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A NOTE ON THE DOMINATION NUMBER OF THE CARTESIAN PRODUCTS OF PATHS AND CYCLES

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Abstract. Using algebraic approach we implement a constant time algorithm for computing the domination numbers of the Cartesian products of paths and cycles. Closed formulas are given for domination numbers $\gamma(P_n \square C_k)$ (for $k \leq 11$, $n \in \mathbb{N}$) and domination numbers $\gamma(C_n \square P_k)$ and $\gamma(C_n \square C_k)$ (for $k \leq 7$, $n \in \mathbb{N}$).

1. Introduction

Domination and its variations have been intensively studied and its algorithmic aspects have been widely investigated [11, 12]. It is well known that the problem of determining the domination number of arbitrary graphs is NP–complete [11]. It is therefore interesting to consider algorithms for some classes of graphs, including Cartesian products of paths and cycles. Exact domination numbers of the Cartesian products of paths $P_n \square P_k$ with fixed k were established in [1, 3, 4, 6, 10, 21]. Formulas were given for k up to 19 [6, 21] and in [1] for $k, n \leq 29$. Recently Chang's conjecture stating that for every $16 \leq k \leq n$, $\gamma(P_n \square P_k) = \left\lfloor \frac{(n+2)(k+2)}{5} \right\rfloor - 4$ was proved in [8].

Domination number of the Cartesian product of cycles $C_n \square C_k$, also known as tori, was studied in [5, 16]. In [16], exact formulas for all $k \leq 4$ and all $n \in \mathbb{N}$ were given. Formula for $k = 5$ appears in [17], referring to an unpublished manuscript by the second author. Domination number of the Cartesian product of more than two cycles are investigated and in a special case, a formula was given [16].

Cartesian products of cycles and paths are considered in [19]: Exact values for $C_n \square P_k$ ($k \leq 4$, $n \in \mathbb{N}$) are calculated and the domination number of $C_n \square P_5$ is

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bounded. Exact values of $\gamma(C_n \Box P_5)$ for some n were given in [20], as they can be deduced from the Roman domination numbers.

A general $O(\log n)$ algorithm based on path algebra approach which can be used to compute various graph invariants on fasciagraphs and rotagraphs has been proposed in [17]. The algorithm of [17] can in most cases, including the computation of distance based invariants [14], the domination numbers [22] and Roman domination numbers [20] be turned into a constant time algorithm, i.e. the algorithm can find closed formulas for arbitrary n. The existence of an algorithm that provides closed formulas for domination numbers on grid graphs has been observed or claimed also in [6, 18].

Here we use the algorithm of [22] to find closed formulas for domination numbers of $P_n \Box C_k$ and provide closed formulas (for $k \leq 11$ and $n \in \mathbb{N}$). Furthermore, closed formulas for the domination numbers of $C_n \Box P_k$ and $C_n \Box C_k$ (for $k \leq 7$ and $n \in \mathbb{N}$) are given. The new formulas are an improvement of known results and provide answer to some open questions from [16, 19].

In the rest of this paper we first summarize the background for the main algorithm from [17] and [22]. The algorithm is precisely presented in Section 3. In Section 4 we summarize results obtained by implementing the algorithm. Also some constructions for minimum dominating sets of investigated graphs are given.

2. Preliminaries

We consider finite undirected and directed graphs. A graph will always mean an undirected graph, a digraph will stand for a directed graph. An edge in an undirected graph will be denoted uv while in directed graph, an arc between vertices u and v will be denoted (u, v). P_n will stand for a path on n vertices and C_n for a cycle on n vertices.

For a graph $G = (V, E)$, a set D is a *dominating set* if every vertex in $V \setminus D$ is adjacent to a vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a *minimum dominating set*, or shortly a γ–set.

The *Cartesian product* of graphs G and H, denoted $G \Box H$, is a graph with vertex set $V(G) \times V(H)$ and two vertices (g, h) and (g', h') are adjacent if $g = g'$ and $hh' \in E(H)$ or $gg' \in E(G)$ and $h = h'$. Examples of the Cartesian product graphs include the grid graphs, which are products of paths $P_n \Box P_k$, and tori, which are products of cycles $C_n \Box C_k$.

Polygraphs can be used in chemical graph theory as a graph theoretical model of polymers. They were first introduced in [2] and also discussed in [9, 13, 15]. Let $G_1, \ldots, G_n$ be arbitrary mutually disjoint graphs and $X_1, \ldots, X_n$ a sequence of sets of edges such that an edge of X_i joins a vertex of $V(G_i)$ with a vertex of $V(G_{i+1})$ ($X_i \subseteq V(G_i) \times V(G_{i+1})$ for $i = 1, \ldots, n$). For convenience we also set $G_{n+1} = G_1$.

