THE RAINBOW DOMINATION NUMBER OF A DIGRAPH

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Abstract. Let $D = (V, A)$ be a finite and simple digraph. A $2$-rainbow dominating function ($2$RDF) of a digraph $D$ is a function $f$ from the vertex set $V$ to the set of all subsets of the set $\{1, 2\}$ such that for any vertex $v \in V$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N^- (v)} f(u) = \{1, 2\}$ is fulfilled, where $N^- (v)$ is the set of in-neighbors of $v$. The weight of a $2$RDF $f$ is the value $\omega (f) = \sum_{v \in V} |f(v)|$. The $2$-rainbow domination number of a digraph $D$, denoted by $\gamma_{2r}(D)$, is the minimum weight of a $2$RDF of $D$. In this paper we initiate the study of rainbow domination in digraphs and we present some sharp bounds for $\gamma_{2r}(D)$.

1. Introduction

Let $D$ be a finite simple digraph with vertex set $V(D) = V$ and arc set $A(D) = A$. A digraph without directed cycles of length 2 is an oriented graph. The order $n = n(D)$ of a digraph $D$ is the number of its vertices. We write $d_D^+(v)$ for the outdegree of a vertex $v$ and $d_D^-(v)$ for its indegree. The minimum and maximum indegree and minimum and maximum outdegree of $D$ are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. If $uv$ is an arc of $D$, then we also write $u \rightarrow v$, and we say that $v$ is an out-neighbor of $u$ and $u$ is an in-neighbor of $v$. For a vertex $v$ of a digraph $D$, we denote the set of in-neighbors and out-neighbors of $v$ by $N^- (v) = N^-D(v)$ and $N^+ (v) = N^+D(v)$, respectively. Let $N^- [v] = N^- (v) \cup \{v\}$ and $N^+ [v] = N^+ (v) \cup \{v\}$. For $S \subseteq V(D)$, we define $N^+[S] = \bigcup_{v \in S} N^+[v]$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by $X$. If $X \subseteq V(D)$ and $v \in V(D)$, then $E(X,v)$ is the set of arcs from $X$ to $v$. Consult [2, 7] for the notation and terminology which are not defined here. For a real-valued function $f : V(D) \rightarrow \mathbb{R}$ the weight of $f$ is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$.

A vertex $v$ dominates all vertices in $N^+[v]$. A subset $S$ of vertices of $D$ is a dominating set if $S$ dominates $V(D)$. The domination number $\gamma(D)$ is the minimum

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cardinality of a dominating set of $D$. The domination number of digraphs was introduced by Chartrand, Harary and Yue [1] as the out-domination number and has been studied by several authors (see, for example [4, 8]). A Roman dominating function (RDF) on a digraph $D$ is a function $f: V \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $v$ for which $f(v) = 0$ has an in-neighbor $u$ for which $f(u) = 2$. The weight of an RDF $f$ is the value $\omega(f) = \sum_{v \in V} f(v)$. The Roman domination number of a digraph $D$, denoted by $\gamma_R(D)$, equals the minimum weight of an RDF on $D$. A $\gamma_R(D)$-function is a Roman dominating function of $D$ with weight $\gamma_R(D)$. The Roman domination number in digraphs was introduced by Kamaraj and Jakkammal [3] and has been studied in [5]. A Roman dominating function $f: V \rightarrow \{0, 1, 2\}$ can be represented by the ordered partition $(V_0, V_1, V_2)$ (or $(V^0_f, V^1_f, V^2_f)$ to refer $f$) of $V$, where $V_i = \{v \in V \mid f(v) = i\}$.

For a positive integer $k$, a $k$-rainbow dominating function (kRDF) of a digraph $D$ is a function $f$ from the vertex set $V(D)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(D)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N^{-}(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The weight of a kRDF $f$ is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The $k$-rainbow domination number of a digraph $D$, denoted by $\gamma_{rk}(D)$, is the minimum weight of a kRDF of $D$. A $\gamma_{rk}(D)$-function is a $k$-rainbow dominating function of $D$ with weight $\gamma_{rk}(D)$. Note that $\gamma_{r1}(D)$ is the classical domination number $\gamma(D)$. A 2-rainbow dominating function (briefly, rainbow dominating function) $f: V \rightarrow \mathcal{P}\{1, 2\}$ can be represented by the ordered partition $(V_0, V_1, V_2, V_{1,2})$ (or $(V^0_f, V^1_f, V^2_f, V^1_{1,2})$ to refer $f$) of $V$, where $V_0 = \{v \in V \mid f(v) = \emptyset\}$, $V_1 = \{v \in V \mid f(v) = \{1\}\}$, $V_2 = \{v \in V \mid f(v) = \{2\}\}$ and $V_{1,2} = \{v \in V \mid f(v) = \{1, 2\}\}$. In this representation, its weight is $\omega(f) = |V_1| + |V_2| + 2|V_{1,2}|$. Since $V_1 \cup V_2 \cup V_{1,2}$ is a dominating set when $f$ is a 2RDF, and since assigning $\{1, 2\}$ to the vertices of a dominating set yields an 2RDF, we have

$$\gamma(D) \leq \gamma_{r2}(D) \leq 2\gamma(D).$$

Our purpose in this paper is to establish some bounds for the rainbow domination number of a digraph.

For $S \subseteq V(D)$, the $S$-out-private neighbors of a vertex $v$ of $S$ are the vertices of $N^+[v] \setminus N^+[S - \{v\}]$. The vertex $v$ is its own out-private neighbor if $v \notin N^+[S - \{v\}]$. The other out-private neighbors are external, i.e., belong to $V - S$. We make use of the following result in this paper.

**Proposition 1.1.** [3] Let $f = (V_0, V_1, V_2)$ be any $\gamma_R(D)$-function of a digraph $D$. Then

(a) If $w \in V_1$, then $N_D^-(w) \cap V_2 = \emptyset$.

(b) Let $H = D[V_0 \cup V_2]$. Then each vertex $v \in V_2$ with $N^-(v) \cap V_2 \neq \emptyset$, has at least two out-private neighbors relative to $V_2$ in the digraph $H$.

