Kragujevac Journal of Mathematics
Volume 37(2) (2013), Pages 287–297.

ULAM STABILITY OF BOUNDARY VALUE PROBLEM

RABHA W. IBRAHIM

Abstract. In this paper we present and discuss different types of Ulam stability: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for Cauchy differential equation of fractional order in the unit disk. Moreover, the generalized Ulam-Hyers stability for univalent solution is introduced and the existence and uniqueness of boundary problem are established.

1. Introduction

The concept of the arbitrary order calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) was invistegated over 300 years ago. Abel in 1823 imposed the generalized tautochrone problem and for the first while employed the fractional calculus techniques in a physical problem. Subsequent Liouville utilized fractional calculus to problems in potential theory. Ever after the fractional calculus has exhausted the attention of many authors in all area of sciences [1].

Fractional differential equations (real and complex) are viewed as models for nonlinear differential equations. Difference of fractional differential equations are applied not only in mathematics but also in physics, dynamical systems, control systems and engineering to create the mathematical modeling of many physical phenomena. In addition, they employed in social science such as food supplement, climate and economics. One of these equations is a super-linear fractional differential equation [2] and heat equation [14].

Newly, the theory of fractional calculus has located pleasant applications in the theory of analytic functions. The classical definitions of fractional operators and their

Key words and phrases. Fractional calculus, Fractional differential equation, Unit disk, Analytic function, Univalent function.

2010 *Mathematics Subject Classification.* 30C45, 39B72, 39B82.

Received: July 28, 2012.

Revised: May 17, 2013.

generalizations have fruitfully been applied in obtaining, for example, the characterization properties, coefficient estimates [3], distortion inequalities [12] and convolution structures for various subclasses of analytic functions and the works in the research monographs.

Srivastava and Owa (1989) [13], gave definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows:

Definition 1.1. The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta,$$

where $0 \le \alpha < 1$ and the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when$(z-\zeta) > 0$.

Definition 1.2. The fractional integral of order $\alpha > 0$ is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta;$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $(\mathbb{C})$ containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Remark 1.1.

$$D_z^\alpha \{z^\mu\} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} \{z^{\mu-\alpha}\}, \ \ \mu > -1;$$

for $0 \le \alpha < 1$ and for $\mu > -1; \ \alpha > 0$,

$$I_z^\alpha \{z^\mu\} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} \{z^{\mu+\alpha}\}.$$

Further properties of these operators can be found in [10] and [13].

2. Methods

In the theory of functional equations there are some special kind of data dependence: Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Hyers- Bourgin, Aoki-Rassias, [4, 10, 11]. In this paper we pose different types of Ulam stability: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for the following Cauchy differential equation of fractional order in the unit disk

$$D_z^\alpha u(z) = f(z, u(z)), \tag{2.1}$$

where $u : U \to \mathbb{B}$ is an analytic function for all z in the unit disk $U := \{z : |z| < 1\}$ and $f : U \times \mathbb{B} \to \mathbb{B}$ is an analytic function in $z \in U$. Hear $\mathbb{B}$ is the space of all analytic and bounded functions in the unit disk. Recently, the existence and uniqueness of

Cauchy differential equations of fractional order in the unit disk are considered and studied in [5–7]. Moreover the Ulam-Hyers stability in complex domain was suggested in [8, 9]. While in real fractional calculus was imposed by [15, 16].

Let $(\mathbb{B}, |.|)$ be a complex Banach space endow with the sup. norm.

