Kragujevac Journal of Mathematics
Volume 37(2) (2013), Pages 245–255.

BOUNDS ON THE DISTANCE LAPLACIAN ENERGY OF GRAPHS

JIESHAN YANG[1], LIHUA YOU[1,*], AND I. GUTMAN[2]

Abstract. Let G be a simple connected graph, v_i its vertex, and D_i the sum of distances between v_i and the other vertices of G. Let $\delta_1, \delta_2, \dots, \delta_n$ be the eigenvalues of the distance matrix $\mathbf{D}$ of G, and $\delta_1^L, \delta_2^L, \dots, \delta_n^L$ the eigenvalues of the distance Laplacian matrix $\mathbf{D}^L$ of G. An earlier much studied quantity $E_D(G) = \sum_{i=1}^n |\delta_i|$ is the distance energy. We now define the distance Laplacian energy as $LE_D(G) = \sum_{i=1}^n \left|\delta_i^L - \frac{1}{n}\sum_{i=1}^n D_i\right|$, and obtain bounds for it.

1. Introduction

Let G be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. The distance matrix $\mathbf{D} = \mathbf{D}(G)$ of G is defined so that its (i,j)-entry d_{ij} is equal to the distance of (= length of a shortest path between) the vertices v_i and v_j [3]. The eigenvalues $\delta_1, \delta_2, \dots, \delta_n$ of $\mathbf{D}(G)$ are said to be the distance eigenvalues of the graph G and form its distance spectrum. The distance eigenvalues obey the following simple relations:

$$\sum_{i=1}^n \delta_i = 0 \qquad \text{and} \qquad \sum_{i=1}^n \delta_i^2 = 2s \tag{1.1}$$

where

$$s = \sum_{1\le i<j\le n} (d_{ij})^2. \tag{1.2}$$

For earlier studies of the distance spectrum see [5–7, 10, 18, 20, 26, 29, 30].

Key words and phrases. Distance (in graph), Distance Laplacian matrix, Distance Laplacian energy.

2010 *Mathematics Subject Classification.* Primary: 05C50. Secondary: 05C35, 15A18.

Received: July 16, 2013.

Revised: October 15, 2013.

The distance degree of the vertex v_i, denoted by D_i, is given by $D_i = \sum_{j=1}^{n} d_{ij}$. In what follows we assume that the vertices of the graph G are labeled so that $D_1 \geq D_2 \geq \cdots \geq D_n$. G is said to be k-distance regular if $D_i = k$ for all i.

The distance Laplacian matrix of a connected graph G has been recently defined by Aouchiche and Hansen [1] as

$$\mathbf{D}^L = \mathbf{D}^L(G) = \mathrm{diag}(D_i) - \mathbf{D}(G)$$

where $\mathrm{diag}(D_i)$ denotes the diagonal matrix of the distance degrees. Since $\mathbf{D}^L$ is real symmetric, all its eigenvalues $\delta_i^L(G)$, $i = 1, 2, \ldots, n$, are real and can be labeled so that $\delta_1^L(G) \geq \delta_2^L(G) \geq \cdots \geq \delta_n^L(G)$. These form the distance Laplacian spectrum of G. If confusion is avoided, we shall write δ_i^L instead of $\delta_i^L(G)$.

Let $\phi_D^G(\lambda)$ and $\phi_L^G(\lambda)$ be, respectively, the characteristic polynomials of the distance matrix and the distance Laplacian matrix. In [1], $\phi_L^G(\lambda)$ has been calculated for some particular graphs, including the complete graph K_n, the complement of an edge $K_n - e$, the complete bipartite graph $K_{a,b}$, and the graph S_n^+, obtained by adding an edge to the star S_n. Results on $\delta_{n-1}^L(G)$, which is similar to the algebraic connectivity, have been obtained [1]. Moreover, the equivalence between the Laplacian spectrum and the distance Laplacian spectrum in the set of connected graphs with diameter 2 has also been demonstrated [1].

The following results from [1] will be needed.

Lemma 1.1. [1] *If $\{\delta_1, \delta_2, \ldots, \delta_n\}$ is the distance spectrum of a k-distance regular graph G, then $\{k - \delta_n, k - \delta_{n-1}, \ldots, k - \delta_1\}$ is the distance Laplacian spectrum of G.*

It is known that Laplacian eigenvalues of a graph interlace the Laplacian eigenvalues of its edge-deleted subgraph. Such an interlacing does not apply to the distance Laplacian spectrum [1]. Instead of it, one has the following.