**Proposition 1.2.** Let $k \geq 1$ be an integer. If $D$ is a digraph of order $n$, then

$$\min\{k, n\} \leq \gamma_{rk}(D) \leq n.$$
In particular, \( \gamma_{rn}(D) = n \).

**Proof.** Let \( f \) be a \( \gamma_{rk}(D) \)-function. If there exists a vertex \( v \) such that \( f(v) = \emptyset \), then the definition yields to \( f(N^-(v)) = \{1, 2, \ldots, k\} \) and thus \( k \leq \gamma_{rk}(D) \). If \( f(v) \) is nonempty for all vertices \( v \in V(D) \), then \( n \leq \gamma_{rk}(D) \), and the first inequality is proved.

Next consider the function \( g \), defined by \( g(v) = \{1\} \) for each \( v \in V(D) \). Then \( g \) is a \( k \)-rainbow dominating function of weight \( n \), and so \( \gamma_{rk}(D) \leq n \).

**Proposition 1.3.** Let \( k \geq 1 \) be an integer. If \( D \) is a digraph of order \( n \), then

\[
\gamma_{rk}(D) \leq n - \Delta^+(D) + k - 1.
\]

**Proof.** Let \( v \) be a vertex of maximum outdegree \( \Delta^+(D) \). Define \( f : V(D) \to \mathcal{P}(\{1, 2, \ldots, k\}) \) by \( f(v) = \{1, 2, \ldots, k\} \), \( f(x) = \emptyset \) if \( x \in N^+(v) \) and \( f(x) = \{1\} \) otherwise. It is easy to see that \( f \) is a \( k \)-rainbow dominating function of \( D \) and thus \( \gamma_{rk}(D) \leq n - \Delta^+(D) + k - 1 \).

Let \( k \geq 1 \) be an integer, and let \( D \) be a digraph of order \( n \geq k \) and maximum outdegree \( \Delta^+(D) = n - 1 \). Since \( n \geq k \), Proposition 1.2 leads to \( \gamma_{rk}(D) \geq k \). Hence it follows from Proposition 1.3 that

\[
k \leq \gamma_{rk}(D) \leq n - \Delta^+(D) + k - 1 = k
\]

and therefore \( \gamma_{rk}(D) = k \). This example shows that Proposition 1.3 is sharp.

2. **Bounds on the rainbow domination number of digraphs**

**Theorem 2.1.** For a digraph \( D \), \( \frac{2}{3} \gamma_R(D) \leq \gamma_{r2}(D) \leq \gamma_R(D) \).

**Proof.** The upper bound is immediate by definition. To prove the lower bound, let \( f \) be a \( \gamma_{r2}(D) \)-function and let \( X_i = \{v \in V(D) \mid i \in f(v)\} \) for \( i = 1, 2 \). We may assume that \( |X_1| \leq |X_2| \). Then \( |X_1| \leq (|X_1| + |X_2|)/2 = \gamma_{r2}(D)/2 \). Define \( g : V(D) \to \{0, 1, 2\} \) by \( g(u) = 0 \) if \( f(u) = \emptyset \), \( g(u) = 1 \) when \( f(u) = \{2\} \) and \( g(u) = 2 \) if \( 1 \in f(u) \). Obviously, \( g \) is a Roman dominating function on \( D \) with \( \omega(g) \leq 2|X_1| + |X_2| \leq \frac{3}{2} \gamma_{r2}(D) \) and the result follows.

**Lemma 2.1.** Let \( k \geq 0 \) be an integer and let \( D \) be a digraph of order \( n \).

(i) If \( \gamma_R(D) = \gamma_{r2}(D) + k \), then there exists a subset \( U \) of \( V(D) \) such that \( |N^+[U]| = n - \gamma_R(D) + 2|U| \). In particular, there exists a subset \( U \) of \( V(D) \) such that \( |N^+[U]| = n - \gamma_{r2}(D) + 2|U| - k \).

(ii) For each \( U \subseteq V(D) \), \( \gamma_R(D) \leq n - (|N^+[U]| - 2|U|) \). In particular, if there exists a subset \( U \) of \( V(D) \) such that \( |N^+[U]| = n - \gamma_{r2}(D) + 2|U| - k \), then \( \gamma_R(D) \leq \gamma_{r2}(D) + k \).

(iii) If \( U^* \) is a subset of \( V(D) \) such that \( |N^+[U^*]| - 2|U^*| \) is maximum, then \( \gamma_R(D) = n - (|N^+[U^*]| - 2|U^*|) \).
Proof. (i) Let \( f = (V_0, V_1, V_2) \) be a \( \gamma_R(D) \)-function. By Proposition 1.1 (a), \( N^+[V_2] = V_0 \cup V_2 \). Since \( \gamma_R(D) = |V_1| + 2|V_2| \), we have
\[
|N^+[V_2]| = |V_0| + |V_2| = (n - |V_1| + |V_2|) + |V_2| = n - (\gamma_R(D) - 2|V_2|).
\]
Hence \( V_2 \) is the desired subset of \( V(D) \).

(ii) Let \( U \) be a subset of \( V(D) \). Clearly, \( f = (N^+[U] - U, V(D) - N^+[U], U) \) is an RDF of \( D \) of weight \( n - (|N^+[U]| - 2|U|) \) and hence \( \gamma_R(D) \leq n - (|N^+[U]| - 2|U|) \).

(iii) It follows from (i) and the choice of \( U^* \) that
\[
|N^+[U^*]| - 2|U^*| \geq n - \gamma_r(D) - k = n - \gamma_R(D),
\]
when \( \gamma_R(D) = \gamma_r(D) + k \).

