertions. We alert the reader that we here extend our previous results in the unit ball of $\mathbb{C}^n$ from [30], but since in this simpler case our arguments do not change much we provide sometimes below sketches of proofs. Nevertheless we found these general results interesting enough to be recorded in a separate paper after the appearance of [1, 2], where the so-called r - lattice from Kobayashi ball in $\mathbb{C}^n$ with some very nice properties for applications were finally found and studied.

We again will need for all proofs various properties of r-lattices of D domain, which we listed in previous section and various properties of Kobayashi balls from recent papers [1] and [2] which we also listed above.

During the past decades the theory of Bergman spaces in strictly pseudoconvex domains was developed in many papers by various authors. This paper considers generalizations of these spaces. About the Bergman space theory in the unit disk and the unit ball we refer reader to books [33, 34]. One of the goals of this paper is to extend some results of standard weighted Bergman spaces in the strictly pseudoconvex domains in $\mathbb{C}^n$ to the case of more general $A(p, q, \alpha)$ classes.

As we noted above using properties of Kobayashi metric balls and r-lattice from [1, 2] we get the following estimates

$$\|f\|^p_{A^p_\alpha} = \int\limits_D |f(z)|^p \delta^\alpha(z) d\nu(z) \asymp \sum_{k=1}^{\infty} \max_{z\in B_D(a_k,r)} |f(z)|^p \delta_\alpha(B_D(a_k, r))$$

$$\asymp \sum_{k=1}^{\infty} \int\limits_{B_D(a_k,R)} |f(z)|^p \delta^\alpha(z) d\nu(z); \tag{4.1}$$

where $0 < p < \infty,\ \alpha > -1;\ R = \frac{1+r}{2}, r \in (0; 1)$.

Motivated be (4.1) we introduce a new space as follows.

Definition 4.1. Let μ be a positive Borel measure in D, $0 < p, q < \infty,\ s > -1$. Fix $r \in (0; \infty)$ and an r-lattice $\{a_k\}_{k=1}^{\infty}$. The analytic space $A(p, q, d\mu)$ is the space of all holomorphic functions f such that

$$\|f\|^q_{A(p,q,d\mu)} = \sum_{k=1}^{\infty} \Big(\int\limits_{B(a_k,r)} |f(z)|^p d\mu(z) \Big)^{\frac{q}{p}} < \infty.$$

If $d\mu = \delta^s(z) d\nu(z)$ then we will denote by $A(p, q, s)$ the space $A(p, q, d\mu)$. This is Banach space for $\min(p, q) \geq 1$. It is clear that $A(p, p, s) = A^p_s$.

Remark 4.1. It is clear now from discussion above and the definition of $A(p, p, s)$ spaces that these classes are independent of $\{a_k\}$ and r. But in general case of $A(p, q, s)$ spaces the answer is unknown. For simplicity we denote $\|f\|_{A(p,q,s,a_k,r)}$ by $\|f\|_{A(p,q,s)}$.

We also have the following estimates using r-lattice

$$\|f\|^q_{A(p,q,s)} = \sum_{k=1}^{\infty}\Big(\int\limits_D \chi_{B_D(a_k,r)}(z)|f(z)|^p\delta^s(z)d\nu(z)\Big)^{\frac{q}{p}}$$
$$\le C\Big(\int\limits_D |f(z)|^p\delta^s(z)d\nu(z)\Big)^{\frac{q}{p}} = C\|f\|^q_{A^p_s}$$

where $q \ge p,\ s > -1$.

So finally we have

$$\|f\|_{A(p,q,s)} \le C\|f\|_{A^p_s}$$

where $q \ge p,\ s > -1$.

Motivated by this estimate we pose the following very natural and more general problem.

Problem: Let μ a positive Borel measure in D and let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence so that $B_D(a_k,r)$ is a r-lattice for strictly pseudoconvex domain D in $\mathbb{C}^n$. Let X be a quazinormed subspace of $H(D)$ and $p,q \in (0;\infty)$. Describe all positive Borel measures such that

$$\|f\|_{A(p,q,d\mu)} \le C\|f\|_X.$$

Definition 4.2. (Muckenhoupt type weights for D domains via Kobayashi balls) A positive locally integrable function $v(z)$ on D is said to belong to $MH(p)$ class if the following condition holds

$$\sup_{\substack{B(z,r)\\ 0<r\le 1}} \left(\frac{1}{|B(z,r)|}\int\limits_{B_D(z,r)} v(w)d\nu(w)\right)\left(\frac{1}{|B(z,r)|}\int\limits_{B_D(z,r)} v^{-\frac{q}{p}}(w)d\nu(w)\right)^{\frac{p}{q}}$$

for any Kobayashi metric ball $B_D(z,r)$; where $1 < p < \infty$, $\frac{1}{p}+\frac{1}{q} = 1$; (we put $|B(z,r)| = |B_D(z,r)|$).

