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SOLUBLE GROUPS WITH $\mathfrak{F}$-PERMUTABLE SUBGROUPS

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Abstract. Let G be a finite group and $\mathfrak{F}$ a class of finite groups. A subgroup H of G is said to be $\mathfrak{F}$-permutable in G if there exists a subgroup T of G such that HT is s-permutable in G and $(H \cap T)H_G/H_G$ is contained in the $\mathfrak{F}$-hypercenter $Z_\infty^{\mathfrak{F}}(G/H_G)$ of G/H_G. By using this new concept, we establish some new criteria for a group G to be soluble.

1. Introduction

Throughout this article, all groups considered are finite and G always denotes a group. The terminologies and notations are standard, as in [4] and [9].

Recall that a subgroup H of G is said to be s-permutable, or π-quasinormal [7] in G if H is permutable with every Sylow subgroup P of G (that is, $HP = PH$). A subgroup H of G is said to be c-supplemented [11] in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the maximal normal subgroup of G contained in H. By using the s-permutability and c-supplementation of subgroups, people have obtained many interesting results; see, for example, [1, 5, 7, 8, 10–12], etc.

In this article, we give the following more generalized concept.

Definition 1.1. Let H be a subgroup of G and $\mathfrak{F}$ a class of finite groups. We say that H is $\mathfrak{F}$-permutable in G if there exists a subgroup T of G such that HT is s-permutable in G and $(H \cap T)H_G/H_G$ is contained in the $\mathfrak{F}$-hypercenter $Z_\infty^{\mathfrak{F}}(G/H_G)$ of G/H_G.

Recall that, for a class $\mathfrak{F}$ of groups, a chief factor H/K of a group G is called $\mathfrak{F}$-central (see [4]) if $[H/K](G/C_G(H/K)) \in \mathfrak{F}$. The symbol $Z_\infty^{\mathfrak{F}}(G)$ denotes the $\mathfrak{F}$-hypercenter of a group G, that is, the product of all such normal subgroups H of G

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whose G-chief factors are $\mathfrak{F}$-central. A subgroup H of G is said to be $\mathfrak{F}$-hypercenter in G if $H \le Z_\infty^{\mathfrak{F}}(G)$. A class $\mathfrak{F}$ of groups is called a formation if it is closed under a homomorphic image and a subdirect product. It is clear that every group G has a smallest normal subgroup (called $\mathfrak{F}$-residual of G and denoted by $G^{\mathfrak{F}}$) with quotient in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if it contains every group G with $G/\Phi(G) \in \mathfrak{F}$. We use $\mathfrak{S}$ to denote the formation of all soluble groups.

Obviously, all s-permutable subgroups and all c-supplemented subgroups are all $\mathfrak{F}$-permutable subgroups. However, the following examples show that the converse is not true.

Example 1.1. Let $G = C_7 \wr C_3 = [K]C_3$ be a regular wreath product, where K is the base group of $C_7 \wr C_3$ and $|C_i| = i$. Then $K = F(G)$ is the Sylow 7- subgroup of G and the subgroup $H = \{(a_1, a_2, 1) \mid a_1, a_2 \in C_7\}$ is maximal in K. It is clear that H is $\mathfrak{F}$-permutable in G. However, H is not s-permutable in G.

Example 1.2. Let $G = A \rtimes B$, where A is a cyclic group of order 5 and $B = \langle\alpha\rangle$, where $\alpha \in Aut(A)$ with $|\alpha| = 4$. It is easy to see that $\langle\alpha^2\rangle$ is $\mathfrak{F}$-permutable in G. However, $\langle\alpha^2\rangle$ is not c-supplemented in G.

2. Preliminaries

Lemma 2.1. *Let A, B and K be subgroups of a group G.*

(1) *If $(|G : A|, |G : B|) = 1$, then $G = AB$ [4, Lemma 3.8.1].*
(2) *If $(|G : A|, |G : B|) = 1$ and K is normal in G, then $K = (K \cap A)(K \cap B)$ [4, Lemma 3.8.2].*
(3) *$K \cap AB = (K \cap A)(K \cap B)$ if and only if $KA \cap KB = K(A \cap B)$ [2, Lemma A.1.2].*

A formation $\mathfrak{F}$ is said to be S-closed (S_n-closed) if it contains all subgroups (all normal subgroups, respectively) of all its groups. The following lemma is well known.