By (ii), we obtain
\[
\gamma_R(D) \leq n - (|N^+[U^*]| - 2|U^*|) \leq n - (n - \gamma_R(D)) = \gamma_R(D)
\]
and the proof is complete. \( \square \)

**Theorem 2.2.** Let \( k \geq 0 \) be an integer and let \( D \) be a digraph of order \( n \). Then \( \gamma_R(D) = \gamma_r(D) + k \) if and only if

(i) there exists no subset \( U \) of \( V(D) \) such that
\[
n - \gamma_r(D) + 2|U| - k + 1 \leq |N^+[U]| \leq n - \gamma_r(D) + 2|U|
\]
and

(ii) there exists a subset \( U \) of \( V(D) \) such that \( |N^+[U]| = n - \gamma_r(D) + 2|U| - k \).

**Proof.** The proof is by induction on \( k \). Let \( k = 0 \). If \( \gamma_R(D) = \gamma_r(D) \), then (i) is trivial and (ii) follows from Lemma 2.1 (i). Now assume that (i) and (ii) hold. It follows from Lemma 2.1 (ii) that \( \gamma_R(D) \leq \gamma_r(D) \). By Theorem 2.1, we have \( \gamma_R(D) = \gamma_r(D) \), as desired. Therefore we may assume that \( k \geq 1 \) and the theorem is true for each \( m \leq k - 1 \).

Let \( \gamma_R(D) = \gamma_r(D) + k \). By Lemma 2.1 (i), it suffices to show that (i) holds. Assume to the contrary that there exists an integer \( 0 \leq i \leq k - 1 \) and a subset \( U \subseteq V(D) \) such that \( n - \gamma_r(D) + 2|U| - i \leq |N^+[U]| \). By choosing \( (U, i) \) so that \( i \) is as small as possible and by the inductive hypothesis we have \( \gamma_R(D) \leq \gamma_r(D) + i < \gamma_r(D) + k \) which is a contradiction. This proves the “only” part of the theorem.

Conversely, assume that (i) and (ii) hold. By Lemma 2.1 (ii) we need to show that \( \gamma_R(D) \geq \gamma_r(D) + k \). Suppose \( i = \gamma_R(D) - \gamma_r(D) \). If \( 0 \leq i \leq k - 1 \), then by the inductive hypothesis there exists a set \( U \subseteq V(D) \) such that \( n - \gamma_r(D) + 2|U| - i = |N^+[U]| \) which contradicts (i). Hence \( \gamma_R(D) = \gamma_r(D) + k \) and the proof is complete. \( \square \)

**Theorem 2.3.** Let \( D \) be a digraph of order \( n \). Then
\[
\gamma_r(D) \geq \left\lfloor \frac{2n}{2 + \Delta^+(D)} \right\rfloor.
\]
Proof. Let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{r2}(D)$-function. Then $\gamma_{r2}(D) = |V_1| + |V_2| + 2|V_{1,2}|$ and $n = |V_0| + |V_1| + |V_2| + |V_{1,2}|$.

Since each vertex of $V_0$ has at least one in-neighbor in $V_{1,2}$ or at least one out-neighbor in $V_2$ and at least one in-neighbor in $V_1$, we deduce that $|V_0| \leq \Delta^+(|V_2| + |V_{1,2}|)$ and $|V_0| \leq \Delta^+(|V_1| + |V_{1,2}|)$. Hence, we conclude that

$$(\Delta^+ + 2)\gamma_{r2}(D) = (\Delta^+ + 2)(|V_1| + |V_2| + 2|V_{1,2}|)$$

$$= (\Delta^+ + 2)(|V_1| + |V_2|) + 4|V_{1,2}| + 2\Delta^+|V_{1,2}|$$

$$\geq 2|V_1| + 2|V_2| + 2|V_0| + 4|V_{1,2}|$$

$$= 2n + 2|V_{1,2}|$$

$$\geq 2n,$$

as desired. \qed

The proof of Theorem 2.3 shows that

$$\gamma_{r2}(D) \geq \left\lceil \frac{2n + 2}{2 + \Delta^+(D)} \right\rceil$$

if there exists a $\gamma_{2r}(D)$-function $f$ such that $V_{r,2} \neq \emptyset$. Let $G$ be a graph and $\gamma_{r2}(G)$ its 2-rainbow domination number. If $G$ is of order $n$ and maximum degree $\Delta$, then Theorem 2.3 implies immediately the known bound $\gamma_{r2}(G) \geq \lceil 2n/(\Delta + 2) \rceil$, given by Sheikholeslami and Volkmann [6]. Next we characterize the digraphs $D$ with $\gamma_{r2}(D) = 2$.

**Proposition 2.1.** Let $D$ be a digraph of order $n \geq 2$. Then $\gamma_{r2}(D) = 2$ if and only if $n = 2$ or $n \geq 3$ and $\Delta^+(D) = n - 1$ or there exist two different vertices $u$ and $v$ such that $V(D) - \{u, v\} \subseteq N^+(u)$ and $V(D) - \{u, v\} \subseteq N^+(v)$.

**Proof.** If $n = 2$ or $n \geq 3$ and $\Delta^+(D) = n - 1$ or there exist two different vertices $u$ and $v$ such that $V(D) - \{u, v\} \subseteq N^+(u)$ and $V(D) - \{u, v\} \subseteq N^+(v)$, then it is easy to see that $\gamma_{r2}(D) = 2$.

Conversely, assume that $\gamma_{r2}(D) = 2$. Let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{r2}(D)$-function. Clearly, $2 = \gamma_{r2}(D) = |V_1| + |V_2| + 2|V_{1,2}|$ and thus $|V_{1,2}| \leq 1$. If $|V_{1,2}| = 1$, then $|V_1| = |V_2| = 0$ and hence $\Delta^+(D) = n - 1$. If $|V_{1,2}| = 0$, then $|V_1|, |V_2| \leq 2$. If $|V_1| = 0$ or $|V_2| = 0$, then we deduce that $n = 2$. In the remaining case that $|V_1| = |V_2| = 1$, we assume that $V_1 = \{u\}$ and $V_2 = \{v\}$. The definition of the 2-rainbow dominating function implies that each vertex of $V(D) - \{u, v\}$ has $u$ and $v$ as an in-neighbor. Consequently, $V(D) - \{u, v\} \subseteq N^+(u)$ and $V(D) - \{u, v\} \subseteq N^+(v)$. \qed

**Proposition 2.2.** Let $D$ be a digraph of order $n$. Then $\gamma_{r2}(D) < n$ if and only if $\Delta^+(D) \geq 2$ or $\Delta^-(D) \geq 2$.