A *polygraph* $\Omega_n = \Omega_n(G_1, \dots G_n; X_1, \dots X_n)$ over monographs $G_1, \dots, G_n$ is defined in the following way:

$$V(\Omega_n) = V(G_1) \cup \dots \cup V(G_n),$$
$$E(\Omega_n) = E(G_1) \cup X_1 \cup \dots \cup E(G_n) \cup X_n.$$

For a polygraph Ω_n and for $i = 1, \dots, n$ we also define

$$D_i = \{u \in V(G_i) \mid \exists v \in G_{i+1} : uv \in X_i\},$$
$$R_i = \{u \in V(G_{i+1}) \mid \exists v \in G_i : uv \in X_i\}.$$

In general, $R_i \cap D_{i+1}$ does not have to be empty. If all graphs G_i are isomorphic to a fixed graph G (i.e. there exists an isomorphism $\varphi_i : V(G_i) \longrightarrow V(G)$ for $i = 1, \dots, n+1$, and $\varphi_{n+1} = \varphi_1$) and all sets X_i are equal to a fixed set $X \subseteq V(G) \times V(G)$ ($(u, v) \in X \iff \big(\varphi_i^{-1}(u), \varphi_{i+1}^{-1}(v)\big) \in X_i$ for all i), we call such a graph *rotagraph*, $\omega_n(G; X)$. A rotagraph without edges between the first and the last copy of G is a *fasciagraph*, $\psi_n(G; X)$. More precisely, in a fasciagraph, $X_n = \emptyset$ and $X_1 = X, \dots, X_{n-1} = X$. In a rotagraph as well as in a fasciagraph, all sets D_i and R_i are equal to fixed sets D and R, respectively ($D_i = \varphi_i^{-1}(D)$ and $R_i = \varphi_{i+1}^{-1}(R)$). Of course, in the case of fasciagraphs, $D_n = \emptyset$ and $R_n = \emptyset$. Observe that Cartesian products of paths $P_n \Box P_k$ are examples of fasciagraphs and that Cartesian products of cycles $C_n \Box C_k$ are examples of rotagraphs. Products of a path and a cycle can be treated either as fasciagraphs or as rotagraphs.

A *semiring* $\mathcal{P} = (P, \oplus, \circ, e^{\oplus}, e^{\circ})$ is a set P on which two binary operations, $\oplus$ and $\circ$ are defined such that:

(a) $(P, \oplus)$ is a commutative monoid with $e^{\oplus}$ as a unit;
(b) $(P, \circ)$ is a monoid with e° as a unit;
(c) $\circ$ is left– and right–distributive over $\oplus$;
(d) $\forall x \in P, x \circ e^{\oplus} = e^{\oplus} = e^{\oplus} \circ x$.

An idempotent semiring is called a *path algebra*. It is easy to see that a semiring is a path algebra if and only if $e^{\circ} \oplus e^{\circ} = e^{\circ}$ holds for e°, the unit of the monoid $(P, \circ)$. An important example of a path algebra for our work is $\mathcal{P}_1 = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$. Here $\mathbb{N}_0$ denotes the set of nonnegative integers and $\mathbb{N}$ the set of positive integers.

Let $\mathcal{P} = (P, \oplus, \circ, e^{\oplus}, e^{\circ})$ be a path algebra and let $\mathcal{M}_n(\mathcal{P})$ be the set of all $n \times n$ matrices over P. Let $A, B \in \mathcal{M}_n(\mathcal{P})$ and define operations $\oplus$ and $\circ$ in the usual way:

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij},$$
$$(A \circ B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \circ B_{kj}.$$

$\mathcal{M}_n(\mathcal{P})$ equipped with above operations is a path algebra with the zero and the unit matrix as units of semiring. In our example $\mathcal{P}_1 = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$, all elements of the zero matrix are ∞, the unit of the monoid $(P, \min)$, and the unit matrix is a diagonal matrix with diagonal elements equal to $e^{\circ} = 0$ and all other elements equal to $e^{\oplus} = \infty$.

