Kragujevac Journal of Mathematics
Volume 37(2) (2013), Pages 299–306.

AN ALTERNATIVE METHOD FOR SOLVING GENERALIZED DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

AMIT CHOUHAN[1], SUNIL DUTT PUROHIT[2], AND SATISH SARASWAT[3]

Abstract. In view of the usefulness and importance of the fractional differential equations in certain physical problems governing reaction-diffusion in complex systems and anomalous diffusion, the authors present an alternative simple method for deriving the solution of the generalized forms of the fractional differential equation and Volterra type differintegral equation. The solutions are obtained in a straightforward manner by the application of Riemann-Liouville fractional integral operator and its interesting properties. As applications of the main results, solutions of certain generalized fractional kinetic equations involving generalized Mittag-Leffler function are also studied. Moreover, results for some particular values of the parameters are also pointed out.

1. Introduction, preliminaries and definitions

In last two decades, fractional differential equations appear more and more frequently for modeling of relevant systems in various fields such as physics, chemistry, biology, economics, engineering, image and signal processing. One may refer to the books [5, 7] and [11], and the recent papers [16] and [9] on the subject. In particular, the solution and application of certain kinetic equations of fractional order are studied by Zaslavsky [18], Saichev and Zaslavsky [10], Saxena et al. [13–16], Haubold et al. [4] using integral transform technique.

The object of this paper is to investigate solution of certain class of generalized fractional differential equations by applying the technique similar to that used by Al-Saqabi and Tuan [1] for solving general differintegral equation of Volterra's type. The method extend the use of Riemann-Liouville fractional calculus operators.

Key words and phrases. Fractional calculus operators, Fractional differintegral equation, Fractional kinetic equation, Generalized Mittag-Leffler function.

2010 *Mathematics Subject Classification.* Primary: 26A33. Secondary: 45D05, 47G20, 33A12 .

Received: October 15, 2012.

In 1903, Mittag-Leffler [6] introduced the following function, in terms of the power series

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \qquad (z, \alpha \in \mathbb{C},\ \Re(\alpha) > 0).$$

Further, a two-index generalization of this function was given by Wiman [17], in the following manner (see also [3])

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad (z, \alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0). \tag{1.1}$$

By means of the series representation a more generalized function $E^{\gamma}_{\alpha,\beta}(z)$ is introduced by Prabhakar [8] as

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)\, n!} z^n, \tag{1.2}$$

where $z, \alpha, \beta, \delta \in \mathbb{C}, \Re(\alpha) > 0$. It is an entire function of order $[\Re(\alpha)]^{-1}$ and for $\delta = 1$, reduces to Mittag-Leffler function $E_{\alpha,\beta}(z)$.

The right sided Riemann Liouville fractional integral operator I^{v}_{a+} and the right sided Riemann Liouville fractional derivative operator D^{v}_{a+} are defined (cf. Samko et al. [11]) for $\Re(v) > 0$ as

$$\left(D^{-v}_{a+} f\right)(x) = \frac{1}{\Gamma(v)} \int_a^x (x-t)^{v-1} f(t)dt, \tag{1.3}$$

and

$$\left(D^{v}_{a+} f\right)(x) = \left(\frac{d}{dx}\right)^n \left(D^{-(n-v)}_{a+} f\right)(x) \qquad (n = [\Re(v)] + 1)\ ,$$

where $[x]$ denotes the greatest integer in the real number x. Further, by virtue of equation (1.2) and (1.3) it is not difficult to show that

$$D^{-\alpha}_{a+}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(t-a)^{\alpha+\beta-1} \tag{1.4}$$

and

$$D^{-\alpha}_{a+}(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}\left[\omega(t-a)^{\rho}\right] = (t-a)^{\mu+\alpha-1} E^{\gamma}_{\rho,\mu+\alpha}\left[\omega(t-a)^{\rho}\right].$$

We recall here the following formula used by Carlitz [2]

$$(a+b)_m = \sum_{r=0}^{m} \binom{m}{r} (a)_r (b)_{m-r}. \tag{1.5}$$

2. Generalized fractional differential equations

Corresponding to the bounded sequence $\{A_k\}_0^\infty$, let the function $f(x)$ be defined as

$$f\left(x\right)=\sum_{k=0}^{\infty}A_k x^k. \tag{2.1}$$

Now, we will give the following two theorems, which exhibit the solutions of generalized fractional differential equations involving the analytic function $f(x)$ defined by (2.1).