**Proof.** Let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{r2}(D)$-function of $D$. The hypothesis $|V_0| + |V_1| + |V_2| + |V_{1,2}| = n > \gamma_{r2}(D) = |V_1| + |V_2| + 2|V_{1,2}|$ implies $|V_0| \geq |V_{1,2}| + 1$. If some vertex $w \in V_0$ has no in-neighbor in $V_{1,2}$, then $|N^-(w) \cap V_1| \geq 1$ and $|N^-(w) \cap V_2| \geq 1$.
which implies that $\Delta^{-}(D) \geq d^{-}(w) \geq 2$. So we may assume each vertex $w \in V_0$ has at least one in-neighbor in $V_{1,2}$. Then we have

$$\sum_{u \in V_{1,2}} d_{D}^{+}(u) \geq |V_0| \geq |V_{1,2}| + 1.$$  

If we suppose on the contrary that $\Delta^{+}(D) \leq 1$, then we arrive at the contradiction

$$|V_{1,2}| \geq \sum_{u \in V_{1,2}} d_{D}^{+}(u) \geq |V_{1,2}| + 1.$$  

If $\Delta^{+}(D) \geq 2$, then Proposition 1.3 implies that $\gamma_{r2}(D) \leq n - \Delta^{+}(D) + 1 < n$. If $\Delta^{-}(D) \geq 2$, then assume that $u$ is a vertex with in-degree $\Delta^{-}(D)$ and let $v, w \in N^{-}(u)$ be two distinct vertices. Then $\{(u, v), (u, v), (v, \emptyset)\}$ is a rainbow dominating function of $D$ implying that $\gamma_{r2}(D) < n$, and the proof is complete. \( \square \)

**Corollary 2.1.** If $D$ is a directed path or directed cycle of order $n$, then $\gamma_{r2}(D) = n$.

Next we characterize the digraphs which attain the lower bound in (1.1).

**Proposition 2.3.** Let $D$ be a digraph on $n$ vertices. Then $\gamma(D) = \gamma_{r2}(D)$ if and only if $D$ has a $\gamma(D)$-set $S$ that partitions into two nonempty subsets $S_1$ and $S_2$ such that $N^{+}(S_1) = V(D) - (S_1 \cup S_2)$ and $N^{+}(S_2) = V(D) - (S_1 \cup S_2)$.

**Proof.** Assume that $\gamma(D) = \gamma_{r2}(D)$ and let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{r2}$-function of $D$. If $\gamma(D) = \gamma_{r2}(D) = n$, then clearly $D$ is empty and the result is immediate. Let $\gamma(D) = \gamma_{r2}(D) < n$. Then the assumption implies that we have equality in $\gamma(D) \leq |V_1| + |V_2| + |V_{1,2}| \leq |V_1| + |V_2| + 2|V_{1,2}| = \gamma_{r2}(D)$. This implies that $|V_{1,2}| = 0$ and hence we deduce that each vertex in $V_0$ has at least one in-neighbor in $V_1$ and one in-neighbor in $V_2$. Therefore, $V(D) - (V_1 \cup V_2) \subseteq N^{+}(V_1)$ and $V(D) - (V_1 \cup V_2) \subseteq N^{+}(V_2)$. If there is an arc $(a, b)$ in $D[V_1 \cup V_2]$, then obviously $(V_1 \cup V_2) - \{b\}$ is a dominating set of $D$ which is a contradiction. Hence $V_1 \cup V_2$ is dominating set of $D$ with $V(D) - (V_1 \cup V_2) = N^{+}(V_1)$ and $V(D) - (V_1 \cup V_2) = N^{+}(V_2)$.

Conversely, assume that $D$ has a minimum dominating set $S$ that partitions into two nonempty subsets $S_1$ and $S_2$ such that $N^{+}(S_1) = V(D) - (S_1 \cup S_2)$ and $N^{+}(S_2) = V(D) - (S_1 \cup S_2)$. It is straightforward to verify that the function $(V(D) - (S_1 \cup S_2), S_1, S_2, \emptyset)$ is a rainbow dominating function of $D$ of weight $\gamma(D)$ and hence $\gamma(D) = \gamma_{r2}(D)$. This completes the proof. \( \square \)

3. Cartesian Product of Directed Cycles

Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs which have disjoint vertex sets $V_1$ and $V_2$ and disjoint arc sets $A_1$ and $A_2$, respectively. The Cartesian product $D_1 \Box D_2$ is the digraph with vertex set $V_1 \times V_2$ and for any two vertices $(x_1, x_2)$ and $(y_1, y_2)$ of $D_1 \Box D_2$, $(x_1, x_2)(y_1, y_2) \in A(D_1 \Box D_2)$ if one of the following holds:

(i) $x_1 = y_1$ and $x_2y_2 \in A(D_2)$;
(ii) $x_1y_1 \in A(D_1)$ and $x_2 = y_2$.  

We denote the vertices of a directed cycle $C_n$ by the integers $\{1, 2, \ldots, n\}$ considered modulo $n$. Note that $N^+(i, j)) = \{(i, j + 1), (i + 1, j)\}$ for any vertex $(i, j) \in V(C_m \square C_n)$, the first and second digit are considered modulo $m$ and $n$, respectively. For any $k \in \{1, 2, \ldots, n\}$, we will denote by $C^k_{2n}$ the subdigraph of $C_m \square C_n$ induced by the vertices $\{(j, k) | j \in \{1, 2, \ldots, m\}\}$. Note that $C^k_{2n}$ is isomorphic to $C_{m}$. Let $f$ be a $\gamma_{r2}(C_m \square C_n)$-function and set $a_k = \sum_{x \in V(C^k_{2n})} |f(x)|$ for any $k \in \{1, 2, \ldots, n\}$. Then $\gamma_{r2}(C_m \square C_n) = \sum_{k=1}^{n} a_k$. It is easy to see that $C_m \square C_n \cong C_{n} \square C_{m}$ for any directed cycles of length $m, n \geq 2$, and so $\gamma_{r2}(C_m \square C_n) = \gamma_{r2}(C_{n} \square C_{m})$.