Theorem 1.1. [1] *Let G be a connected graph of order n, with $m \geq n$ edges, and let $\tilde{G}$ be a connected graph obtained by deleting an edge from G. Let $\{\delta_1^L, \delta_2^L, \ldots, \delta_n^L\}$ and $\{\tilde{\delta}_1^L, \tilde{\delta}_2^L, \ldots, \tilde{\delta}_n^L\}$ be, respectively, the distance Laplacian spectra of G and $\tilde{G}$. Then $\tilde{\delta}_i^L \geq \delta_i^L$ holds for all $1 \leq i \leq n$.*

Corollary 1.1. [1] *Let G be a connected graph on n vertices. Then $\tilde{\delta}_i^L(G) \geq \delta_i^L(K_n)$ for all $1 \leq i \leq n-1$, and $\tilde{\delta}_n^L(G) = \delta_n^L(K_n) = 0$.*

The distance energy of a connected graph G was defined in [12] as

$$E_D(G) = \sum_{i=1}^{n} |\delta_i|.$$

For more results on $E_D(G)$, we refer readers to the references [2, 4, 7, 10–13, 21–24, 27, 28, 30].

In this paper, we define the distance Laplacian energy $LE_D(G)$, and show that it preserves the main features of distance energy.

2. Distance Laplacian energy

Our intention is to conceive a graph-energy-like quantity [17], defined in terms of distance Laplacian eigenvalues, that would preserve the main features of the distance energy. Bearing in mind relations (1.1), we first introduce the auxiliary "eigenvalues" ξ_i , defined as

$$\xi_i = \delta_i^L - \frac{1}{n}\sum_{j=1}^{n} D_j.$$

The trace of a matrix $\mathbf{X} = (x_{ij})_{n\times n}$ is defined as $tr(\mathbf{X}) = \sum_{i=1}^{n} x_{ii}$. It is also equal to the sum of eigenvalues of $\mathbf{X}$.

Lemma 2.1. *Let G be a connected graph of order n. Then $\sum_{i=1}^{n} \xi_i = 0$ and $\sum_{i=1}^{n} \xi_i^2 = 2S$, where*

$$S = s + \frac{1}{2}\sum_{i=1}^{n}\left(D_i - \frac{1}{n}\sum_{j=1}^{n} D_j\right)^2$$

and where s is given by Eq. (1.2).

Proof. Note that

$$\sum_{i=1}^{n}\delta_i^L = tr(\mathbf{D}^L) = \sum_{i=1}^{n} D_i$$

and

$$\sum_{i=1}^{n}(\delta_i^L)^2 = tr[(\mathbf{D}^L)^2] = \sum_{i=1}^{n} D_i^2 + \sum_{i,j=1}^{n}(d_{ij})^2 = \sum_{i=1}^{n} D_i^2 + 2s$$

from which we have

$$\sum_{i=1}^{n}\xi_i = \sum_{i=1}^{n}\left(\delta_i^L - \frac{1}{n}\sum_{j=1}^{n} D_j\right) = \sum_{i=1}^{n}\delta_i^L - \sum_{j=1}^{n} D_j = 0$$

and

$$\begin{aligned}\sum_{i=1}^{n}\xi_i^2 &= \sum_{i=1}^{n}\left(\delta_i^L - \frac{1}{n}\sum_{j=1}^{n} D_j\right)^2\\ &= \sum_{i=1}^{n}(\delta_i^L)^2 - \frac{2}{n}\sum_{j=1}^{n} D_j\sum_{i=1}^{n}\delta_i^L + \frac{1}{n}\left(\sum_{j=1}^{n} D_j\right)^2\\ &= \sum_{i=1}^{n} D_i^2 + 2s - \frac{2}{n}\left(\sum_{j=1}^{n} D_j\right)^2 + \frac{1}{n}\left(\sum_{j=1}^{n} D_j\right)^2\\ &= 2s + \sum_{i=1}^{n}\left(D_i - \frac{1}{n}\sum_{j=1}^{n} D_j\right)^2 = 2S.\end{aligned}$$

□

Note that the equality $S=s$ holds if and only if G is distance regular.