Let $\mathcal{P}$ be a path algebra and let G be a labeled digraph, that is a digraph together with a labeling function ℓ which assigns to every arc of G an element of P. Let $V(G) = \{v_1, v_2, \dots, v_n\}$. The labeling ℓ of G can be extended to paths in the following way: For a path $Q = (v_{i_0}, v_{i_1})(v_{i_1}, v_{i_2}) \dots (v_{i_{k-1}}, v_{i_k})$ of G let

$$\ell(Q) = \ell\left(v_{i_0}, v_{i_1}\right) \circ \ell\left(v_{i_1}, v_{i_2}\right) \circ \dots \circ \ell\left(v_{i_{k-1}}, v_{i_k}\right).$$

Let S_{ij}^k be the set of all paths of order k from v_i to v_j in G and let $A(G)$ be the matrix defined by:

$$A(G)_{ij} = \begin{cases} \ell\left(v_i, v_j\right); & \text{if } (v_i, v_j) \text{ is an arc of } G \\ e^{\oplus}; & \text{otherwise} \end{cases}$$

It is well-known that

$$\left(A(G)^k\right)_{ij} = \bigoplus_{Q \in S_{ij}^k} \ell(Q).$$

3. The domination number of fasciagraphs and rotagraphs

Let us now summarize known results for determining the domination number of fasciagraphs and rotagraphs. The algorithm which computes different graph invariants on fasciagraphs and rotagraphs in $O(\log n)$ time was proposed in [17] and then improved to run in $O(C)$ time for domination number [22] and also for some other graph invariants in [14, 20, 22, 23].

Let $\omega_n(G; X)$ be a rotagraph and $\psi_n(G; X)$ a fasciagraph. Set $U = D_i \cup R_i = D \sqcup R$ and let $N = 2^{|U|}$. Define a labeled digraph $\mathcal{G} = \mathcal{G}(G; X)$ as follows: The vertex set of $\mathcal{G}$ is formed by the subsets of U, denoted V_i. An arc joins a subset V_i with a subset V_j if V_i is not in a "conflict" with V_j. Here a conflict of V_i with V_j means that using V_i and V_j as a part of a solution in consecutive copies of G would violate the problem assumption. In the special case we are considering here, i.e. when computing the domination number, we introduce an arc between vertices V_i and V_j if $V_i \cap R \cap D \cap V_j = \emptyset$.

Now consider for a moment $\psi_3(G; X)$ and let $V_i \subseteq D_1 \cup R_1$ and $V_j \subseteq D_2 \cup R_2$ (of course $R_1 = R_2 = R$ and $D_1 = D_2 = D$). Let $\gamma_{i,j}(G; X)$ stand for the size of minimum dominating set of $G_2 \setminus ((V_i \cap R_1) \cup (D_2 \cap V_j))$. Then we define a labeling of $\mathcal{G}$, $\ell : E(\mathcal{G}) \longrightarrow \mathbb{N}_0 \cup \{\infty\}$, in the following way:

$$\ell(V_i, V_j) = |V_i \cap R| + \gamma_{i,j}(G; X) + |D \cap V_j| - |V_i \cap R \cap D \cap V_j|. \tag{3.1}$$

The following algorithm, first proposed in [17], computes the domination number of a fasciagraph or a rotagraph in $O(\log n)$ time.

Algorithm 1 [17]

(a) Let $\mathcal{P}_1 = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$ be a path algebra.
(b) Label $\mathcal{G}(G; X)$ with the labeling, defined in (3.1).
(c) In $\mathcal{M}(\mathcal{P})$ calculate $A\left(\mathcal{G}\right)^n$.
(d) Let $\gamma\left(\psi_n(G; X)\right) = \left(A\left(\mathcal{G}\right)^n\right)_{00}$ and $\gamma\left(\omega_n(G; X)\right) = \min_i \left(A\left(\mathcal{G}\right)^n\right)_{ii}$.

This algorithm can be improved: computing the powers of $A\left(\mathcal{G}\right)^n = A_n$ in $O(C)$ time is possible using special structure of the matrices in some cases, including the domination numbers.

Lemma 3.1. [22] *Let $k = |V(\mathcal{G}(G; X))|$ and $K = |V(G)|$. Then there is an index $q \leq (2K+2)^{k^2}$ such that $D_q = D_p + C$ for some index $p < q$ and some constant matrix C. Let $P = q - p$. Then for every $r \geq p$ and every $s \geq 0$ we have*

$$A_{r+sP} = A_r + sC\,.$$

Hence, if we assume that the size of G is a given constant (and n is a variable), the algorithm will run in constant time. But it is important to emphasize that the algorithm is useful for practical purposes only if the number of vertices of the monograph G is relatively small, since the time complexity is in general exponential in the number of vertices of the monograph G. Therefore some additional improvements are welcome. One can also omit straightforward implementation of the algorithm. Instead of calculating whole matrices $A(G)^n$, calculating only those rows which are important for the result and checking the difference of the new row against the previously stored rows until a constant difference is detected yields a correct result because of the following lemma.