Theorem 2.1. *If $\upsilon,\mu,c > 0$, then there exists the unique solution of the fractional differential equation*

$$N\left(t\right)+c^{\upsilon}D_{a+}^{-\upsilon}N\left(t\right)=N_a(t-a)^{\mu-1}f\left[-c^{\upsilon}(t-a)^{\upsilon}\right], \tag{2.2}$$

and it is given by

$$N\left(t\right)=N_a(t-a)^{\mu-1}\sum_{m=0}^{\infty}\sum_{k=0}^{m}A_k\frac{\Gamma\left(\upsilon k+\mu\right)}{\Gamma\left(\upsilon m+\mu\right)}[-c^{\upsilon}(t-a)^{\upsilon}]^m.$$

Proof. Multiplying both sides of (2.2) by $(-c^{\upsilon})^m D_{a+}^{-m\upsilon}$and summing up from $m=0$ to ∞, it yields to

$$\sum_{m=0}^{\infty}(-c^{\upsilon})^m D_{a+}^{-m\upsilon}N\left(t\right)-\sum_{m=0}^{\infty}(-c^{\upsilon})^{m+1}D_{a+}^{-(m+1)\upsilon}N\left(t\right)$$
$$=N_a\sum_{m=0}^{\infty}(-c^{\upsilon})^m D_{a+}^{-m\upsilon}(t-a)^{\mu-1}f\left[-c^{\upsilon}(t-a)^{\upsilon}\right],$$

using (2.1), we obtain

$$N\left(t\right)=N_a\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}A_k\left(-c^{\upsilon}\right)^{m+k}D_{a+}^{-m\upsilon}(t-a)^{\upsilon k+\mu-1},$$

using (1.4) above equation becomes

$$N\left(t\right)=N_a(t-a)^{\mu-1}\sum_{m=0}^{\infty}\sum_{k=0}^{m}A_k\frac{\Gamma\left(\upsilon k+\mu\right)}{\Gamma\left(m\upsilon+\mu\right)}[-c^{\upsilon}\left(t-a\right)^{\upsilon}]^m.$$

which completes the proof of Theorem 2.1. □

Theorem 2.2. *If $\upsilon,\mu,c>0$ and $\alpha>0$, then there exists the unique solution of the Volterra's type fractional differintegral equation*

$$D_{a+}^{\alpha}\,N\left(t\right)+c^{\upsilon}D_{a+}^{-\upsilon}N\left(t\right)=N_a(t-a)^{\mu-1}f\left[-c^{\upsilon}(t-a)^{\alpha+\upsilon}\right], \tag{2.3}$$

with the initial conditions

$$D_{a+}^{\alpha-r-1}N_a=b_r; \qquad (r=0,1,2,\dots,n-1), \tag{2.4}$$

and given by

$$N\left(t\right)=N_a(t-a)^{\alpha+\mu-1}\sum_{m=0}^{\infty}\sum_{k=0}^{m}A_k\frac{\Gamma\left[(\alpha+\upsilon)k+\mu\right]}{\Gamma\left[(\alpha+\upsilon)\,m+\alpha+\mu\right]}\left[-c^{\upsilon}\left(t-a\right)^{\alpha+\upsilon}\right]^m$$
$$+\sum_{r=0}^{n-1}b_r(t-a)^{\alpha-r-1}E_{\alpha+\upsilon,\alpha-r}\left[-c^{\upsilon}(t-a)^{\alpha+\upsilon}\right].$$

Proof. By applying the fractional integral operator $D_{a+}^{-\alpha}$ to the both sides of (2.3), and using the formula [11]

$$D_{a+}^{-\alpha}\,D_{a+}^{\alpha}N\left(t\right)=N\left(t\right)-\sum_{r=0}^{n-1}\frac{(t-a)^{\alpha-r-1}}{\Gamma\left(\alpha-r\right)}D_{a+}^{\alpha-r-1}N_a,$$

where n be an integer such that $n=[\alpha]+1$, we obtain

$$N\left(t\right)+c^{\upsilon}D_{a+}^{-(\alpha+\upsilon)}N\left(t\right)=N_aD_{a+}^{-\alpha}(t-a)^{\mu-1}f\left[-c^{\upsilon}(t-a)^{\alpha+\upsilon}\right]$$
$$+\sum_{r=0}^{n-1}\frac{(t-a)^{\alpha-r-1}}{\Gamma\left(\alpha-r\right)}D_{a+}^{\alpha-k-1}N_a. \tag{2.5}$$