Definition 2.1. Let G be a connected graph of order n. Then the distance Laplacian energy of G, denoted by $LE_D(G)$, is defined as $\sum_{i=1}^{n}|\xi_i|$, i.e.,

$$LE_D(G)=\sum_{i=1}^{n}\left|\delta_i^L-\frac{1}{n}\sum_{j=1}^{n}D_j\right|.$$

Example 2.1. $LE_D(K_n)$, $LE_D(K_n-e)$.

Since $\phi_L^{K_n}(\lambda)=\lambda(\lambda-n)^{n-1}$ [1], the distance Laplacian spectrum of K_n is $\{n^{[n-1]},0\}$, where $\omega^{[t]}$ means that ω is an eigenvalue with multiplicity t. Thus,

$$LE_D(K_n)=\sum_{i=1}^{n}\left|\delta_i^L(K_n)-\frac{1}{n}\sum_{j=1}^{n}D_j(K_n)\right|=\sum_{i=1}^{n}\left|\delta_i^L(K_n)-\frac{n(n-1)}{n}\right|=2(n-1).$$

Since $\phi_L^{K_n-e}(\lambda)=\lambda(\lambda-n-2)(\lambda-n)^{n-2}$ [1], the distance Laplacian spectrum of K_n-e is $\{n+2,n^{[n-2]},0\}$. By direct calculation, $D_1=D_2=n$, $D_3=D_4=\cdots=D_n=n-1$, and therefore $\frac{1}{n}\sum_{j=1}^{n}D_j=\frac{n^2-n+2}{n}=n+\frac{2}{n}-1$. Thus,

$$\begin{aligned}LE_D(K_n-e)&=\sum_{i=1}^{n}\left|\delta_i^L(K_n-e)-\frac{1}{n}\sum_{j=1}^{n}D_j(K_n-e)\right|\\&=\sum_{i=1}^{n}\left|\delta_i^L(K_n-e)-\left(n+\frac{2}{n}-1\right)\right|=2\left(n+\frac{2}{n}-1\right).\end{aligned}$$

Lemma 2.2. *If G is k-distance regular, then $LE_D(G)=E_D(G)$.*

Proof. Since G is k-distance regular, then $k=D_i=\frac{1}{n}\sum_{j=1}^{n}D_j$ for $i=1,2,\dots,n$. By Lemma 1.1,

$$\xi_i=\delta_i^L-\frac{1}{n}\sum_{j=1}^{n}D_j=(k-\delta_{n+1-i})-k=-\delta_{n+1-i}$$

for $i=1,2,\dots,n$.

Thus, the result follows from the definitions of the distance energy and the distance Laplacian energy. □

3. Estimating the distance Laplacian energy

Theorem 3.1. *Let G be a connected graph of order n, $n\geq 2$. Then*

$$2\sqrt{S}\leq LE_D(G)\leq\sqrt{2nS}. \tag{3.1}$$

Moreover, the left–hand side equality holds if and only if G is a connected graph with at most two positive distance Laplacian eigenvalues $p>\frac{1}{n}\sum_{i=1}^{n}D_i$ and $q=\frac{1}{n}\sum_{i=1}^{n}D_i\geq n$. The right–hand side equality holds if and only if G is a connected graph whose distance Laplacian spectrum is $\left\{(\frac{2}{n}\sum_{i=1}^{n}D_i)^{[n-1]},0\right\}$.

Proof. Consider the non-negative term $T = \sum_{i=1}^{n}\sum_{j=1}^{n}(|\xi_i| - |\xi_j|)^2$. By direct calculation,

$$T = 2n\sum_{i=1}^{n}|\xi_i|^2 - 2\left(\sum_{i=1}^{n}|\xi_i|\right)\left(\sum_{j=1}^{n}|\xi_j|\right) = 2n\cdot 2S - 2LE_D(G)^2 = 4nS - 2LE_D(G)^2.$$

Since $T \geq 0$, $4nS - 2LE_D(G)^2 \geq 0$, which implies $LE_D(G) \leq \sqrt{2nS}$ for $S > 0$.

From $\left(\sum_{i=1}^{n}\xi_i\right)^2 = 0$ and the fact that $S > 0$, we have

$$\begin{aligned} 2S = \sum_{i=1}^{n}\xi_i^2 &= \left(\sum_{i=1}^{n}\xi_i\right)^2 - 2\sum_{1\leq i<j\leq n}\xi_i\,\xi_j \\ &= -2\sum_{1\leq i<j\leq n}\xi_i\,\xi_j = 2\left|\sum_{1\leq i<j\leq n}\xi_i\xi_j\right| \leq 2\sum_{1\leq i<j\leq n}|\xi_i||\xi_j|. \end{aligned}$$

Thus,

$$\begin{aligned} LE_D(G)^2 &= \left(\sum_{i=1}^{n}|\xi_i|\right)^2 \\ &= \sum_{i=1}^{n}|\xi_i|^2 + 2\sum_{1\leq i<j\leq n}|\xi_i||\xi_j| \geq 2S + 2S = 4S \end{aligned}$$

which yields $LE_D(G) \geq 2\sqrt{S}$.