	$\gamma(P_n \Box C_m)$
$m = 5k$	$k(n+2)$
$m = 5k+1$	$\frac{(8k+3)(n+2)}{8}$
$m = 5k+2$	$\frac{(2k+1)(n+2)}{2}$
$m = 5k+3$	$(k+1)(n+2)$
$m = 5k+4$	$(k+1)(n+2)$

Table 1. Upper bounds for the domination numbers of the Cartesian products of paths and cycles for $n \geq m$.

k	$\gamma(P_n \square C_k)$	
3	$\left\lceil \frac{3n}{4} \right\rceil + 1$;	if $n \equiv 0 \pmod 4$
	$\left\lceil \frac{3n}{4} \right\rceil$;	otherwise
4	n	
5	3;	if $n = 2$
	4;	if $n = 3$
	$n + 2$;	otherwise
6	$\left\lceil \frac{4n}{3} \right\rceil$;	if $n \equiv 1 \pmod 3$
	$\left\lceil \frac{4n}{3} \right\rceil + 1$;	otherwise
7	$\left\lceil \frac{3n}{2} \right\rceil + 1$;	if $n \equiv 1 \pmod 2$
	$\left\lceil \frac{3n}{2} \right\rceil + 2$;	otherwise
8	4;	if $n = 2$
	6;	if $n = 3$
	8;	if $n = 4$
	$\left\lfloor \frac{9n}{5} \right\rfloor + 1$;	if $n \equiv 5 \pmod{10}$
	$\left\lfloor \frac{9n}{5} \right\rfloor + 2$;	otherwise
9	5;	if $n = 2$
	7;	if $n = 3$
	10	if $n = 4$
	$2n + 2$;	otherwise
10	$2n + 2$;	if $n \leq 5$
	$2n + 3$;	if $6 \leq n \leq 9$
	$2n + 4$	otherwise
11	$\left\lceil \frac{19n}{8} \right\rceil + 1$;	if $n \in \{1, 2, 4, 6\}$ or $n \equiv 3 \pmod 8$
	$\left\lceil \frac{19n}{8} \right\rceil + 2$;	otherwise

Table 2. Domination numbers of $P_n \square C_k$ for a fixed k.

Lemma 3.2. [20] *Assume that the j–th row of A_{n+P} and A_n differ for a constant, $a_{ji}^{(n+P)} = a_{ji}^{(n)} + C$ for all i. Then $\min_i a_{ji}^{(n+P)} = \min_i a_{ji}^{(n)} + C$.*

4. Summary

We implemented the algorithm as described above and got results presented in the sequel. Calculations of $\gamma(P_n \Box C_k)$ for a fixed k were implemented as fasciagraph and calculations of $\gamma(C_n \Box P_k)$ and $\gamma(C_n \Box C_k)$ for a fixed k were implemented as rotagraphs.

k	$\gamma(C_n \Box P_k)$		$\gamma(C_n \Box C_k)$	
2	$\lceil \frac{n}{2} \rceil + 1$;	if $n \equiv 2 \pmod 4$		
	$\lceil \frac{n}{2} \rceil$;	otherwise		
3	$\lceil \frac{3n}{4} \rceil$		$\lceil \frac{3n}{4} \rceil$	
4	$n+1$;	if $n \in \{5, 9\}$	n	
	n;	otherwise		
5	4;	if $n = 3$	n;	if $n \equiv 0 \pmod 5$
	$\lceil \frac{6n}{5} \rceil + 1$;	if $n \equiv 3, 5, 9 \pmod{10}$	$n+2$;	if $n \equiv 3 \pmod 5$
	$\lceil \frac{6n}{5} \rceil$;	otherwise	$n+1$;	otherwise
6	9;	if $n = 6$	$\lceil \frac{4n}{3} \rceil + 1$;	$n \equiv 2, 3, 8, 9, 11, 14,$
	$\lceil \frac{10n}{7} \rceil + 1$;	$n \equiv 2, 6, 7, 9, 13 \pmod{14}$		$15, 17 \pmod{18}$
	$\lceil \frac{10n}{7} \rceil$;	otherwise	$\lceil \frac{4n}{3} \rceil$;	otherwise
7	6;	if $n = 3$	$\lceil \frac{3n}{2} \rceil$;	$n \equiv 0, 5, 9 \pmod{14}$
	16;	if $n = 9$	$\lceil \frac{3n}{2} \rceil + 2$;	$n \equiv 2, 8, 12 \pmod{14}$
	36;	if $n = 21$	$\lceil \frac{3n}{2} \rceil + 1$;	otherwise

Table 3. Domination numbers of the Cartesian products of paths and cycles for a fixed k.