Now by applying the operator $(-c^{\upsilon})^mD_{a+}^{-m(\alpha+\upsilon)}$ to the both sides of (2.5) and summing up from $m=0$ to ∞, we get

$$\sum_{m=0}^{\infty}(-c^{\upsilon})^mD_{a+}^{-m(\alpha+\upsilon)}N\left(t\right)-\sum_{m=0}^{\infty}(-c^{\upsilon})^{m+1}D_{a+}^{-(m+1)(\alpha+\upsilon)}N\left(t\right)$$
$$=N_a\sum_{m=0}^{\infty}(-c^{\upsilon})^mD_{a+}^{-\alpha-m(\alpha+\upsilon)}(t-a)^{\mu-1}f\left[-c^{\upsilon}(t-a)^{\alpha+\upsilon}\right]$$
$$+\sum_{m=0}^{\infty}\sum_{r=0}^{n-1}(D_{a+}^{\alpha-r-1}N_0)(-c^{\upsilon})^mD_{a+}^{-m(\alpha+\upsilon)}\frac{(t-a)^{\alpha-r-1}}{\Gamma\left(\alpha-r\right)}.$$

Using (1.4), (2.4) and applying equation (1.1), we obtain

$$N\left(t\right)=N_a\sum_{m=0}^{\infty}(-c^{\upsilon})^mD_{a+}^{-\alpha-m(\alpha+\upsilon)}(t-a)^{\mu-1}f\left[-c^{\upsilon}(t-a)^{\alpha+\upsilon}\right]$$
$$+\sum_{r=0}^{n-1}b_r(t-a)^{\alpha-r-1}E_{\alpha+\upsilon,\alpha-r}\left[-c^{\upsilon}(t-a)^{\alpha+\upsilon}\right].$$

Finally by making use of equations (2.1) and (1.4), we obtain

$$N\left(t\right)=N_a(t-a)^{\alpha+\mu-1}\sum_{m=0}^{\infty}\sum_{k=0}^{m}A_k\frac{\Gamma\left[(\alpha+\upsilon)k+\mu\right]}{\Gamma\left[(\alpha+\upsilon)\,m+\alpha+\mu\right]}\left[-c^{\upsilon}\left(t-a\right)^{\alpha+\upsilon}\right]^m$$
$$+\sum_{r=0}^{n-1}b_r(t-a)^{\alpha-r-1}E_{\alpha+\upsilon,\alpha-r}\left[-c^{\upsilon}(t-a)^{\alpha+\upsilon}\right].$$

Which completes the proof of Theorem 2.2. □

Remark 2.1. It is interesting to observe that, when $\alpha = 0$ and $b_r = 0$ Theorem 2.2 reduces to Theorem 2.1.

3. Generalized fractional kinetic equation

In this section, we consider some consequences and applications of the main results. By assigning suitable special values to the arbitrary sequence A_k, our main results (Theorems 2.1 and 2.2) can be applied to derive solutions of certain generalized fractional kinetic equation. If $N(t)$ is the number density of a given species at time t and N_a is the number density of that species at time $t = a$, then following results holds.

Theorem 3.1. *If $\upsilon > 0$, $\mu > 0, c > 0$, then there exists the unique solution of the fractional kinetic equation*

$$N(t) - N_a(t-a)^{\mu-1}E^{\gamma}_{\upsilon,\mu}\left[-c^{\upsilon}(t-a)^{\upsilon}\right] = -c^{\upsilon}D^{-\upsilon}_{a+}N(t),$$

given by

$$N(t) = N_a(t-a)^{\mu-1}E^{\gamma+1}_{\upsilon,\mu}\left[-c^{\upsilon}(t-a)^{\upsilon}\right], \tag{3.1}$$

where the Mittag-Leffler function defined by (1.2).