Definition 2.1. The equation (2.1) is Ulam-Hyers stable if there exists a real number $c > 0$ such that for each $\epsilon > 0$ and for each solution in the holomorphic space $v \in \mathcal{H}(U; \mathbb{B})$ of

$$|D_z^\alpha v(z) - f(z, v(z))| \le \epsilon, \quad z \in U \tag{2.2}$$

there exists a solution $u \in \mathcal{H}(U; \mathbb{B})$ of (2.1) with

$$|v(z) - u(z)| \le c\epsilon, \quad z \in U.$$

Definition 2.2. The equation (2.1) is generalized Ulam-Hyers stable if there exists $\psi \in \mathcal{H}(\mathbb{R}_+; \mathbb{R}_+)$ such that for each solution in the holomorphic space $v \in \mathcal{H}(U; \mathbb{B})$ of (2.2) there exists a solution $u \in \mathcal{H}(U; \mathbb{B})$ of (2.1) with

$$|v(z) - u(z)| \le \psi(\epsilon), \quad z \in U.$$

Definition 2.3. The equation (2.1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in \mathcal{H}(U; \mathbb{R}_+)$ if there exists a real number $c > 0$ such that for each $\epsilon > 0$ and for each solution in the holomorphic space $v \in \mathcal{H}(U; \mathbb{B})$ of

$$|D_z^\alpha v(z) - f(z, v(z))| \le \epsilon\varphi(z), \quad z \in U \tag{2.3}$$

there exists a solution $u \in \mathcal{H}(U; \mathbb{B})$ of (2.1) with

$$|v(z) - u(z)| \le c\epsilon\varphi(z), \quad z \in U.$$

Remark 2.1. A function $v \in \mathcal{H}(U; \mathbb{B})$ is a solution of (2.2) if and only if there exists a function $g \in \mathcal{H}(U; \mathbb{B})$ (depends on v) such that

(i) $|g(z)| \le \epsilon, \quad z \in U,$
(ii) $D_z^\alpha v(z) = f(z, v(z)) + g(z), \quad z \in U.$

We need the following result in the sequel:

Lemma 2.1. [5] *For $0 < \alpha < 1$ and f is analytic, then*

$$I_z^\alpha D_z^\alpha f(z) = f(z), \quad f(0) = 0.$$

3. Ulam stability

Let us consider the equation (2.1) and the inequality (2.2). We have the following result.

Theorem 3.1. *Assume that*

(i) $f \in \mathcal{H}[U \times \mathbb{B}; \mathbb{B}]$,
(ii) *there exists $\ell > 0$, $\ell \neq \Gamma(\alpha + 1)$ such that $|f(z, u(z)) - f(z, v(z))| \le \ell|u - v|$, $z \in U$, $u, v \in \mathbb{B}$. Then the equation (2.1) is generalized Ulam-Hyers stable.*

Proof. Let $v \in \mathcal{H}(U;\mathbb{B})$ be a solution of (2.2). Let us denote by $u \in \mathcal{H}(U;\mathbb{B})$ the unique solution of the Cauchy problem [5]

$$D_z^\alpha u(z) = f(z,u(z)), \quad z \in U$$

$$u(0) = v(0).$$

By using Lemma 2.1, we pose

$$u(z) = v(0) + \frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta,u(\zeta))(z-\zeta)^{\alpha-1}d\zeta$$

and

$$|v(z) - v(0) - \frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta,v(\zeta))(z-\zeta)^{\alpha-1}d\zeta| \le \epsilon.$$

From these relations we obtain,

$$\begin{aligned}
|v(z)-u(z)| &= \left|v(z) - v(0) - \frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta,u(\zeta))(z-\zeta)^{\alpha-1}d\zeta\right| \\
&\le \left|v(z) - v(0) - \frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta,v(\zeta))(z-\zeta)^{\alpha-1}d\zeta\right| \\
&\quad + \left|\frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta,v(\zeta))(z-\zeta)^{\alpha-1}d\zeta - \frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta,u(\zeta))(z-\zeta)^{\alpha-1}d\zeta\right| \\
&\le \epsilon + \frac{\ell|v(z)-u(z)|}{\Gamma(\alpha+1)}
\end{aligned}$$

or

$$|v(z)-u(z)| \le \frac{\epsilon}{1-\frac{\ell}{\Gamma(\alpha+1)}} := c\epsilon, \quad \ell \ne \Gamma(\alpha+1).$$

Hence (2.1) is Ulam-Hyers stable. By putting $\psi(\epsilon) = c\epsilon$, $\psi(0) = 0$ yields that Eq. (2.1) is generalized Ulam-Hyers stable. □