For two fixed real parameters $a > 0$ and $b > -1$ and for a f function a locally integrable function in D and $\{a_k\}$ - r - lattice of D we consider the following integral operator (Bergman-type operator)

$$\Big(S^{a,b}_{a_k,r}f\Big)(z) = \Big[\delta^a(z)\Big]\int\limits_{B_D(a_k,r)} (\delta^b(w))(f(w))K_{n+1+a+b}(r,w)d\nu(w), \quad z \in D,$$

where $K_{n+1+a+b}$ is a Bergman kernel of type t, $t = a+b+n+1$, see [35].

We will study this Bergman-type operator on D that we just defined above. We note again for all proofs of assertions below we will need properties of r-lattice, which we listed in previous sections, and various properties of Kobayashi balls from recent papers [1] and [2] which we also listed above.

All theorems of this section in very particular case of unit ball can be seen in [30]. Moreover arguments are similar, so we omit some proofs below. Almost everywhere we can replace K_t by $K^{t_1}_{n+1}$, $t_1 = \frac{t}{n+1}$.

Theorem 4.1. *Let* $0 < q, p < \infty$, $0 < s \le p < \infty, \beta > -1$. *Let* μ *be a positive Borel measure on* D. *Then we have the following assertion*

$$\|f\|_{A(q,p,d\mu)} \le C\|f\|_{A^s_\beta}$$

if and only if

$$\mu(B_D(a_k, r)) \le C(\delta(a_k))^{\frac{q(n+1+\beta)}{s}}. \tag{4.2}$$

Theorem 4.2. *Let* $\{a_k\}_{k\in\mathbb{N}}$ *be a sequence forming an* r*-lattice for* D. *Let also* $0 < s < \infty$, $1 \le p < \infty$, $t \in (-1; \infty)$. *Then we have the following inequality*

$$\sum_{k=1}^{\infty}\Big(\int\limits_{B_D(a_k,r)} |S^{a,b}_{a_k,r} f(z)|^p d\nu_t(z)\Big)^{\frac{s}{p}} \le C\sum_{k=1}^{\infty}\Big(\int\limits_{B_D(a_k,r)} |f(z)|^p d\nu_t(z)\Big)^{\frac{s}{p}}. \tag{4.3}$$

for some $t \in (t_0; t_1)$, $t_0 = t_0(a, b, p, s)$, $t_1 = t_1(a, b, p, s)$.

Remark 4.2. These estimates (4.2) and (4.3) for unit ball can be found in a paper [30] for simpler case $p = q, p = s$ last theorems are proved in [34] in unit ball.

Theorem 4.3. *Let* $0 < q, s < \infty$, $q \ge s$, $\alpha > -1$. *Let* $\{a_k\}_{k=1}^{\infty}$ *be a sequence forming* r*-lattice in* D. *Let* μ *be a positive Borel measure in* D. *Then*

$$\int\limits_D |f(z)|^q d\mu(z) \le C\int\limits_D\Big(\int\limits_{B_D(z,r)} |f(w)|^s d\nu_\alpha(w)\Big)^{\frac{q}{s}} d\nu(z)$$

if and only if

$$\mu(B_D(a_k, r)) \le C(\delta(a_k))^{q(\frac{n+1+\alpha}{s}+\frac{n+1}{q})}, \tag{4.4}$$

for some constant $C > 0$, $k \in \mathbb{N}$.