**Theorem 3.1.** For $n \geq 2$, $\gamma_{r2}(C_2 \square C_n) = \begin{cases} n & \text{if } n \text{ is even} \\ n + 1 & \text{if } n \text{ is odd.} \end{cases}

**Proof.** First let $n$ be even. Define $f : V(C_2 \square C_n) \to \mathcal{P}\{\{1, 2\}\}$ by $f((1, 2i - 1)) = \{1\}$ for $1 \leq i \leq \frac{n}{2}$, $f((2, 2i)) = \{2\}$ for $1 \leq i \leq \frac{n}{2}$ and $f(x) = \emptyset$ otherwise. Obviously, $f$ is a 2RDF of $C_2 \square C_n$ of weight $n$ and hence $\gamma_{r2}(C_2 \square C_n) \leq n$. On the other hand, since $\Delta^+(C_2 \square C_n) = 2$, it follows from Theorem 2.3 that $\gamma_{r2}(C_2 \square C_n) = n$.

Now let $n$ be odd. We claim that $\gamma_{r2}(C_2 \square C_n) \geq n + 1$. Assume to the contrary that $\gamma_{r2}(C_2 \square C_n) \leq n$. Let $f$ be a $\gamma_{r2}(C_2 \square C_n)$-function and set $a_k = \sum_{x \in V(C^k_{2n})} |f(x)|$ for any $k \in \{1, 2, \ldots, n\}$. If $a_k = 0$ for some $k$, say $k = 3$, then $f((1, 3)) = f((2, 3)) = \emptyset$ and to dominate the vertices $(1, 3)$ and $(2, 3)$ we must have $f((1, 2)) = \{1\}$ and $f((2, 2)) = \{1, 2\}$, respectively. Then the function $g : V(C_2 \square C_n) \to \mathcal{P}\{\{1, 2\}\}$ by $g((1, 2)) = \{1\}, g((2, 2)) = \emptyset, g((2, 1)) = g((2, 3)) = \{2\}$ and $g(x) = f(x)$ for $x \in V(C_2 \square C_n) - \{(1, 2), (2, 2), (2, 1), (2, 3)\}$ is a 2RDF of $C_2 \square C_n$ of weight less than $\omega(f)$ which is a contradiction. Thus $a_k \geq 1$ for each $k$. By assumption $a_k = 1$ for each $k$.

We may assume, without loss of generality, that $f((1, 2)) = \{1\}$. To dominate $(2, 2)$, we must have $f((2, 1)) = \{2\}$. Since $a_3 = 1$ and $f((2, 2)) = \emptyset$, to dominate $(1, 3)$, we must have $f((2, 3)) = \{2\}$. Repeating this process we obtain $f((1, 2i)) = \{1\}$ for $1 \leq i \leq \frac{n-1}{2}$ and $f((2, 2i - 1)) = \{2\}$ for $1 \leq i \leq \frac{n+1}{2}$ and $f(x) = \emptyset$ otherwise. But then the vertex $(1, 1)$ is not dominated, a contradiction. Thus $\gamma_{r2}(C_2 \square C_n) \geq n + 1$.

Define $g : V(C_2 \square C_n) \to \mathcal{P}\{\{1, 2\}\}$ by $g((1, 1)) = \{1\}, g((1, 2i)) = \{1\}$ for $1 \leq i \leq \frac{n-1}{2}$ and $g((2, 2i - 1)) = \{2\}$ for $1 \leq i \leq \frac{n+1}{2}$ and $g(x) = \emptyset$ otherwise. Clearly, $g$ is a 2RDF of $C_2 \square C_n$ of weight $n + 1$ and hence $\gamma_{r2}(C_2 \square C_n) = n + 1$.

**Theorem 3.2.** For $n \geq 2$, $\gamma_{r2}(C_3 \square C_n) = 2n$.

**Proof.** First we prove $\gamma_{r2}(C_3 \square C_n) \leq 2n$. Define $g : V(C_3 \square C_n) \to \mathcal{P}\{\{1, 2\}\}$ as follows:

- If $n \equiv 0 \pmod{3}$, then $g((1, 3i + 1)) = g((2, 3i + 2)) = g((3, 3i + 3)) = \{1\}, g((1, 3i + 3)) = g((2, 3i + 1)) = g((3, 3i + 2)) = \{2\}$ for $0 \leq i \leq \frac{n}{3} - 1$ and $g(x) = \emptyset$ otherwise,

- if $n \equiv 1 \pmod{3}$, then $g((3, n)) = \{1\}, g((2, n)) = \{2\}, g((1, 3i + 1)) = g((2, 3i + 2)) = g((3, 3i + 3)) = \{1\}, g((1, 3i + 3)) = g((2, 3i + 1)) = g((3, 3i + 2)) = \{2\}$ for $0 \leq i \leq \frac{n-1}{3} - 1$ and $g(x) = \emptyset$ otherwise,

- if $n \equiv 2 \pmod{3}$, then $g((1, n)) = g((1, n - 1)) = g((3, n)) = \{1\}, g((2, n - 1)) = \{2\}, g((1, 3i + 1)) = g((2, 3i + 2)) = g((3, 3i + 3)) = \{1\}, g((1, 3i + 3)) = g((2, 3i + 1)) = \emptyset$, otherwise.