Theorem 3.2. *Assume that*

(i) $f \in \mathcal{H}[U \times \mathbb{B};\mathbb{B}]$,
(ii) *there exists* $\ell > 0$, $\ell \ne \Gamma(\alpha+1)$ *such that*

$$|f(z,u(z)) - f(z,v(z))| \le \ell|u-v|, \quad z \in U,\ u,v \in \mathbb{B}.$$

(iii) *there exists* $\varphi \in \mathcal{H}(U;\mathbb{R}_+)$ *that satisfies (2.3). Then the equation (2.1) is Ulam-Hyers-Rassias stable.*

Proof. Let $v \in \mathcal{H}(U;\mathbb{B})$ be a solution of (2.3). In virtue of Remark 2.1,

$$|v(z) - v(0) - \frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta,v(\zeta))(z-\zeta)^{\alpha-1}d\zeta| \le \epsilon\varphi(z).$$

From this relation we impose,

$$\begin{aligned}
|v(z)-u(z)| &= \Big|v(z)-v(0)-\int_0^z \frac{f(\zeta,u(\zeta))(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)}d\zeta\Big| \\
&\le \Big|v(z)-v(0)-\int_0^z \frac{f(\zeta,v(\zeta))(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)}d\zeta\Big| \\
&\quad + \Big|\frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta,v(\zeta))(z-\zeta)^{\alpha-1}d\zeta - \frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta,u(\zeta))(z-\zeta)^{\alpha-1}d\zeta\Big| \\
&\le \epsilon\varphi(z) + \frac{\ell|v(z)-u(z)|}{\Gamma(\alpha+1)}
\end{aligned}$$

or

$$|v(z)-u(z)| \le \frac{\epsilon\varphi(z)}{1-\frac{\ell}{\Gamma(\alpha+1)}} := c\epsilon\varphi(z).$$

($\ell \neq \Gamma(\alpha+1)$). Hence problem (2.1) is Ulam-Hyers-Rassias stable. □

In the next results, we investigate the generalized Ulam-Hyers stability for univalent solution of Cauchy problem. It will be taken in two cases depending on α.

Theorem 3.3. *Let $0 < \alpha < 1$. If the problem (2.1) has a univalent (one-to-one) solution in U, then it is generalized Ulam-Hyers stable.*

Proof. Let v be univalent solution of the problem (1) in U [5]. In view of Theorem 3 in [12], we have

$$|D_z^\alpha v(z)| \le \frac{r^{1-\alpha}}{\Gamma(1-\alpha)}\int_0^1 \frac{1+rt}{(1-t)^\alpha(1-rt)^3}dt,$$

$$(|z| = r < 1,\ z \in U,\ 0 < \alpha < 1).$$

Assume that $|f| \le M < \infty$; thus we obtain

$$|D_z^\alpha v(z) - f(z,v)| \le \frac{r^{1-\alpha}}{\Gamma(1-\alpha)}$$
$$\int_0^1 \frac{1+rt}{(1-t)^\alpha(1-rt)^3}dt + M := \epsilon.$$

To show that (2.1) is generalized Ulam-Hyers stable, a computation gives

$$\begin{aligned}
|v(z)-u(z)| &= \Big| \int_0^z f(\zeta, v(\zeta))(z-\zeta)^{\alpha-1} d\zeta - \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta, u(\zeta))(z-\zeta)^{\alpha-1} d\zeta \Big| \\
&\le \frac{2M}{\Gamma(\alpha+1)} \Big| \int_0^z (z-\zeta)^{\alpha-1} d\zeta \Big| \\
&= \frac{2M}{\Gamma(\alpha+1)} \\
&= \frac{2(\epsilon - \frac{r^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 \frac{1+rt}{(1-t)^{\alpha}(1-rt)^3} dt)}{\Gamma(\alpha+1)} \\
&:= \psi(\epsilon),
\end{aligned}$$

for sufficient small r. Hence (2.1) is generalized Ulam-Hyers stable. □

Theorem 3.4. *Let $1 < \alpha < 2$. If the problem (2.1) has a univalent solution in U, then it is generalized Ulam-Hyers stable.*