Theorem 4.4. *Let* $0 < r < \infty, f \in H(D)$, $\{a_k\}$ *be a sequence forming* r*-lattice in* D. *The following two statements hold.*

(i) *If* $0 < s < \infty, \alpha > -1, v \in MH(p), p > 1$ *then*

$$\sum_{k=1}^{\infty}\left(\int\limits_{B_D(a_k,r)} \left(S^{0,\alpha}_{a_k,r} f\right)^p v(z)d\nu(z)\right)^{\frac{s}{p}} \le C\sum_{k=1}^{\infty}\left(\int\limits_{B_D(a_k,r)} |f(z)|^p v(z)d\nu(z)\right)^{\frac{s}{p}};$$

(ii) *If* $v^p \in MH(\frac{p}{q}), p > q$ *and*

$$\left(\int\limits_{B_D(a_k,r)} |f(z)|^p d\nu(z)\right)\left(\delta^{-\frac{(n+1)(p-q)}{q}}(a_k)\right)\times\left(\int\limits_{B_D(a_k,r)} (v^{-p}(z)^{\frac{q}{p-q}})d\nu(z)\right)^{\frac{q-p}{q}}$$
$$\leq \int\limits_{B_D(a_k,r)} |f(z)|^p v^{-p}(z)d\nu(z),$$

then

$$\left(\int\limits_D |f(z)|^p v^p(z)d\nu(z)\right)^{\frac{q}{p}} \leq \sum_{k=1}^{\infty}\left(\delta^{(n+1)(p-q)(\frac{1}{p}-\frac{1}{q})}(a_k)\right)$$
$$\times\left(\int\limits_{B_D(a_k,r)} |f(z)|^p v^{-p}(z)d\nu(z)\right)^{\frac{q}{p}}.$$

Remark 4.3. Note in functional spaces in R^n these assertions are known (see [30] for this). For unit ball case they can be seen in [30].

Remark 4.4. Theorem 4.3 can be extended if we replace $d\nu(z)$ by $d\nu_\beta(z); \beta > -1$. We leave this to readers.

Theorem 4.5. *Let* μ *a positive Borel measure on* D *and* $\{a_k\}$ *be a Kobayashi sampling sequence forming* r*-lattice. Let* $\alpha > -1, f_i \in H(D), 0 < p_i < q_i < \infty, i = 1, \ldots, m$ *so that* $\sum_{i=1}^{m}\left(\frac{1}{q_i}\right) = 1$. *Then*

$$\int\limits_D \prod_{i=1}^{m} |f_i(z)|^{p_i} d\mu(z) \leq C \prod_{i=1}^{m}\left[\sum_{k=1}^{\infty}\left(\int\limits_{B(a_k,R)} |f_i(z)|^{p_i}\delta^{\alpha}(z)d\nu(z)\right)^{q_i}\right]^{\frac{1}{q_i}}$$

if and only if

$$\mu(B_D(a_k, r)) \leq C\delta^{m(n+1+\alpha)}(a_k), \tag{4.5}$$

for every $k = 1, 2, 3, \ldots,\quad R = \left(\frac{1+r}{2}\right),\ r > 0.$

Remark 4.5. Assertion of Theorem 4.5 can be found in paper [30] for case of unit ball in $\mathbb{C}^n$. For $q_i = 1, p_i = p, m = 1$ it can be seen in [34] in unit ball.

Theorem 4.6. *Let* $0 < p < q < \infty,\quad \alpha > 0$. *Let* $\{a_k\}_{k=1}^{\infty}$ *be a sampling sequence (a sequence forming a* r*-lattice for* D*). Let* μ *be a positive Borel measure on* D*. Then* *the following two statements are equivalent*

$$\int\limits_D\left(\int\limits_D\Big((\delta(z))^n K_{2n}(z,\lambda)\Big)^{1+\alpha q}d\mu(z)\right)^{\frac{q}{q-p}}\delta^{\alpha qn-1}(\lambda)d\nu(\lambda)<\infty;$$

$$\sum_{k=1}^{\infty}\Big(\delta^{\frac{p}{q-p}(-(n+n\alpha q))}(a_k)\Big)\Big(\mu(B_D(a_k,r))^{\frac{q}{q-p}}\Big)<\infty,\ n\in\mathbb{N}.$$

As it was mentioned above we intend to give in this paper much more general versions of our earlier results proved before in case of unit ball in $\mathbb{C}^n$ in bounded strictly pseudoconvex domains with smooth boundary. We heavily use for this purpose the new vital technique which was developed in very recent vital papers [1, 2], where the so-called r-lattice was introduced and studied for bounded strictly pseudoconvex domains. We note that all proofs of Theorems 4.1–4.5 will not be given in this paper because of certain real similarities in arguments we used in case of unit ball before and here below. Note also again here as before in case of unit ball all our proofs are heavily based on nice properties of r-lattice, which we listed in previous sections, we mentioned above and which will not be mentioned again below.