Hence, $\gamma_{r2}(C_3 \square C_n) = 2n$, and the proof is complete.
$g((3, 3i + 2)) = \{2\} \text{ for } 0 \leq i \leq \frac{n-2}{3} \text{ and } g(x) = \emptyset \text{ otherwise. It is easy to see that}
\text{in each case, } g \text{ is a 2RDF of } C_3 \square C_n \text{ of weight } 2n \text{ and hence } \gamma_r(2) (C_3 \square C_n) \leq 2n.$

Now we show that $\gamma_r(2) (C_3 \square C_n) \geq 2n$. First we prove that for any $\gamma_r(2) (C_3 \square C_n)$-function $f$, $\sum_{x \in V(C_n^\bullet)} |f(x)| \geq 1$ for each $k$. Let to the contrary that $\sum_{x \in V(C_n^\bullet)} |f(x)| = 0$ for some $k$, say $k = n$. Then we must have $f((1, n-1)) = f((2, n-1)) = f((3, n-1)) = \{1, 2\}$. Then the function $f_1$ defined by $f_1((1, n-1)) = f_1((2, n-1)) = f_1((3, n-1)) = \{1\}$, $f_1((1, n)) = f_1((2, n)) = \{2\}$ and $f_1(x) = f(x)$ otherwise, is a 2RDF of $G$ of weight less than $\omega(f)$, a contradiction.

Let $g$ be a $\gamma_r(2) (C_3 \square C_n)$-function such that the size of the set $S = \{k | a_k = \sum_{x \in V(C_n^\bullet)} |g(x)| = 1 \text{ and } 1 \leq i \leq n\}$ is minimum. We claim that $|S| = 0$ implying that $\gamma_r(2) (C_3 \square C_n) = \omega(g) \geq 2n$. Assume to the contrary that $|S| \geq 1$. Suppose, without loss of generality, that $a_n = 1$ and that $g((1, n)) = \{1\}$ and $g((2, n)) = g((3, n)) = \emptyset$. To dominate $(2, n)$ and $(3, n)$ we must have $2 \in g((2, n-1))$ and $g((3, n-1)) = \{1, 2\}$, respectively. If $n = 3$, then clearly $a_1 \geq 2$ and so $\gamma_r(2) (C_3 \square C_n) \geq 2n$. Let $n \geq 4$. Consider two cases.

Case 1. $a_{n-2} = 1$.

As above we have $a_{n-3} \geq 3$. Thus $a_{n-3} + a_{n-2} + a_{n-1} + a_n \geq 8$. If $n = 4$, we are done. Suppose $n \geq 5$. Since $a_{n-4} \geq 1$, $1 \in \bigcup_{i=1}^3 g((i, n-4))$ or $2 \in \bigcup_{i=1}^3 g((i, n-4))$.

Let $1 \in \bigcup_{i=1}^3 g((i, n-4))$ (the case $2 \in \bigcup_{i=1}^3 g((i, n-4))$ is similar).

If $1 \in g((3, n-4))$, then define $h : V(C_3 \square C_n) \rightarrow \mathcal{P}(\{1, 2\})$ by $h((1, n-3)) = h((1, n-1)) = h((2, n-3)) = h((3, n-1)) = \{2\}$, $h((1, n)) = h((2, n)) = h((2, n-2)) = h((3, n-2)) = \{1\}$, $h((1, n-2)) = h((2, n-1)) = h((3, n-3)) = h((3, n)) = \emptyset$ and $h(x) = g(x)$ otherwise.

If $1 \in g((2, n-4))$, then define $h : V(C_3 \square C_n) \rightarrow \mathcal{P}(\{1, 2\})$ by $h((1, n-3)) = h((2, n-1)) = h((3, n-1)) = h((3, n-3)) = \{2\}$, $h((1, n)) = h((1, n-2)) = h((2, n-2)) = h((3, n)) = \{1\}$, $h((2, n-2)) = h((1, n-1)) = h((2, n-3)) = h((2, n)) = \emptyset$ and $h(x) = g(x)$ otherwise.

If $1 \in g((1, n-4))$, then define $h : V(C_3 \square C_n) \rightarrow \mathcal{P}(\{1, 2\})$ by $h((2, n-3)) = h((2, n-1)) = h((3, n-3)) = h((3, n-1)) = \{2\}$, $h((1, n)) = h((1, n-2)) = h((3, n-2)) = h((3, n)) = \{1\}$, $h((2, n-2)) = h((1, n-1)) = h((3, n-3)) = h((2, n)) = \emptyset$ and $h(x) = g(x)$ otherwise.

Clearly, $h$ is a 2RDF of $C_3 \square C_n$ for which $|\{k | \sum_{x \in V(C_n^\bullet)} |h(x)| = 1 \text{ and } 1 \leq i \leq n\}| < |\{k | \sum_{x \in V(C_n^\bullet)} |g(x)| = 1 \text{ and } 1 \leq i \leq n\}|$ which contradicts the choice of $g$.

Case 2. $a_{n-2} \geq 2$.

Then $a_{n-2} + a_{n-1} + a_n \geq 6$. Since $a_{n-3} \geq 1$, $1 \in \bigcup_{i=1}^3 g((i, n-3))$ or $2 \in \bigcup_{i=1}^3 g((i, n-3))$. Assume $1 \in \bigcup_{i=1}^3 g((i, n-3))$ (the case $2 \in \bigcup_{i=1}^3 g((i, n-3))$ is similar).

If $1 \in g((3, n-3))$, then define $h : V(C_3 \square C_n) \rightarrow \mathcal{P}(\{1, 2\})$ by $h((1, n-2)) = h((2, n-2)) = h((1, n-1)) = \{2\}$, $h((1, n)) = h((3, n)) = h((3, n-1)) = \{1\}$, $h((1, n-1)) = h((2, n)) = h((3, n-2)) = \emptyset$ and $h(x) = g(x)$ otherwise.


If $1 \in g((2, n-3))$, then define $h : V(C_3 \square C_n) \to \mathcal{P}\{\{1, 2\}\}$ by $h((1, n-2)) = h((2, n-1)) = h((3, n-2)) = \{2\}$, $h((1, n)) = h((3, n)) = h((3, n-1)) = \{1\}$, $h((1, n-1)) = h((2, n-2)) = h((2, n)) = \emptyset$ and $h(x) = g(x)$ otherwise.