Lemma 2.2. *Let G be a group and $A \le G$. Let $\mathfrak{F}$ be a non-empty saturated formation. Then*

(1) *If A is normal in G, then $AZ_\infty^{\mathfrak{F}}(G)/A \le Z_\infty^{\mathfrak{F}}(G/A)$.*
(2) *If $\mathfrak{F}$ is S-closed, then $Z_\infty^{\mathfrak{F}}(G) \cap A \le Z_\infty^{\mathfrak{F}}(A)$.*
(3) *If $\mathfrak{F}$ is S_n-closed and A is normal in G, then $Z_\infty^{\mathfrak{F}}(G) \cap A \le Z_\infty^{\mathfrak{F}}(A)$.*
(4) *If $G \in \mathfrak{F}$, then $Z_\infty^{\mathfrak{F}}(G) = G$.*

Lemma 2.3. *Suppose that H be a subgroup of G and H is s-permutable in G. Then*

(1) *If $H \le K \le G$, then H is s-permutable in K.*
(2) *If N is a normal subgroup of G, then HN is s-permutable in G and HN/N is s-permutable in G/N.*
(3) *H is subnormal in G.*

In view of Lemmas 2.2 and 2.3, we can get the following lemma easily.

Lemma 2.4. *Let G be a group and $H \leq M \leq G$.*

(1) *If H is $\mathfrak{F}$-permutable in G and $\mathfrak{F}$ is S-closed, then H is $\mathfrak{F}$-permutable in M.*
(2) *Suppose that $H \trianglelefteq G$. Then M/H is $\mathfrak{F}$-permutable in G/H if and only if M is $\mathfrak{F}$-permutable in G.*
(3) *If $H \trianglelefteq G$, then for every $\mathfrak{F}$-permutable subgroup E of G with $(|H|, |E|) = 1$, HE/H is $\mathfrak{F}$-permutable in G/H.*

Lemma 2.5. [3, Theorem A] *Suppose that G has a Hall π-subgroup, where π is a set of odd primes. Then all Hall π-subgroups of G are conjugate.*

Lemma 2.6. [10, Lemma A] *If P is an s-permutable p-subgroup of a group G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.7. [8, Lemma 2.4] *Suppose that H is s-permutable in G, and let P be a Sylow p-subgroup of H. If $H_G = 1$, then P is s-permutable in G.*

3. Main results

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime dividing the order of G. If all maximal subgroups of P are $\mathfrak{S}$-permutable in G, then G is soluble.*

Proof. Suppose that the assertion is false and let G be a counterexample of minimal order. Then by the well known Feit-Thompson's theorem, we have that $p = 2$. We now proceed the proof by the following steps.

(1) $O_2(G) = 1$.

Assume that $L = O_2(G) \neq 1$. Obviously, P/L is a Sylow 2-subgroup of G/L. Let M/L be a maximal subgroup of P/L. Then M is a maximal subgroup of P. By the hypothesis and Lemma 2.4(2), M/L is $\mathfrak{S}$-permutable in G/L. The minimal choice of G implies that G/L is soluble. Consequently, G is soluble. This contradiction shows that step (1) holds.

(2) $O_{2'}(G) = 1$.

Assume that $E = O_{2'}(G) \neq 1$. Then, obviously, PE/E is a Sylow 2-subgroup of G/E. Suppose that M/E is a maximal subgroup of PE/E. Then there exists a maximal subgroup P_1 of P such that $M = P_1E$. By the hypothesis and Lemma 2.4(3), $M/E = P_1E/E$ is $\mathfrak{S}$-permutable in G/E. The minimal choice of G implies that G/E is soluble. By the well known Feit-Thompson's theorem, we know that E is soluble. It follows that G is soluble, a contradiction.

(3) P is not cyclic.

If P is cyclic, then G is 2-nilpotent by [9, Theorem 10.1.9]. This implies that G is soluble, a contradiction.

(4) If $1 \neq N \trianglelefteq G$, then N is not soluble and $G = PN$.

If N is soluble, then $O_2(N) \neq 1$ or $O_{2'}(N) \neq 1$. Since $O_2(N)$ char $N \trianglelefteq G$, $O_2(N) \leq O_2(G)$. Analogously $O_{2'}(N) \leq O_{2'}(G)$. Hence $O_2(G) \neq 1$ or $O_{2'}(G) \neq 1$, which contradicts step (1) or step (2). Therefore N is not soluble. Assume that $PN < G$.

By Lemma 2.4(1), every maximal subgroup of P is $\mathfrak{S}$-permutable in PN. Thus PN satisfies the hypothesis. By the minimal choice of G, PN is soluble and so N is. This contradiction shows that $G = PN$.

(5) G has a unique minimal normal subgroup, N say (where N maybe is G).