Bearing in mind the above considerations, we note that equality in the left–hand side inequality (3.1) holds if and only if there is at most one positive-valued and at most one negative-valued ξ_i, $i = 1, 2, \ldots, n$. By Lemma 1.1, we have $\xi_n = \delta_n^L - \frac{1}{n}\sum_{j=1}^{n}D_j = -\frac{1}{n}\sum_{j=1}^{n}D_j < 0$ and $\delta_i^L \geq n > 0$ for $i = 1, 2, \ldots, n-1$. This implies that $\xi_1 = \xi_2 = \cdots = \xi_r > 0$ and $\xi_{r+1} = \cdots = \xi_{n-1} = 0$, where $r \in \{0, 1, 2, \cdots, n-1\}$. The case $r = 0$ means that $\xi_1 = \xi_2 = \ldots = \xi_{n-1} = 0$. Hence, in the set $\{\delta_1^L, \delta_2^L, \ldots, \delta_{n-1}^L\}$ there are at most two positive–valued elements $p > \frac{1}{n}\sum_{j=1}^{n}D_j$ and $q = \frac{1}{n}\sum_{j=1}^{n}D_j \geq n$.

In addition, we note that equality in the right–hand side inequality (3.1) holds if and only if $T = 0$, i.e., $|\xi_i| = |\xi_j|$ for $i, j = 1, 2, \ldots, n$. Thus $\delta_i^L = \frac{2}{n}\sum_{j=1}^{n}D_j$, $i = 1, 2, \ldots, n-1$, since

$$|\xi_n| = \left|\delta_n^L - \frac{1}{n}\sum_{i=1}^{n}D_i\right| = \frac{1}{n}\sum_{i=1}^{n}D_i$$

and $\delta_i^L \geq n > 0$ for $i = 1, 2, \ldots, n-1$. □

Note that K_n is a graph with exactly one positive distance Laplacian eigenvalue $n > \frac{1}{n}\sum_{i=1}^{n}D_i = n-1$. Therefore, $LE_D(K_n) = 2\sqrt{S}$. In addition, for $n \geq 3$,

$K_n - e$ is a graph with exactly two positive distance Laplacian eigenvalues $n+2$, n. Therefore, $LE_D(K_n - e) \neq 2\sqrt{S}$ since $n \neq \frac{1}{n}\sum_{i=1}^{n} D_i = n + \frac{2}{n} - 1$.

Corollary 3.1. *Let G be a connected graph of order n. Then $LE_D(G) \geq \sqrt{2n(n-1)}$.*

Proof. By Theorem 3.1 and $d_{ij} \geq 1$ for $i, j = 1, 2, \ldots, n$,

$$LE_D(G) \geq 2\sqrt{S} = \sqrt{\sum_{1\leq i<j\leq n} (d_{ij})^2 + \frac{1}{2}\sum_{i=1}^{n}\left(D_i - \frac{1}{n}\sum_{j=1}^{n} D_j\right)^2}$$

$$\geq 2\sqrt{\sum_{1\leq i<j\leq n} (d_{ij})^2} \geq 2\sqrt{\binom{n}{2}} \geq \sqrt{2n(n-1)}.$$

□

The diameter of G, denoted by $\mathrm{diam}(G)$, is the maximum distance between any two vertices of G.