If $1 \in g((1, n-3))$, then define $h : V(C_3 \square C_n) \to \mathcal{P}\{\{1, 2\}\}$ by $h((2, n-2)) = h((3, n-2)) = h((3, n-1)) = \{2\}$, $h((1, n)) = h((1, n-1)) = h((2, n)) = \{1\}$, $h((3, n)) = h((1, n-2)) = h((2, n-1)) = \emptyset$ and $h(x) = g(x)$ otherwise.

Clearly, $h$ is a 2RDF of $C_3 \square C_n$ for which $|\{k \mid \sum_{x \in V(C_m^k)} |g(x)| = 1 \text{ and } 1 \leq i \leq n\}| < |\{k \mid \sum_{x \in V(C_m^k)} |g(x)| = 1 \text{ and } 1 \leq i \leq n\}|$ which is a contradiction again.

Therefore, $|\{k \mid a_k = \sum_{x \in V(C_m^k)} |g(x)| = 1 \text{ and } 1 \leq i \leq n\}| = 0$ and hence $\gamma_{r2}(C_3 \square C_n) = \omega(g) \geq 2n$. Thus $\gamma_{r2}(C_3 \square C_n) = 2n$ and the proof is complete. □

**Proposition 3.1.** If $m = 2r$ and $n = 2s$ for some positive integers $r, s$, then $\gamma_{r2}(C_m \square C_n) = \frac{mn}{2}$.

**Proof.** Define $f : V(C_m \square C_n) \to \mathcal{P}\{\{1, 2\}\}$ by $f((2i - 1, 2j - 1)) = \{1\}$ for each $1 \leq i \leq r$ and $1 \leq j \leq s$, $f((2i, 2j)) = \{2\}$ for each $1 \leq i \leq r$ and $1 \leq j \leq s$ and $f(x) = \emptyset$ otherwise. It is easy to see that $f$ is a 2RDF of $C_m \square C_n$ with weight $\frac{mn}{2}$ and so $\gamma_{r2}(C_m \square C_n) \leq \frac{mn}{2}$. Now the results follows from Theorem 2.3. □

### 4. Strong product of directed cycles

Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs which have disjoint vertex sets $V_1$ and $V_2$ and disjoint arc sets $A_1$ and $A_2$, respectively. The strong product $D_1 \otimes D_2$ is the digraph with vertex set $V_1 \times V_2$ and for any two vertices $(x_1, x_2)$ and $(y_1, y_2)$ of $D_1 \otimes D_2$, $(x_1, x_2)(y_1, y_2) \in A(D_1 \otimes D_2)$ if one of the following holds:

(i) $x_1y_1 \in A(D_1)$ and $x_2y_2 \in A(D_2)$;
(ii) $x_1 = y_1$ and $x_2y_2 \in A(D_2)$;
(iii) $x_1y_1 \in A(D_1)$ and $x_2 = y_2$.

We denote the vertices of a directed cycle $C_n$ by the integers $\{1, 2, \ldots, n\}$ considered modulo $n$. There is an arc $xy$ from $x$ to $y$ in $C_n$ if and only if $y = x + 1 \pmod{n}$ and $N^+(i, j) = \{(i, j + 1), (i + 1, j), (i + 1, j + 1)\}$ for any vertex $(i, j) \in V(C_m \square C_n)$, the first and second digit are considered modulo $m$ and $n$, respectively. We use the notation defined in Section 3.

**Lemma 4.1.** For positive integers $m, n \geq 2$, $\gamma_{r2}(C_m \otimes C_n) \geq \lceil \frac{mn}{2} \rceil$.

**Proof.** Observe that the vertices of $C_m^k$ are dominated by vertices of $C_m^{k-1}$ or $C_m^k$, $k = 1, 2, \ldots, n$. Especially, the vertices of $C_m^1$ are dominated by $C_m^1$ and $C_m^n$. We show that $\sum_{k=1}^{n} a_k \geq \lceil \frac{mn}{2} \rceil$. In order to this, we show that $a_k + a_{k+1} \geq m$ for each $k = 1, 2, \ldots, n$, where $a_{n+1} = a_1$. First let $a_{k+1} = 0$. Then to rainbowly dominate $(i, k + 1)$ for each $1 \leq i \leq m$, we must have $|f((i - 1, k))| + |f((i, k))| \geq 2$. Then $2a_k = \sum_{i=1}^{m} (|f((i - 1, k))| + |f((i, k))|) \geq 2m$ and hence $a_k + a_{k+1} \geq m$. If $a_{k+1} = t$,
then it is not hard to see that $a_k \geq m - t$ and so $a_k + a_{k+1} \geq m$. Therefore,

$$2\gamma_{r,2}(C_m \otimes C_n) = 2 \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (a_k + a_{k+1}) \geq nm$$

where $a_{n+1} = a_1$. This implies that $\gamma_{r,2}(C_m \otimes C_n) \geq \lceil\frac{mn}{2}\rceil$.

\textbf{Proposition 4.1.} If $m = 2r$ and $n = 2s$ for some positive integers $r, s$, then $\gamma_{r,2}(C_m \otimes C_n) = \lceil\frac{mn}{2}\rceil$.

\textit{Proof.} Define $f : V(D) \to \mathcal{P}(\{1, 2\})$ by $f((2i - 1, 2j - 1)) = \{1, 2\}$ for each $1 \leq i \leq r$ and $1 \leq j \leq s$, and $f(x) = \emptyset$ otherwise. It is easy to see that $f$ is a 2RDF of $C_m \otimes C_n$ with weight $\frac{mn}{2}$ and so $\gamma_{r,2}(C_m \otimes C_n) \leq \frac{mn}{2}$. Now the results follows from Lemma 4.1. \hfill \Box

\textbf{Proposition 4.2.} If $m = 4r$ and $n = 2s + 1$ for some positive integers $r, s$, then $\gamma_{r,2}(C_m \otimes C_n) = \lceil\frac{mn}{2}\rceil$.