Proof. Let v be univalent solution of the problem (2.1) in U [5]. Let $\alpha = \beta + 1$, $0 < \beta < 1$. In view of Theorem 4 in [12], we have

$$|D_z^{\alpha} v(z)| \le \frac{r^{-\beta}}{\Gamma(1-\beta)} (rF(2,1,1-\beta; r))',$$

$$(|z| = r \neq 0,\ z \in U,\ 0 < \beta < 1),$$

where $F(2,1,1-\beta; r)$ is a hypergeometric function. Assume that $|f| \le M < \infty$; thus we obtain

$$|D_z^{\alpha} v(z) - f(z,v)| \le \frac{r^{-\beta}}{\Gamma(1-\beta)} (rF(2,1,1-\beta; r))' + M := \epsilon.$$

Now a calculation implies

$$\begin{aligned}
|v(z)-u(z)| &= \Big| \int_0^z f(\zeta, v(\zeta))(z-\zeta)^{\alpha-1} d\zeta - \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta, u(\zeta))(z-\zeta)^{\alpha-1} d\zeta \Big| \\
&\le \frac{2M}{\Gamma(\alpha+1)} \Big| \int_0^z (z-\zeta)^{\alpha-1} d\zeta \Big| \\
&= \frac{2M}{\Gamma(\alpha+1)} \\
&= \frac{2(\epsilon - \frac{r^{-\beta}}{\Gamma(1-\beta)} (rF(2,1,1-\beta; r))'}{\Gamma(\alpha+1)} \\
&:= \psi(\epsilon),
\end{aligned}$$

for sufficient small r. Thus (2.1) is generalized Ulam-Hyers stable. □

4. Boundary value problem

In this section, we establish the existence and uniqueness of solution for the following Cauchy problem

$$D_z^\alpha u(z) = f(z, u(z)), \tag{4.1}$$

such that

$$\Big(au(0) + bu(1) = cu(\xi), \quad z, \xi \in U,\ a + b \neq c\Big).$$

This boundary condition appears in certain problems of physics where the controllers at the boundary points dissipate or add energy according to a censor located at an intermediate position.

Now, we state a known result due to Krasnoselskii which is needed to prove the existence of at least one solution of (4.1).

Lemma 4.1. *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that*

(i) *$Ax + By \in M$ whenever $x, y \in M$*
(ii) *A is compact and continuous*
(iii) *B is a contraction mapping. Then there exists $w \in M$ such that $w = Aw+Bw$.*

Theorem 4.1. *Let $f \in \mathcal{H}[U \times \mathbb{B}; \mathbb{B}]$ be a holomorphic function satisfying*

$$\|f(z, u) - f(z, v)\| \leq L\|u - v\|, \quad z \in U,\ u, v \in \mathbb{B}, \tag{4.2}$$

and for $\mu > 0$

$$\|f(z, u)\| \leq \mu, \quad \forall\, z \in \overline{U},\ u \in \mathbb{B}. \tag{4.3}$$

Then the problem (4.1) has a unique solution provided

$$L < \frac{\Gamma(\alpha + 1)}{\Big(1 + \frac{|b|+|c|)}{|a+b-c|}\Big)}.$$

Proof. Define $P : \mathcal{H} \to \mathcal{H}$ by

$$\begin{aligned}(Pu)(z) = \frac{1}{\Gamma(\alpha)} \int_0^z & f(\zeta, u(\zeta))(z - \zeta)^{\alpha-1} d\zeta \\ & - \frac{1}{(a + c - c)\Gamma(\alpha)} \Big[b \int_0^1 f(\zeta, u(\zeta))(1 - \zeta)^{\alpha-1} d\zeta \\ & - c \int_0^\xi f(\zeta, u(\zeta))(\xi - \zeta)^{\alpha-1} d\zeta\Big].\end{aligned}$$