Proofs are rather sketchy (see [30]).

The proof of Theorem 4.1. Suppose (4.2) holds then we have using properties of r-lattice, which we listed in previous sections (lemma's 2.1-2.5)

$$\left(\sum_{k=1}^{\infty}\left[\int\limits_{B_D(a_k,r)}|f(z)|^q d\mu(z)\right]^{\frac{p}{q}}\right)^{\frac{s}{p}}\le C\left(\sum_{k=1}^{\infty}\max_{z\in B_D(a_k,r)}|f(z)|^p\delta^{\frac{p(n+1+\beta)}{s}}(a_k)\right)^{\frac{s}{p}}$$
$$\le C\sum_{k=1}^{\infty}\max_{z\in B_D(a_k,r)}|f(z)|^s\delta^{(n+1+\beta)}(a_k)$$
$$\le C\int\limits_D|f(z)|^s\delta^{\beta}(z)d\nu(z)\le C\|f\|^s_{A^s_\beta(D)},$$

where $\beta>-1, 0<s<\infty$.

Conversely using appropriate test function $f_k(z)$ and estimates from below of Bergman-type kernel K_s from [1, 2] and using also properties of r-lattice, which we listed in previous section, for test function

$$f_k(z)=\left(\delta^{n+\beta+1}(a_k)K_t(z,a_k)\right)^{\frac{1}{s}},\quad z\in D, k=1,2,\dots,t=2(n+\beta+1)$$

(we can even replace K_t by $K^{t_1}_{n+1}$ here and below, $t_1=\frac{t}{n+1}$) and noting that

$$\left(\int\limits_{B_D(a_k,r)}|f(z)|^q d\mu(z)\right)^{\frac{1}{q}}\le C_1\left[\sum_{k=1}^{\infty}\left(\int\limits_{B_D(a_k,r)}|f(z)|^q d\mu(z)\right)^{\frac{p}{q}}\right]^{\frac{1}{p}}\le c\|f\|_{A^s_\beta}. \tag{4.6}$$

we complete the proof. Indeed putting f_k into (4.6) and using the fact that $\sup_k \|f_k\|_{A^s_\beta} \leq C$ which follows from Theorem 2.1 (see also [35]) we will get what we need. The proof is complete. $\square$

The proof of Theorem 4.2. It is based on Schur test (see also [30] for $S^{a,b}_{a_k,r}$ operator). If $p = 1$ then the Theorem follows from Fubini's theorem. Let $p \in (1;\infty), \frac{1}{p} + \frac{1}{q} = 1$. Then using Theorem 2.1 properties of r-lattice, which we listed in previous section we have

$$\int\limits_{B_D(a_k,r)} \Big(\delta^{\widetilde{s}q}(w)\Big)(\delta(z))^a(\delta(w))^b K_{n+1+a+b}(z,w)d\nu(w) \leq C_1\Big(\delta^{\widetilde{s}q}(z)\Big);$$

$z \in B_D(a_k,r)$, and also

$$\int\limits_{B_D(a_k,r)} \Big(\delta^{\widetilde{s}p}(w)\Big)(\delta(w))^a(\delta(z))^b K_t(z,w)d\nu(w) \leq C_2\Big(\delta^{\widetilde{s}p}(z)\Big),$$

$z \in B_D(a_k,r)$, $(K_{n+1+a+b}(z,w)) = K_t(z,w)$; $t = n+1+a+b$. It remains to use Schur test choosing appropriate $\widetilde{s}$. We refer the reader to [34] for Schur test. $\square$

The proof of Theorem 4.3. Let (4.4) holds. We have for same $\{a_k\}$ sequence and using properties of r-lattice, which we listed in previous sections (lemma's 2.1-2.4)

$$\begin{aligned}\int\limits_D |f(w)|^q d\mu(w) &\leq \sum_{k=1}^{\infty} \sup_{w\in B_D(a_k,r)} |f(w)|^q \mu(B_D(a_k,r)) \\ &\leq C\sum_{k=1}^{\infty}\Bigg(\sup_{w\in B_D(a_k,r)}|f(w)|^s\Bigg)^{\frac{q}{s}} \delta^{q(\frac{n+1+\alpha}{s}+\frac{n+1}{q})}(a_k).\end{aligned}$$