It is important to emphasize that calculations for fasciagraphs are a lot less time consuming than those for rotagraphs because

$$\gamma\left(\psi_n(G;X)\right) = \left(A\left(\mathcal{G}\right)^n\right)_{00} \quad \text{and} \quad \gamma\left(\omega_{\mathrm{n}}(\mathrm{G};\mathrm{X})\right) = \min_{\mathrm{i}}\left(\mathrm{A}\left(\mathcal{G}\right)^{\mathrm{n}}\right)_{\mathrm{ii}}.$$

In other words, only one element of the matrix $A\left(\mathcal{G}\right)^n$ is sufficient for the result on fasciagraphs and a minimum over all diagonal elements of the same matrix is needed for rotagraphs.

Theorem 4.1. *Domination numbers of the Cartesian products of paths and cycles, where the size of one factor is fixed, can be in computed constant time, i.e. independently of the size of the second factor. Closed expressions for $\gamma(P_n \Box C_k)$, $\gamma(C_n \Box P_k)$, and $\gamma(C_n \Box C_k)$ are given in Table 2 and in Table 3 for some values of k.*

We also present constructions of γ-sets in some cases (see Fig. 1 and Fig. 2). The patterns are enclosed in dashed boxes. In case of the graph $P_n \Box C_{11}$ the pattern is shifted hence the period is $11 \cdot 8 = 88$.

We conclude with a couple of remarks. Since every vertex of the Cartesian product of two cycles can dominate at most five vertices, it follows that $\gamma(C_n \Box C_k) \geq \frac{nk}{5}$. Adapting similar idea as in case of $P_n \Box C_{10}$ to graphs $C_{5m} \Box C_{5l}$ we have $\gamma(C_n \Box C_k) \leq \frac{nk}{5}$ by construction. Therefore $\gamma(C_{5m} \Box C_{5l}) = 5ml$. This result could also be deduced from results on Roman domination presented in [7].

By obvious reasoning we can extend constructions from Fig. 2 to obtain upper bounds for graphs $P_n \Box C_m$, see Table 1. We conjecture that the bounds are exact values for $m = 5k, 5k+1, 5k+2, 5k+4$. Clearly as $\gamma(C_n \Box C_m) \leq \gamma(P_n \Box C_m)$, these are also upper bounds for the Cartesian product of cycles.

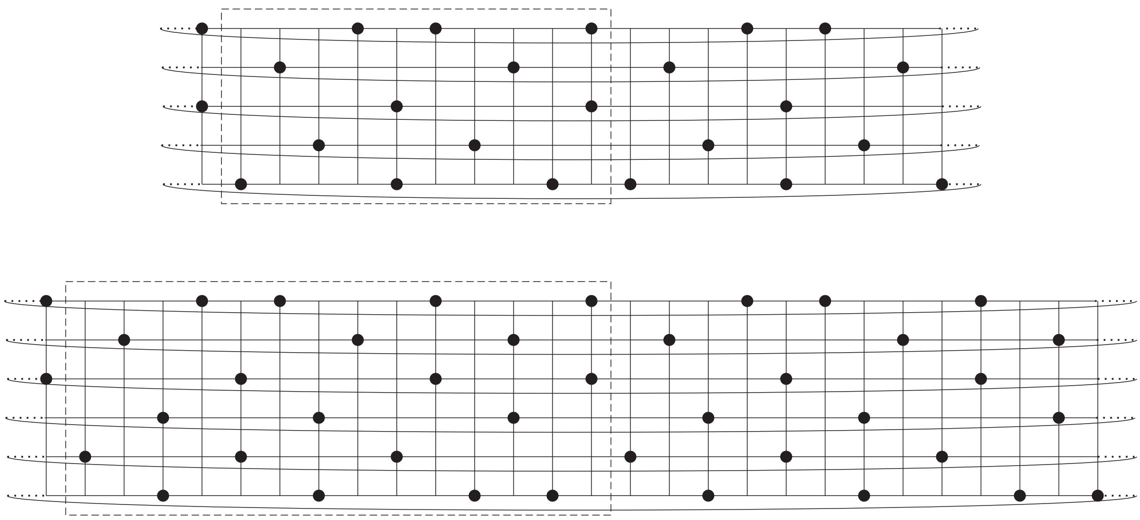

FIGURE 1. Periodical behaviour of γ-sets of $C_n \Box P_k$ for $5 \leq k \leq 6$.