By using (4.3), we obtain

$$\begin{aligned}\|(Pu)(z)\| &\leq \int_0^z \frac{\|f(\zeta,u(\zeta))\||z-\zeta|^{\alpha-1}}{\Gamma(\alpha)}d\zeta\\ &\quad+\frac{1}{|a+b-c|\Gamma(\alpha)}\Big[|b|\int_0^1 \|f(\zeta,u(\zeta))\||1-\zeta|^{\alpha-1}d\zeta\\ &\quad+|c|\int_0^\xi \|f(\zeta,u(\zeta))\||\xi-\zeta|^{\alpha-1}d\zeta\Big]\\ &\leq \frac{\mu}{\Gamma(\alpha+1)}+\frac{\mu(|b|+|c|)}{|a+b-c|\Gamma(\alpha+1)}:=r.\end{aligned}$$

Hence $PB_r \subset B_r$, where $B_r = \{u \in \mathbb{B} : \|u\| \leq r\}$. Now for $u, v \in \mathbb{B}$, we pose

$$\begin{aligned}\|(Pu)(z)-(Pv)(z)\| &\leq \frac{1}{\Gamma(\alpha)}\int_0^z \|f(\zeta,u(\zeta))-f(\zeta,v(\zeta))\||z-\zeta|^{\alpha-1}d\zeta\\ &\quad+\frac{1}{|a+b-c|\Gamma(\alpha)}\Big[|b|\int_0^1 \|f(\zeta,u(\zeta))-f(\zeta,v(\zeta))\||1-\zeta|^{\alpha-1}d\zeta\\ &\quad+|c|\int_0^\xi \|f(\zeta,u(\zeta))-f(\zeta,v(\zeta))\||\xi-\zeta|^{\alpha-1}d\zeta\Big]\\ &\leq \frac{L\|u-v\|}{\Gamma(\alpha+1)}+\frac{L\|u-v\|(|b|+|c|)}{|a+b-c|\Gamma(\alpha+1)}\\ &=\|u-v\|\Big[\frac{L}{\Gamma(\alpha+1)}+\frac{L(|b|+|c|)}{|a+b-c|\Gamma(\alpha+1)}\Big]\\ &=\|u-v\|\frac{L}{\Gamma(\alpha+1)}\Big(1+\frac{|b|+|c|)}{|a+b-c|}\Big).\end{aligned}$$

Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Theorem 4.2. *Assume that (4.2) and (4.3) hold with*

$$\frac{L}{\Gamma(\alpha+1)}\Big(\frac{|b|+|c|)}{|a+b-c|}\Big)<1.$$

Then the problem (4.1) has at least one solution on U.

Proof. Define two operators A and B as follows:

$$(Au)(z)=\frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta,u(\zeta))(z-\zeta)^{\alpha-1}d\zeta,$$

$$(Bu)(z)=-\frac{1}{(a+c-c)\Gamma(\alpha)}\Big[b\int_0^z f(\zeta,u(\zeta))(1-\zeta)^{\alpha-1}d\zeta-c\int_0^\xi f(\zeta,u(\zeta))(\xi-\zeta)^{\alpha-1}d\zeta\Big].$$

For $r=\frac{\mu(|a+b-c|+|b|+|c|)}{|a+b-c|\Gamma(\alpha+1)}$ and $u, v \in B_r$, we find that $\|(Au)(z)+(Bv)(z)\| \leq r$, which implies that $(Au)(z)+(Bv)(z) \in B_r$. It follows from the assumption (4.2) that B is a contraction mapping for

$$\frac{L}{\Gamma(\alpha+1)}\Big(\frac{|b|+|c|)}{|a+b-c|}\Big) < 1.$$

Now we prove the compactness of the operator A. Since f is bounded on the compact $\overline{U} \times B_r$ we have