\textit{Proof.} Let

\begin{align*}
W_1 &= \{(4i + 1, 1) \mid 0 \leq i \leq r - 1\}, \\
W_2 &= \{(4i + 3, 1) \mid 0 \leq i \leq r - 1\}, \\
Z_1^1 &= \{(4i + 2, 2j) \mid 0 \leq i \leq r - 1 \text{ and } 1 \leq j \leq s\}, \\
Z_1^2 &= \{(4i + 4, 2j) \mid 0 \leq i \leq r - 1 \text{ and } 1 \leq j \leq s\}, \\
Z_2^1 &= \{(4i + 4, 2j + 1) \mid 0 \leq i \leq r - 1 \text{ and } 1 \leq j \leq s\}, \\
Z_2^2 &= \{(4i + 2, 2j + 1) \mid 0 \leq i \leq r - 1 \text{ and } 1 \leq j \leq s\}.
\end{align*}

Define $f : V(C_m \otimes C_n) \to \mathcal{P}(\{1, 2\})$ by $f(x) = \{1\}$ for $x \in W_1 \cup Z_1^1 \cup Z_2^1$, $f(x) = \{2\}$ for $x \in W_2 \cup Z_1^2 \cup Z_2^2$, and $f(x) = \emptyset$ otherwise. It is easy to see that $f$ is a 2RDF of $C_m \otimes C_n$ with weight $\frac{mn}{2}$ and so $\gamma_{r,2}(C_m \otimes C_n) \leq \frac{mn}{2}$. It follows from Lemma 4.1 that $\gamma_{r,2}(C_m \otimes C_n) = \lceil\frac{mn}{2}\rceil$. \hfill \Box

\textbf{Proposition 4.3.} If $m = 4r + 2$ and $n = 2s + 1$ for some positive integers $r, s$, then $\lceil\frac{mn}{2}\rceil \leq \gamma_{r,2}(C_m \otimes C_n) \leq \lceil\frac{mn}{2}\rceil + 1$.

\textit{Proof.} Let

\begin{align*}
C_1 &= \{(4r + 2, 2s + 1) \cup (4i + 4, 2s + 1) \mid 0 \leq i \leq r - 1\}, \\
C_2 &= \{(4r + 1, 2s + 1) \cup (4i + 2, 2s + 1) \mid 0 \leq i \leq r - 1\}, \\
C_{1,2} &= \{(4r + 1, 2j + 1) \mid 0 \leq j \leq s - 1\}, \\
Z_1^1 &= \{(4i + 1, 2j + 1) \mid 0 \leq i \leq r - 1 \text{ and } 0 \leq j \leq s - 1\}, \\
Z_1^2 &= \{(4i + 1, 2j) \mid 0 \leq i \leq r - 1 \text{ and } 1 \leq j \leq s\}, \\
Z_2^1 &= \{(4i + 3, 2j) \mid 0 \leq i \leq r - 1 \text{ and } 1 \leq j \leq s\}, \\
Z_2^2 &= \{(4i + 3, 2j + 1) \mid 0 \leq i \leq r - 1 \text{ and } 0 \leq j \leq s - 1\}.
\end{align*}
Define $f : V(C_m \otimes C_n) \to \mathcal{P}\{1, 2\}$ by $f(x) = \{1\}$ for $x \in C_{1,2}$, $f(x) = \{1\}$ for $x \in C_1 \cup Z_1^1 \cup Z_2^1$, $f(x) = \{2\}$ for $x \in C_2 \cup Z_1^2 \cup Z_2^2$, and $f(x) = \emptyset$ otherwise. It is easy to see that $f$ is a 2RDF of $C_m \otimes C_n$ with weight $\frac{mn}{2} + 1$ and so $\gamma(C_m \otimes C_n) \leq \frac{mn}{2} + 1$. It follows from Lemma 4.1 that $\lceil \frac{mn}{2} \rceil \leq \gamma(C_m \otimes C_n) \leq \lceil \frac{mn}{2} \rceil + 1$. □

**Lemma 4.2.** If $m = 4r + 3$ and $n = 4s + 3$ for some non-negative integers $r, s$, then $\gamma(C_m \otimes C_n) = \lceil \frac{mn}{2} \rceil$.

**Proof.** Let

\[
\begin{align*}
R_1^1 &= \{(4i + 1, 4j + 1) \mid 0 \leq i \leq r \text{ and } 0 \leq j \leq s\}, \\
R_1^2 &= \{(4i + 1, 4j + 3) \mid 0 \leq i \leq r \text{ and } 0 \leq j \leq s\}, \\
R_2^1 &= \{(4i + 2, 4j + 2) \mid 0 \leq i \leq r \text{ and } 0 \leq j \leq s\}, \\
R_2^2 &= \{(4i + 2, 4j + 4) \mid 0 \leq i \leq r \text{ and } 0 \leq j \leq s - 1\}, \\
R_3^1 &= \{(4i + 3, 4j + 3) \mid 0 \leq i \leq r \text{ and } 0 \leq j \leq s\}, \\
R_3^2 &= \{(4i + 3, 4j + 1) \mid 0 \leq i \leq r \text{ and } 0 \leq j \leq s\}, \\
R_4^1 &= \{(4i + 4, 4j + 4) \mid 0 \leq i \leq r - 1 \text{ and } 0 \leq j \leq s - 1\}, \\
R_4^2 &= \{(4i + 4, 4j + 2) \mid 0 \leq i \leq r - 1 \text{ and } 0 \leq j \leq s\}.
\end{align*}
\]

Define $f : V(C_m \otimes C_n) \to \mathcal{P}\{1, 2\}$ by $f(x) = \{1\}$ for $x \in R_1^1 \cup R_2^1 \cup R_3^1 \cup R_4^1$, $f(x) = \{2\}$ for $x \in R_1^2 \cup R_2^2 \cup R_3^2 \cup R_4^2$, and $f(x) = \emptyset$ otherwise. It is easy to see that $f$ is a 2RDF of $C_m \otimes C_n$ with weight $\lceil \frac{mn}{2} \rceil + 1$ and so $\gamma(C_m \otimes C_n) \leq \lceil \frac{mn}{2} \rceil$. By Lemma 4.1 we obtain $\gamma(C_m \otimes C_n) = \lceil \frac{mn}{2} \rceil$. □

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