Then we have $\delta(w) \asymp \delta(z), z \in B_D(w,r)$, see [1, 2] and hence

$$\int\limits_{B_D(a_k,R)} |f(z)|^s d\nu(z) \leq C \int\limits_{B_D(a_k,2R)} \Big(\int\limits_{B_D(z,r)} |f(\widetilde{w})|^s d\nu_\alpha(\widetilde{w})\Big)\frac{d\nu(z)}{\delta^{n+1+\alpha}(z)}.$$

Hence we have now $t = q(\frac{n+1+\alpha}{s} + \frac{n+1}{q})$

$$\begin{aligned}\int\limits_D |f(z)|^q d\mu(z) &\leq C\sum_{k=1}^{\infty}\Bigg(\int\limits_{B_D(a_k,R)} |f(z)|^s d\nu(z)\frac{1}{\delta^{n+1}(a_k)}\Bigg)^{\frac{q}{s}}(\delta(a_k))^t \\ &\leq C\sum_{k=1}^{\infty}\Bigg(\int\limits_{B_D(a_k,R)} |f(z)|^s d\nu(z)\Bigg)^{\frac{q}{s}}\Bigg(\delta^{n+1+\frac{q\alpha}{s}}(a_k)\Bigg) \\ &\leq C\sum_{k=1}^{\infty}\Bigg(\int\limits_{B_D(a_k,R)}\int\limits_{B_D(z,r)} |f(\widetilde{w})|^s d\nu_\alpha(\widetilde{w})\frac{d\nu(z)}{\delta^{n+1}(z)}\Bigg)^{\frac{q}{s}}\Bigg(\delta^{n+1}(a_k)\Bigg).\end{aligned}$$

By Holder's inequality we have, using properties of r-lattice (lemma's 2.1-2.5)

$$\left(\int\limits_{B_D(a_k,R)}\int\limits_{B_D(z,r)}|f(\widetilde{w})|^s d\nu_\alpha(\widetilde{w})\frac{d\nu(z)}{\delta^{n+1}(z)}\right)^{\frac{q}{s}}\leq\Big(R=\frac{1+r}{2}\Big)$$

$$\leq\int\limits_{B_D(a_k,R)}\left(\int\limits_{B_D(z,r)}|f(\widetilde{w})|^s d\nu_\alpha(\widetilde{w})\right)^{\frac{q}{s}}\Big(\delta^{-(n+1)}(a_k)\Big)d\nu(z).$$

Combining all estimates we get the desired results. We show the reverse. We have for $\{a_k\}, z\in D, k=1,2,\dots$ and β which is big enough. Let

$$f_k(z)=\Big(\delta^{\beta-\frac{n+1+\alpha}{s}-\frac{n+1}{q}}(a_k)\Big)\Big[K_{n+1}(z,a_k)\Big]^{\widetilde{\beta}};\quad \tau=\beta q-q\frac{n+1+\alpha}{s}-(n+1),\widetilde{\beta}=\frac{\beta}{n+1}.$$

Then by Theorem 2.1 and Lemma's 2.1-2.5 we have

$$\int\limits_D\left(\int\limits_{B_D(w,r)}|f_k(z)|^s d\nu_\alpha(z)\right)^{\frac{q}{s}}d\nu(w)\leq\left[C\Big(\delta^\tau(a_k)\Big)\Big(\frac{1}{\delta^\tau(a_k)}\Big)\right]\leq const;$$

Then we have

$$\int\limits_D|f_k(z)|^q d\mu(z)\geq\Big(\mu(B_D(a_k,r))\Big)\Big[\delta^{-q(\frac{n+1+\alpha}{s}+\frac{n+1}{q})}(a_k)\Big].$$

The rest is clear (see also [30]). ☐

Note that in proofs we repeat arguments from [30].

The proof of Theorem 4.5. First suppose that (4.5) holds. Then using properties of r-lattices, which we listed in previous sections and Kobayashi balls we have

$$\int\limits_D\prod_{i=1}^m|f_i(z)|^{p_i}d\mu(z)\leq C\sum_{k=1}^\infty\Big(\mu(B_D(a_k,r))\Big)\prod_{i=1}^m\sup_{z\in B_D(a_k,r)}|f_i(z)|^{p_i},$$

i.e.