$$\begin{aligned}\|(Au)(z_1)-(Au)(z_2)\| &= \frac{1}{\Gamma(\alpha)}\Big(\int_0^{z_1} f(\zeta,u(\zeta))[(z_1-\zeta)^{\alpha-1}-(z_2-\zeta)^{\alpha-1}]d\zeta \\ &\quad + \int_{z_1}^{z_2} f(\zeta,u(\zeta))(z_2-\zeta)^{\alpha-1}d\zeta\Big) \\ &\le \frac{\mu}{\Gamma(\alpha+1)}|2(z_1-z_2)^{\alpha}+z_1^{\alpha}-z_2^{\alpha}|\end{aligned}$$

which is independent of u. So A is relatively compact on B_r. Hence, By Arzela Ascoli Theorem, A is compact on B_r. Thus all the assumptions of Lemma 4.1 are satisfied and the conclusion of Lemma 4.1 implies that the problem (4.1) has at least one solution on U. □

5. Applications

In order to show the effectiveness of the generalized Ulam-Hyers stable and Ulam-Hyers-Rassias stable for fractional Cauchy problem in the unit disk, we present some well known examples.

Example 5.1. Consider the following linear fractional differential equation:

$$D_z^{\alpha}u(z) = -u(z), \quad 1 < \alpha \le 2 \tag{5.1}$$

such that

$$u(0) = 1, \quad u'(0) = 0.$$

This problem has a solution of the form

$$u(z) = E_{\alpha}(-z^{\alpha}),$$

where

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}$$

is the Mittag-Leffler function of order α. In view of Theorem 3.1, problem (5.1) is generalized Ulam-Hyers stable for all $1 < \alpha \le 2$.

Example 5.2. Consider the following linear fractional differential equation:

$$D_z^{\alpha}u(z) - au(z) = \lambda g(z), \tag{5.2}$$

such that $1 < \alpha \le 2$, $\lambda > 0$, and

$$u(0) = 1, \quad u'(0) = 0.$$

Assume that $g \in \mathcal{H}(U; \mathbb{R}_+)$ such that

$$|D_z^\alpha v(z) - av(z)| \leq \lambda g(z)$$

for some function $v \in \mathcal{H}(U; \mathbb{B})$. By letting $f(z,u) = au(z)$, if $0 < a < 1$, $1 < \alpha \leq 2$ then in view of Theorem 3.2, the problem (5.2) is Ulam-Hyers-Rassias stable.

Example 5.3. Consider the following non-linear problem

$$D_z^{\frac{1}{2}} u(z) = \frac{1}{16}\Big(z \sin u(z) - u(z)\cos(z)\Big), \tag{5.3}$$

such that

$$u(0) + u(1) = u(\frac{1}{2}).$$

Here $a = b = c = 1$, since

$$\begin{aligned}\Big|f(z,u(z)) - f(z,v(z))\Big| &= \frac{1}{16}\Big|\Big(z\sin u(z) - u(z)\cos(z)\Big) - \Big(z \sin v(z) - v(z)\cos(z)\Big)\Big| \\ &\leq \frac{1}{16}\Big|z(\sin u(z) - \sin v(z)) + \cos(z)(v(z) - u(z))\Big| \\ &\leq \frac{1}{8}|u-v|;\end{aligned}$$

thus (4.3) is satisfied with $L = \frac{1}{8}$. Furthermore

$$\frac{1}{8} = L < \frac{\Gamma(\alpha+1)}{\Big(1 + \frac{|b|+|c|)}{|a+b-c|}\Big)} = \frac{\Gamma(\frac{3}{2})}{3} = \frac{0.886}{3} = 0.295333..,$$

in view of Theorem 4.1, problem (5.3) has a unique solution. In addition, it is generalized Ulam-Hyers stable (Theorem 3.1).

6. Conclusion

From above, we conclude that Lipschitze (Theorems 3.1 and 3.2) and univalent (Theorems 3.3 and 3.4) conditions yield the generalized Ulam-Hyers stable and Ulam-Hyers-Rassias stable for the Cauchy differential equations of fractional order in the unit disk. By employing the Krasnoselskii and Banach fixed point theorems, the existence and uniqueness of boundary value problem are established.

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Institute of Mathematical Sciences,
University Malaya, 50603,
Malaysia
E-mail address: rabhaibrahim@yahoo.com