By step (4), we see that $G = PN$ for every non-identity normal subgroup N of G. It follows that G/N is soluble. Since $\mathfrak{S}$ is closed under subdirect product, G has a unique minimal normal subgroup, N say.

(6) $Z_\infty^{\mathfrak{S}}(G) = 1$.

If $Z_\infty^{\mathfrak{S}}(G) \neq 1$, then we may take a minimal normal subgroup N of G which contained in $Z_\infty^{\mathfrak{S}}(G)$. Obviously, N is an elementary Abelian r-subgroup for some prime r, which contradicts steps (1) and (2).

(7) Final contradiction.

Let P_1 be a maximal subgroup of P. By the hypothesis, there exists a subgroup K_1 of G such that P_1K_1 is s-permutable in G and

$$(P_1 \cap K_1)(P_1)_G/(P_1)_G \subseteq Z_\infty^{\mathfrak{S}}(G/(P_1)_G).$$

In view of steps (1) and (6), we get $P_1 \cap K_1 = 1$. This means that $4 \nmid |K_1|$. Hence by [9, Theorem 10.1.9], K_1 has a normal Hall $2'$-subgroup M_1. Evidently, M_1 is also a Hall $2'$-subgroup of P_1K_1. Obviously, there exists a Sylow 2-subgroup $(K_1)_2$ of K_1 such that $P_1(K_1)_2$ is a Sylow 2-subgroup P_1K_1. If $(P_1K_1)_G = 1$, then $P_1(K_1)_2$ is s-permutable in G by Lemma 2.7. Assume that $|(K_1)_2| = 1$. Then P_1 is s-permutable in G. In view of Lemma 2.6, $P_1 \trianglelefteq PO^p(G) = G$, and so $P_1 \leq (P_1K_1)_G = 1$. This shows that P is cyclic, a contradiction. Hence we have $|(K_1)_2| = 2$. Then $P_1(K_1)_2$ is a Sylow 2-subgroup of G. Applying Lemma 2.6 again, $P_1(K_1)_2$ is normal in G, which contradicts $(P_1K_1)_G = 1$. Therefore, $(P_1K_1)_G \neq 1$. By steps (4) and (5), $N \leq P_1K_1$. Since $N \trianglelefteq G$, $N \trianglelefteq P_1K_1$. It is easy to see that $M_1 \cap N$ is also a Hall $2'$-subgroup of N. Since $G = PN$, we have

$$|G : M_1 \cap N| = |PN : M_1 \cap N| = \frac{|P||N|}{|N \cap P||M_1 \cap N|} = |N : M_1 \cap N||P : P \cap N|$$

is a 2-number. This implies that $M_1 \cap N$ is a Hall $2'$-subgroup of G. Thus $M_1 \cap N = M_1$ is a Hall $2'$-subgroup of N and also a Hall $2'$-subgroup of G. For any element $x \in G$, both M_1^x and M_1 are Hall $2'$-subgroups of N. Since any two Hall $2'$-subgroups of a group are conjugate by Lemma 2.5, M_1^x and M_1 are conjugate in N. Let $H = N_G(M_1)$. By Frattini argument, $G = NH$. Since $(|N : N \cap P|, |N : M_1|) = 1$, $N = (N \cap P)M_1$ by Lemma 2.1(1). Hence $G = (N \cap P)H$. It follows that

$$P = P \cap (N \cap P)H = (N \cap P)(P \cap H).$$

Since $(|G : P|, |G : M_1|) = 1$, we have $G = PM_1 = PH$ by Lemma 2.1(1). If $P \cap H = P$, then $P \leq H$ and so $G = H$ has a non-identity normal Hall $2'$-subgroup M_1, which contradicts $O_{2'}(G) = 1$. Thus $P \cap H < P$ and so there exists a maximal subgroup P_2 of P such that $P \cap H \leq P_2$. Then $P = (N \cap P)(P \cap H) = (N \cap P)P_2$. By the hypothesis, there exists a subgroup K_2 of G such that P_2K_2 is s-permutable in G and $P_2 \cap K_2 = 1$. Using the same argument as above, we can see that K_2 has a non-identity normal Hall $2'$-subgroup M_2 such that M_2 is a Hall $2'$-subgroup of N and also a Hall $2'$-subgroup of G. Moreover, $N \leq P_2K_2$. Hence

$$G = PM_2 = PK_2 = (N \cap P)P_2K_2 = P_2K_2.$$

Since both M_1 and M_2 are Hall $2'$-subgroups of G, by Lemma 2.5 there exists an element $g \in P$ such that $M_2^g = M_1$. Since $(|H : P \cap H|, |H : M_1|) = 1$, $H = (P \cap H)M_1$ by Lemma 2.1(1). Therefore,

$$G = (P_2K_2)^g = P_2N_G(M_2^g) = P_2N_G(M_1) = P_2H = P_2(P \cap H)M_1 = P_2M_1.$$