$$\int\limits_D\prod_{i=1}^m|f_i(z)|^{p_i}d\mu(z)\leq C\sum_{k=1}^\infty\frac{\mu(B_D(a_k,r))}{\delta^{m(n+1+\alpha)}(a_k)}\prod_{i=1}^m\int\limits_{B_D(a_k,R)}|f_i(w)|^{p_i}\delta^\alpha(w)d\nu(w)$$

$$\leq C\sum_{k=1}^\infty\prod_{i=1}^m\int\limits_{B_D(a_k,R)}|f_i(w)|^{p_i}\delta^\alpha(w)d\nu(w).$$

Using the condition $\sum\limits_{i=1}^m\Big(\frac{1}{q_i}\Big)=1$, Holder's inequality for m functions we get that we need. The reverse follows from chain of equalities and estimates based again on properties of r-lattice, which we listed in previous section. Indeed we have as above for f_i test function

$$f_i(z) = \Big(\delta^{\frac{n+1+\alpha}{p_i}}(a_k)\Big)\Big(K_{\frac{2(n+1+\alpha)}{p_i}}(a_k, z)\Big); \quad i = 1, 2, \dots, m.$$

By properties of r-lattice, which we listed in previous sections (lemma's 2.1-2.4) we have

$$\int_D \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \geq \int_{B_D(a_k,r)} \Big(\delta^{m(n+1+\alpha)}(a_k)\Big)\Big(K_\tau(a_k, z)\Big) d\mu(z)$$
$$\geq \frac{\mu(B_D(a_k, r))}{\delta^{m(n+1+\alpha)}(a_k)},$$

where $\tau = (2m)(n + 1 + \alpha)$.

Hence we get what we need. Indeed we have the following estimates

$$\prod_{i=1}^m \Bigg(\sum_{k=1}^\infty \Big(\int_{B_D(a_k,R)} |f_i(z)|^{p_i} \delta^\alpha(z) d\nu(z)\Big)^{q_i}\Bigg)^{\frac{1}{q_i}} \leq \Big(R = \frac{1+r}{2}\Big)$$
$$\leq \prod_{i=1}^m \sum_{k=1}^\infty \int_{B_D(a_k,R)} |f_i(z)|^{p_i} (\delta^\alpha(z)) d\nu(z)$$
$$\leq C \prod_{i=1}^m \int_D |f_i(z)|^{p_i} (\delta^\alpha(z)) d\nu(z)$$
$$\leq \Bigg(\int_D \delta^\alpha(z) \delta^{n+1+\alpha}(a_k) K_{\tau_1}(a_k, z) d\nu(z)\Bigg)^m \leq c;$$

where $\tau_1 = 2(n + 1 + \alpha)$. □

The careful analysis of proofs we provided above shows various similarities with our previous mentioned work in the unit ball.Nevertheless bounded strictly pseudoconvex domains are much more general as domains than the unit balls. We provided the complete proof of only one assertion from Theorem 4.4 and the proof of the rest we leave to readers (see [30]).

The proof of Theorem 4.4. Using again properties of r-lattice, which we listed in previous sections we have

$$M = \int_{B_D(a_k,r)} \Bigg[\int_{B_D(a_k,r)} |f(w)| \delta^\alpha(w) (K_{n+1+\alpha}(z, w)) d\nu(w)\Bigg]^p v(z) d\nu(z)$$
$$\leq \Bigg(\delta^{-(n+1)p}(a_k) \int_{B_D(a_k,r)} v(z) d\nu(z) \int_{B_D(a_k,r)} |f(w)| d\nu(w)\Bigg)^p.$$

Using Holder's inequality we get

$$\left(\int\limits_{B_D(a_k,r)} |f(w)| d\nu(w)\right)^p \leq \left(\int\limits_{B_D(a_k,r)} |f(w)|^p v(w) d\nu(w)\right) \times \left(\int\limits_{B_D(a_k,r)} v^{-\frac{q}{p}} d\nu(w)\right)^{\frac{p}{q}}.$$

Since $v \in MH(p)$ we have now

$$M \leq C_1 \int\limits_{B_D(a_k,r)} |f(w)|^p v(w) d\nu(w).$$

□

Remark 4.6. For $q = p, v = c$ these estimates in Theorem 4.4 are well-known in literature see for this [30] and references there.

The proof of Theorem 4.6 will be omitted. We refer the reader to [30] to recover the proof. Note also the proof of Theorem 4.6 is very similar to proof of Theorem 5.3 see [30] the unit ball case and needs only some modifications. The complete proof will be provided in our next paper which is in preparation.

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