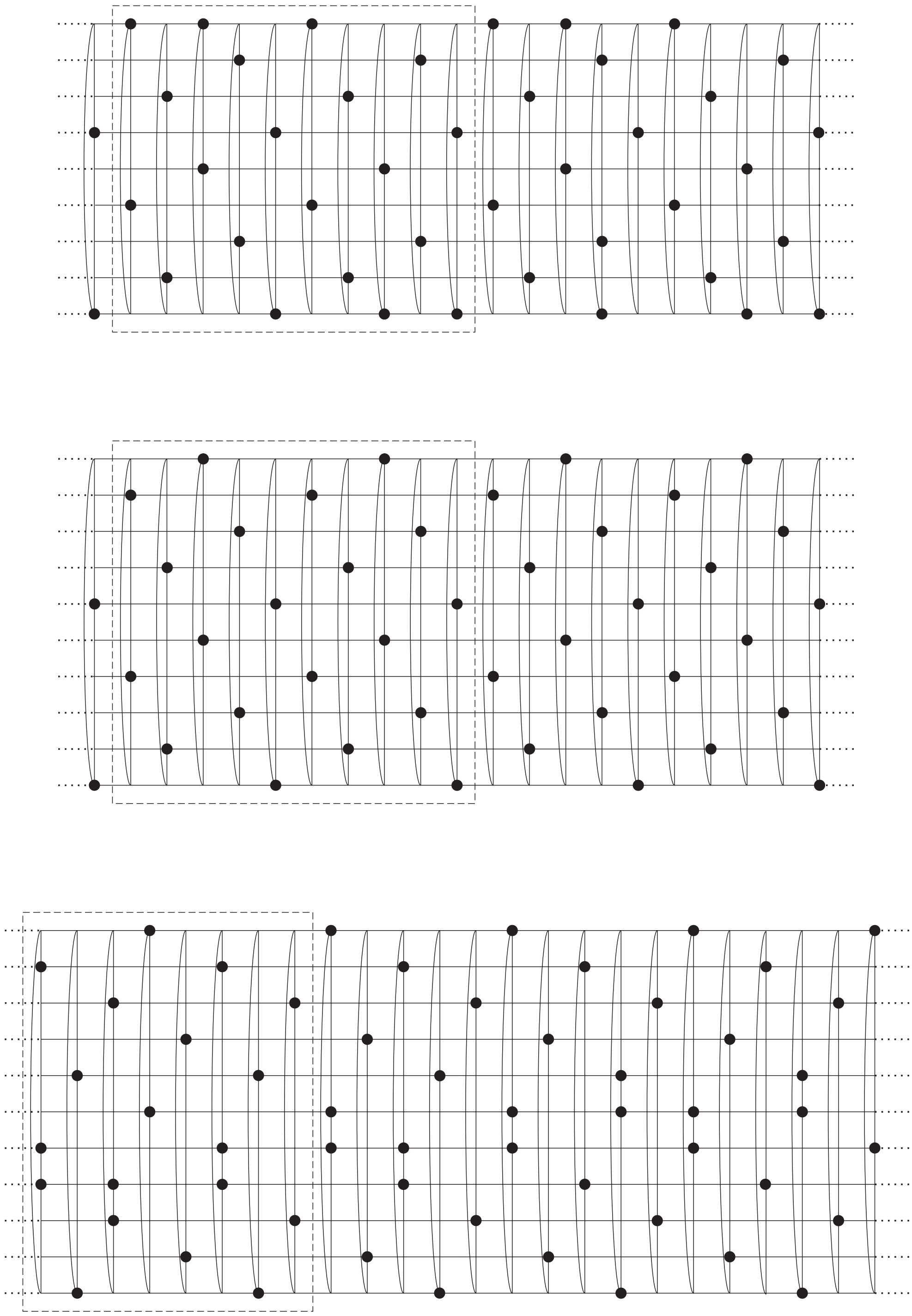

Figure 2. Periodical behaviour of γ-sets of $P_n \square C_k$ for $9 \leq k \leq 11$.

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