Proof. Let $A_k = \frac{(\gamma)_k}{\Gamma(\upsilon k+\mu)k!}$, then by virtue of Theorem 2.1, we have

$$N(t) = N_a(t-a)^{\mu-1}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{(\gamma)_k}{\Gamma(\upsilon k+\mu)\,k!}\frac{\Gamma(\upsilon k+\mu)}{\Gamma(m\upsilon+\mu)}\left[-c^{\upsilon}(t-a)^{\upsilon}\right]^m,$$

using (1.5) we get

$$N(t) = N_a(t-a)^{\mu-1}\sum_{m=0}^{\infty}\frac{(\gamma+1)_m}{\Gamma(\upsilon m+\mu)\,m!}\left[-c^{\upsilon}(t-a)^{\upsilon}\right]^m,$$

finally on using (1.2), we can easily get (3.2). □

When $a = 0$, then (3.1) reduces to the following result given earlier by Saxena et al. [12].

Corollary 3.1. *If $\upsilon > 0$, $c > 0$, $\mu > 0$, then the unique solution of the integral equation*

$$N(t) - N_0 t^{\mu-1}E^{\gamma}_{\upsilon,\mu}\left[-c^{\upsilon}t^{\upsilon}\right] = -c^{\upsilon}D^{-\upsilon}_{0+}N(t),$$

is given by

$$N(t) = N_0 t^{\mu-1}E^{\gamma+1}_{\upsilon,\mu}\left[-c^{\upsilon}t^{\upsilon}\right]. \tag{3.2}$$

Theorem 3.2. *If $v > 0, \alpha > 0,\ \mu\ >\ 0, c\ >\ 0$, then there exists the unique solution of the fractional kinetic differintegral equation*

$$D_{a+}^{\alpha}\ N\left(t\right) - N_a(t-a)^{\mu-1}E_{\alpha+v,\mu}^{\gamma}\left[-c^v(t-a)^{\alpha+v}\right] = -c^v D_{a+}^{-v}N\left(t\right), \tag{3.3}$$

with the initial conditions

$$D_{a+}^{\alpha-r-1}N_a = b_r \qquad (r = 0, 1, 2, \ldots, n-1),$$

given by

$$\begin{aligned} N\left(t\right) = {} & N_a(t-a)^{\alpha+\mu-1}E_{\alpha+v,\alpha+\mu}^{\gamma+1}\left[-c^v(t-a)^{\alpha+v}\right] \\ & + \sum_{r=0}^{n-1} b_r(t-a)^{\alpha-r-1}E_{\alpha+v,\alpha-r}\left[-c^v(t-a)^{\alpha+v}\right]. \end{aligned} \tag{3.4}$$

Proof. Let $A_k = \frac{(\gamma)_k}{\Gamma[(\alpha+v)k+\mu]k!}$, then by virtue of Theorem 2.2, we have

$$\begin{aligned} N\left(t\right) = {} & N_a(t-a)^{\alpha+\mu-1}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{(\gamma)_k}{\Gamma\left[(\alpha+v)k+\mu\right]k!}\frac{\Gamma\left[(\alpha+v)k+\mu\right]}{\Gamma\left[(\alpha+v)\,m+\alpha+\mu\right]} \\ & \times\left[-c^v\left(t-a\right)^{\alpha+v}\right]^m + \sum_{r=0}^{n-1} b_r(t-a)^{\alpha-r-1}E_{\alpha+v,\alpha-r}\left[-c^v(t-a)^{\alpha+v}\right]. \end{aligned}$$

By applying (1.5), we obtain

$$\begin{aligned} N\left(t\right) = {} & N_a(t-a)^{\alpha+\mu-1}\sum_{m=0}^{\infty}\frac{(\gamma+1)_m}{\Gamma\left[(\alpha+v)\,m+\alpha+\mu\right]m!}\left[-c^v\left(t-a\right)^{\alpha+v}\right]^m \\ & + \sum_{r=0}^{n-1} b_r(t-a)^{\alpha-r-1}E_{\alpha+v,\alpha-r}\left[-c^v(t-a)^{\alpha+v}\right]. \end{aligned}$$

Finally, on making use of (1.2), one can easily arrive at (3.4). □

Remark 3.1. Again, if we set $\alpha = 0$ and $b_r = 0$ Theorem 3.2 reduces to Theorem 3.1.