It follows that $|G| = |P_2||M_1| < |P||M_1| = |G|$. The final contradiction completes the proof. □

Corollary 3.1. *Let M be a maximal subgroup of a group G with $|G : M| = r$, where r is a prime. Let p be the smallest prime dividing $|M|$. If there exists a Sylow p-subgroup P of M such that every maximal subgroup of P is $\mathfrak{S}$-permutable in G, then G is soluble.*

Proof. If $|G|$ is odd number, then G is soluble by the well known Feit-Thompson's theorem. Now we assume that $2||G|$. If $r = 2$, then M is normal in G. By Lemma 2.4(1), every maximal subgroup of P is $\mathfrak{S}$-permutable in M. Theorem 3.1 implies that M is soluble. It follows that G is soluble. If $r \neq 2$, then $p = 2$ and P is a Sylow 2-subgroup of G. By using our Theorem 3.1, we obtain that G is soluble. □

Theorem 3.2. *A group G is soluble if and only if every Sylow subgroup of G is $\mathfrak{S}$-permutable in G.*

Proof. The necessity is obvious. We need only prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order. Then:

(1) $P_G = 1$ for any prime p dividing $|G|$ and any Sylow p-subgroup P of G.

If there exists a Sylow p-subgroup P of G such that $P_G \neq 1$, then by Lemma 2.4(1), it is easy to see that G/P_G satisfies the hypothesis of the theorem. Hence the minimal choice of G implies that G/P_G is soluble, and so G is soluble, a contradiction.

(2) $Z_\infty^{\mathfrak{S}}(G) = 1$.

If $Z_\infty^{\mathfrak{S}}(G) \neq 1$, then we may take a minimal normal subgroup N of G which is contained in $Z_\infty^{\mathfrak{S}}(G)$. Obviously, N is abelian. With the same argument as step (1), we have that G is soluble, a contradiction.

(3) If $1 \neq N \trianglelefteq G$, then G/N is soluble.

Let M/N be a Sylow p-subgroup of G/N, where $p||G/N|$. Then, obviously $M/N = PN/N$, where P is a Sylow p-subgroup of G. By the hypothesis, there exists a subgroup K of G such that PK is s-permutable in G and $P \cap K = 1$. Hence

$$(|PK \cap N : N \cap K|, |PK \cap N : N \cap P|) = 1.$$

By Lemma 2.1(1), $PK \cap N = (P \cap N)(K \cap N)$. In view of Lemma 2.1(3), $PN \cap KN = N(P \cap K) = N$. This implies that $(PN/N) \cap (KN/N) = 1$. By Lemma

2.3(2), $(PN/N)(NK/N) = (PK)N/N$ is s-permutable in G/N. Therefore, $M/N = PN/N$ is $\mathfrak{S}$-permutable in G/N. This shows that G/N satisfies the hypothesis of the theorem. The minimal choice of G implies that G/N is soluble.

(4) Final contradiction.

Since $\mathfrak{S}$ is closed under subdirect product, by step (3), G has only one minimal normal subgroup, N say. For any prime p dividing the order of N, we claim that every Sylow p-subgroup N_p of N is complemented in N. In fact, let P be a Sylow p-subgroup of G such that $N_p \leq P$. Then, obviously, $N_p = N \cap P$. By the hypothesis, there exists a subgroup K of G such that PK is s-permutable in G and $P \cap K = 1$. Obviously, K is a p-complement of PK. If $(PK)_G = 1$, P is s-permutable in G by Lemma 2.7. It follows that $P \trianglelefteq PO^p(G) = G$ from Lemma 2.6, a contradiction. The unique minimal normality of N implies that $N \leq PK$. Since $(|PK : K|, |PK : P|) = 1$, $N = (N \cap P)(N \cap K) = N_p(N \cap K)$ by Lemma 2.1(2). Then $N_p \cap (N \cap K) = (P \cap N) \cap (N \cap K) = 1$. This shows that every Sylow p-subgroup of N is complemented in N. Hence N is soluble by Hall's theorem [6], which induces that G is soluble. This contradiction completes the proof. □

Corollary 3.2. [11, Theorem 2.4] *A group G is soluble if and only if every Sylow subgroup of G is c-supplemented in G.*

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