Corollary 3.2. *Let G be a connected (n, m)-graph with* $\mathrm{diam}(G) = 2$ *. Then*

$$LE_D(G) \geq 2\sqrt{2n^2 - 2n - 3m}.$$

Proof. Since $\mathrm{diam}(G) = 2$, $d_{ij} = 1$ if $v_iv_j \in E(G)$ and $d_{ij} = 2$ if $v_iv_j \notin E(G)$. By Theorem 3.1, we have

$$LE_D(G) \geq 2\sqrt{S} = 2\sqrt{\sum_{1\leq i<j\leq n} (d_{ij})^2 + \frac{1}{2}\sum_{i=1}^{n}\left(D_i - \frac{1}{n}\sum_{i=1}^{n} D_i\right)^2}$$

$$\geq 2\sqrt{\sum_{1\leq i<j\leq n} (d_{ij})^2} = 2\sqrt{m \cdot 1^2 + \left[\frac{n(n-1)}{2} - m\right] \cdot 2^2}$$

$$= 2\sqrt{2n^2 - 2n - 3m}.$$

□

Lemma 3.1. [14] *Let $a_1, a_2, \ldots, a_n$ be non-negative numbers. Then*

$$n\left[\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{1/n}\right] \leq n\sum_{i=1}^{n} a_i - \left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2$$

$$\leq n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{1/n}\right].$$

Theorem 3.2. *Let G be a connected graph with n vertices, $\mathbf{I}_n$ the unit matrix of order n, and*

$$\Delta = \left| \det \left(\mathbf{D}^L - \frac{1}{n} \sum_{i=1}^{n} D_i \, \mathbf{I}_n \right) \right| .$$

Then

$$\sqrt{2S + n(n-1)\Delta^{2/n}} \le LE_D(G) \le \sqrt{2(n-1)S + n\,\Delta^{2/n}}. \tag{3.2}$$

Proof. The bounds (3.2) are special cases of a result from [8]. For the sake of completeness, we sketch their proof.

Let $a_i = \xi^2$, $i = 1, 2, \dots, n$, and

$$K = n \left[\frac{1}{n} \sum_{i=1}^{n} \xi_i^2 - \left(\prod_{i=1}^{n} \xi_i^2 \right)^{1/n} \right] \ge 0.$$

Then by Lemma 3.1,

$$K \le n \sum_{i=1}^{n} \xi_i^2 - \left(\sum_{i=1}^{n} |\xi_i| \right)^2 \le (n-1)K$$

that is, $K \le 2nS - LE_D(G)^2 \le (n-1)K$.

Since

$$K = n \left[\frac{2}{n} S - \left(\prod_{i=1}^{n} |\xi_i| \right)^{2/n} \right] = 2S - n\Delta^{2/n}$$

and

$$2S - n\Delta^{2/n} \le 2nS - LE_D(G)^2 \le (n-1)(2S - n\Delta^{2/n})$$

we arrive at the desired result. □

Remark 3.1. The upper bound of $LE_D(G)$ in Theorem 3.2 is always better than the one in Theorem 3.1. By $K \ge 0$, we have

$$2S = \sum_{i=1}^{n} \xi_i^2 = \sum_{i=1}^{n} |\xi_i|^2 \ge n \left(\prod_{i=1}^{n} |\xi_i|^2 \right)^{1/n} = n\,\Delta^{2/n}.$$

Thus,

$$\sqrt{2(n-1)S + n\Delta^{2/n}} \le \sqrt{2nS}.$$

From the proof of Theorem 3.1, the equality

$$\sqrt{2(n-1)S + n\Delta^{2/n}} = \sqrt{2nS}$$

holds if and only if $|\xi_1| = |\xi_2| = \ldots = |\xi_n|$, i.e., if and only if G is a connected graph whose distance Laplacian spectrum is

$$\left\{ \left(\frac{2}{n} \sum_{i=1}^{n} D_i \right)^{[n-1]}, 0 \right\}.$$

Emulating a method invented by Koolen and Moulton [15, 16], and originally applied to the ordinary graph energy, we can formulate the following.

Theorem 3.3. *Let G be a connected graph of order n. Then*

$$LE_D(G) \leq \frac{1}{n} \sum_{i=1}^{n} D_i + \sqrt{(n-1) \left[2S - \left(\frac{1}{n} \sum_{i=1}^{n} D_i \right)^2 \right]}. \tag{3.3}$$

Proof. By Lemma 1.1, $\xi_n = 0 - \frac{1}{n} \sum_{i=1}^{n} D_i = -\frac{1}{n} \sum_{i=1}^{n} D_i$. Consider the non-negative term $T' = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (|\xi_i| - |\xi_j|)^2$. By direct calculation,

$$\begin{aligned} T' &= 2(n-1) \sum_{i=1}^{n-1} |\xi_i|^2 - 2 \left(\sum_{i=1}^{n-1} |\xi_i| \right) \left(\sum_{j=1}^{n-1} |\xi_j| \right) \\ &= 2(n-1) \left[2S - \left(\frac{1}{n} \sum_{i=1}^{n} D_i \right)^2 \right] - 2 \left(LE_D(G) - \frac{1}{n} \sum_{i=1}^{n} D_i \right)^2 \geq 0 \end{aligned}$$

which straightforwardly leads to inequality (3.3). □

Definition 3.1. [19] Let $(a) = (a_1, a_2, \ldots, a_r)$ and $(b_1, b_2, \ldots, b_s)$ be nonincreasing sequences of real numbers. Then (a) majorizes (b) if (a) and (b) satisfy the conditions:

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i \quad \text{for} \quad k = 1, 2, \ldots, \min\{r, s\}, \text{ and} \tag{1}$$

$$\sum_{i=1}^{r} a_i = \sum_{i=1}^{s} b_i\,. \tag{2}$$

Remark 3.2. The upper bound for $LE_D(G)$ in Theorem 3.3 is always better than the one in Theorem 3.1. By applying the Cauchy–Schwartz inequality to the two $(n-1)$-dimensional vectors $(1, 1, \ldots, 1)$ and $(|\xi_2|, |\xi_3|, \ldots, |\xi_n|)$, we get

$$\left(\sum_{i=2}^{n} |\xi_i| \right)^2 \leq (n-1) \sum_{i=2}^{n} |\xi_i|^2$$

that is,

$$(LE_D(G) - |\xi_1|)^2 \leq (n-1)(2S - |\xi_1|^2)$$

and therefore

$$LE_D(G) \leq |\xi_1| + \sqrt{(n-1)(2S - |\xi_1|^2)}. \tag{3.4}$$

It was proven in [25] that the spectrum of a positive semidefinite Hermitian matrix majorizes its main diagonal (when both are rearranged in nonincreasing order). By Lemma 1.1, $\mathbf{D}^L$ is a positive semidefinite matrix since $\delta_i^L(G) \geq 0$ for all $1 \leq i \leq n$. Noting that $\mathbf{D}^L$ is also symmetric, we have $\delta_1^L \geq D_1$ and $\xi_1 = \delta_1^L - \frac{1}{n}\sum_{i=1}^n D_i \geq D_1 - \frac{1}{n}\sum_{i=1}^n D_i \geq 0$. Thus Eq. (3.4) can be written as

$$LE_D(G) \leq \xi_1 + \sqrt{(n-1)(2S - \xi_1^2)}.$$

Define the function $f(x) = x + \sqrt{(n-1)(2S - x^2)}$ for $x \geq 0$. By simple calculus, it can be shown that $f(x)$ is decreasing in the internal $\left[\sqrt{2S/n}, +\infty\right]$, and increasing in the internal $\left[0, \sqrt{2S/n}\right]$. Then

$$f\left(\sqrt{\frac{1}{n}\sum_{i=1}^n D_i}\right) = \frac{1}{n}\sum_{i=1}^n D_i + \sqrt{(n-1)\left[2S - \left(\frac{1}{n}\sum_{i=1}^n D_i\right)^2\right]}$$
$$\leq \sqrt{2nS} = f\left(\sqrt{2S/n}\right) = f(x)_{\max}.$$

Acknowledgement: The research of the first and second authors was supported by the National Natural Science Foundation of China (No. 10901061), the Zhujiang Technology New Star Foundation of Guangzhou (No. 2011J2200090), and the Program on International Cooperation and Innovation, Department of Education, Guangdong Province (No. 2012gjhz0007).

References

[1] M. Aouchiche and P. Hansen, *Two Laplacians for the distance matrix of a graph*, Linear Algebra Appl. **439** (2013), 21–33.

[2] S. B. Bozkurt, A. D. Güngör and B. Zhou, *Note on the distance energy of graphs*, MATCH Commun. Math. Comput. Chem. **64** (2010), 129–134.

[3] F. Buckley and F. Harary, *Distance in Graphs*, Addison–Wesley, Redwood, 1990.

[4] G. Caporossi, E. Chasset and B. Furtula, *Some conjectures and properties on distance energy*, Les Cahiers du GERAD **G-2009-64** (2009), V+1–7.

[5] R. L. Graham and L. Lovász, *Distance matrix polynomials of trees*, Adv. Math. **29** (1978), 60–88.

[6] R. L. Graham and H. O. Pollack, *On the addressing problem for loop switching*, Bell System Techn. J. **50** (1971), 2495–2519.