Corollary 3.2. *If $0 < \alpha \leq 1$, then the solution of* (3.3) *is given by*

$$\begin{aligned} N\left(t\right) = {} & N_a(t-a)^{\alpha+\mu-1}E_{\alpha+v,\alpha+\mu}^{\gamma+1}\left[-c^v(t-a)^{\alpha+v}\right] \\ & +\ b_0(t-a)^{\alpha-1}E_{\alpha+v,\alpha}\left[-c^v(t-a)^{\alpha+v}\right]. \end{aligned}$$

Corollary 3.3. *If $1 < \alpha \leq 2$, the solution of* (3.3) *is given by*

$$
\begin{aligned}
N(t) = {} & N_a(t-a)^{\alpha+\mu-1}E_{\alpha+v,\alpha+\mu}^{\gamma+1}\left[-c^v(t-a)^{\alpha+v}\right] \\
& + b_0(t-a)^{\alpha-1}E_{\alpha+v,\alpha}\left[-c^v(t-a)^{\alpha+v}\right] \\
& + b_1(t-a)^{\alpha}E_{\alpha+v,\alpha-1}\left[-c^v(t-a)^{\alpha+v}\right]. \qquad (3.5)
\end{aligned}
$$

We conclude with the remark that, in this paper, we have used an alternative, simple and effective method for solving fractional differential and Volterra's type fractional differintegral equations. It has been shown that by selecting bounded sequence, we found solutions of generalized Kinetic equations, as special cases of the main results. Further, as the applications of these results, one can easily derive a number of graphical representation for solutions by setting numerical values to the parameters used in the main results.

References

[1] B.N. Al-Saqabi and V.K. Tuan, *Solution of a fractional differ integral equation*, Integral transform and Spec. Funct. **4** (1996), 321–326.

[2] L. Carlitz, *Some expansion and convolution formulas related to Mac Mohan's master theorems*, SIAM J. Math. Annal. **8(2)** (1977), 320–336.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental functions*, McGraw-Hill, New York 1955.

[4] H.J. Haubold, A.M. Mathai and R.K. Saxena, *Further solutions of fractional reaction-diffusion equations in terms of the H-function*, Journal of Comput. Appl. Maths. **235(5)** (2011), 1311–1316.

[5] E. Hilfer (ed), *Applications of Fractional calculus in Physics*, World Scientific, 2000.

[6] G.M. Mittag-Leffler, *Sur la nouvelle fonction $E_\alpha(x)$*, C. R. Acad. Sci. Paris **137** (1903), 554–558.

[7] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[8] T.R. Prabhakar, *A singular integral equation with a generalized Mittag-Leffler function in its kernel*, Yokohama. Math. J. **19** (1971), 7–15.

[9] S.D. Purohit and S.L. Kalla, *On fractional partial differential equations related to quantum mechanics*, J. Phys. A: Math. Theor., **44**(4) (2011) (8pp.)

[10] A. Saichev and M. Zaslavsky, *Fractional kinetic equations: solutions and applications*, Chaos **7** (1997), 753–764.

[11] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Switzerland, 1993.

[12] R.K. Saxena, A.M. Mathai and H.J. Haubold, *On generalized fractional kinetic equations*, Physica A **344** (2004a), 653–664.

[13] R.K. Saxena, A.M. Mathai and H.J. Haubold, *Fractional reaction-diffusion equations*, Astrophysics and Space Sci. **305** (2006a), 289–296.

[14] R.K. Saxena, A.M. Mathai and H.J. Haubold, *Solution of generalized fractional reaction-diffusion equations*, Astrophysics and Space Sci. **305** (2006b), 305–313.
[15] R.K. Saxena, A.M. Mathai and H.J. Haubold, *Reaction-diffusion systems and nonlinear waves*, Astrophysics and Space Sci. **305** (2006c), 297–303.
[16] R.K. Saxena, A.M. Mathai and H.J. Haubold, *Solutions of certain fractional kinetic equations and a fractional diffusion equation*, J. Math. Phys. **51** (2010), 103–506.
[17] A. Wiman, *Uber de fundamental satz in der theorie der funktionen $E_\alpha(x)$*, Acta Math. **29** (1905), 191–201.
[18] G.M. Zaslavsky, *Fractional kinetic equation for Hamiltonian chaos*, Physica D **76** (1994), 110–122.

[1]Department of Mathematics,
JIET Group of Institutions, Jodhpur-342002,
INDIA
E-mail address: amit.maths@gmail.com

[2]Department of Basic Sciences (Mathematics),
College of Technology and Engineering,
M.P. University of Agriculture and Technology, Udaipur-313001
INDIA
E-mail address: sunil_a_purohit@yahoo.com

[3]Department of Mathematics,
Govt. College Kota, Kota-324001,
INDIA
E-mail address: satish_ktt@yahoo.co.in