[7] A. D. Güngör and S. B. Bozkurt, *On the distance spectral radius and distance energy of graphs*, Linear Multilinear Algebra **59** (2011), 365–370.

[8] I. Gutman, *Bounds for all graph energies*, Chem. Phys. Lett. **528** (2012), 72–74.

[9] I. Gutman, B. Zhou, *Laplacian energy of a graph*, Linear Algebra Appl. **414** (2006), 29–37.

[10] A. Ilić, *Distance spectra and distance energy of integral circulant graphs*, Linear Algebra Appl. **433** (2010), 1005–1014.

[11] G. Indulal, *Sharp bounds on the distance spectral radius and the distance energy of graphs*, Linear Algebra Appl. **430** (2009), 106–113.

[12] G. Indulal, I. Gutman and A. Vijaykumar, *On the distance energy of a graph*, MATCH Commun. Math. Comput. Chem. **60** (2008), 461–472.

[13] G. Indulal and A. Vijayakumar, *A note on energy of some graphs*, MATCH Commun. Math. Comput. Chem. **59** (2008), 269–274.

[14] H. Kober, *On the arithmetic and geometric means and the Hölder inequality*, Proc. Am. Math. Soc. **59** (1958), 452–459.

[15] J. Koolen and V. Moulton, *Maximal energy graphs*, Adv. Appl. Math. **26** (2001), 47–52.

[16] J. H. Koolen, V. Moulton and I. Gutman, *Improving the McClelland inequality for total* π*-electron energy*, Chem. Phys. Lett. **320** (2000), 213–216.

[17] X. Li, Y. Shi and I. Gutman, *Graph Energy*, Springer, New York, 2012.

[18] H. Lin, Y. Hong, J. Wang and J. Shu, *On the distance spectrum of graphs*, Linear Algebra Appl. **439** (2013), 1662–1669.

[19] A. W. Marshall and I. Olkin, *Inequalities: Theorey of Majorization and Its Applications*, Academic Press, New York, 1979.

[20] R. Merris, *The distance spectrum of a tree*, J. Graph Theory **14** (1990), 365–369.

[21] H. S. Ramane, I. Gutman and D. S. Revankar, *Distance equienergetic graphs*, MATCH Commun. Math. Comput. Chem. **60** (2008), 473–484.

[22] H. S. Ramane, D. S. Revankar, I. Gutman, S. B. Rao, B. D. Acharya and H. B. Walikar, *Estimating the distance energy of graphs*, Graph Theory Notes New York **55** (2008), 27–32.

[23] H. S. Ramane, D. S. Revankar, I. Gutman, S. B. Rao, B. D. Acharya and H. B. Walikar, *Bounds for the distance energy of a graph*, Kragujevac J. Math. **31**

(2008), 59–68.

[24] H. S. Ramane, D. S. Revankar, I. Gutman and H. B. Walikar, *Distance spectra and distance energies of iterated line graphs of regular graphs*, Publ. Inst. Math. (Beograd) **85** (2009), 39–46.

[25] I. Schur, *Über eine Klasse von Mittelbildungen mit Anwendungen auf die Determinantentheorie*, Sitzungsber. Berl. Math. Ges. **22** (1923), 9–20.

[26] D. Stevanović and A. Ilić, *Distance spectral radius of trees with fixed maximum degree*, Electron. J. Linear Algebra **20** (2010), 168–179.

[27] D. Stevanović and G. Indulal, *The distance spectrum and energy of the compositions of regular graphs*, Appl. Math. Lett. **22** (2009), 1136–1140.

[28] D. Stevanović, M. Milošević, P. Híc and M. Pokorný, *Proof of a conjecture on distance energy of complete multipartite graphs*, MATCH Commun. Math. Comput. Chem. **70** (2013), 157–162.

[29] R. Subhi and D. Powers, *The distance spectrum of the path P_n and the first distance eigenvector of connected graphs*, Linear Multilinear Algebra **28** (1990), 75–81.

[30] B. Zhou and A. Ilić, *On distance spectral radius and distance energy of graphs*, MATCH Commun. Math. Comput. Chem. **64** (2010), 261–280.

[1]School of Mathematical Sciences,
South China Normal University,
Guangzhou, 510631,
China
E-mail address: ylhua@scnu.edu.cn

[2]Faculty of Science,
University of Kragujevac,
P. O. Box 60, 34000 Kragujevac
Serbia
E-mail address: gutman@kg.ac.rs