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SOME NEW CHARACTERIZATIONS OF THE CONVEX FUNCTIONS

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ABSTRACT. In this paper we will present some new characterizations of the convex functions and some consequences.

1. INTRODUCTION

The convex functions has a very important role in the mathematical analysis. The first rigorous viewpoint about this concept are presented by J. L. W. E. Jensen in [4]. After more evolutions, today we accept the folowing definition:

Definition 1.1. Let $I \subset \mathbb{R}$ be an interval. The function $f : I \to \mathbb{R}$ is convex if

$$f(ax + (1 - a)y) \leq af(x) + (1 - a)f(y),$$

for all $x, y \in I$ and $a \in [0, 1]$.

Starting from this point, the branches of convex functions has been a remarkable development. Some results could be founded in two classic books [12, 13] or in the new monographs like [9] or [1]. More, Borwein and Vanderwerff presents in the first chapter of their book (see [2]) a very interesting arguments for to study the convex functions.

In this context it was a very interesting challenge for more mathematicians to find another characterizations of the convex functions. Today are knows more results like Hermite-Hadamard inequalities, Popoviciu inequalities or others. In this paper, the authors wants to answer to this challenge and they came with some new characterizations of the convex functions.

Firstly, we remind a very useful result which could be founded in [1] or [9]. Let I be a nondegenerate interval from $\mathbb{R}$.

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Theorem 1.1. *Let $f : I \to \mathbb{R}$ be a function. For some $a \in I$, we consider the function $s_a : I \setminus \{a\} \to \mathbb{R}$ defined by*

$$s_a(x) = \frac{f(x) - f(a)}{x - a},$$

for all $x \in I \setminus \{a\}$. If the function f is convex then the function s_a is nondecreasing.

A consequence of this theorem is represented by the next corollary.

Corollary 1.1. *Let $f : I \to \mathbb{R}$ be a convex function and $a, b, c, d \in I$ so that $a \neq b$, $c \neq d$, $a \leq c$ and $b \leq d$. Then*

$$\frac{f(a) - f(b)}{a - b} \leq \frac{f(c) - f(d)}{c - d}.$$

2. New characterizations of the convex functions

The main results of our paper is represented by the Proposition 2.1 and 2.2 which are follow in this paragraphs. Firstly, for some $u \in I$ we denote $\underline{I}(u) = (-\infty, u] \cap I$ and $\overline{I}(u) = [u, \infty) \cap I$.

Proposition 2.1. *Let $f : I \to \mathbb{R}$ be a function. The next statements are equivalent*

(i) *The function f is convex;*

(ii) *For any $t \in (0, 1)$ and $y \in I$, the function $g : I \to \mathbb{R}$ defined by*

$$g(x) = tf(x) + (1 - t)f(y) - f(tx + (1 - t)y),$$

for all $x \in I$, is nonincreasing on $\underline{I}(y)$ and nondecreasing on $\overline{I}(y)$;

(iii) *For any $n \in \mathbb{N}, n \geq 1$, $p_0, p_1, \ldots, p_n \in (0, \infty)$ and $x_1, x_2, \ldots, x_n \in I$ the function $h : I \to \mathbb{R}$ defined by*

$$h(x) = p_0 f(x) + \sum_{i=1}^{n} p_i f(x_i) - \left(\sum_{i=0}^{n} p_i\right) f\left(\frac{p_0 x + p_1 x_1 + \cdots + p_n x_n}{p_0 + p_1 + \cdots + p_n}\right)$$

for all $x \in I$, is nonincreasing on $\underline{I}(x_0)$ and nondecreasing on $\overline{I}(x_0)$, where $x_0 = \frac{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n}{p_1 + p_2 + \cdots + p_n}$;

(iv) *For any $n \in \mathbb{N}^*$, $n \geq 1$, $p_0, p_1, \ldots, p_n \in (0, \infty)$ and $x_1, \ldots, x_n \in I$, the function $l : I \to \mathbb{R}$ defined by*

$$l(x) = p_0 f(x) + \left(\sum_{i=1}^{n} p_i\right) \cdot f\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}\right) - \left(\sum_{i=0}^{n} p_i\right) \cdot f\left(\frac{p_0 x + p_1 x_1 + \cdots + p_n x_n}{p_0 + p_1 + \cdots + p_n}\right),$$

for all $x \in I$, is nonincreasing on $\underline{I}(x_0)$ and nondecreasing on $\overline{I}(x_0)$, where x_0 has same value like to the previous point.

Proof. We will proove the equivalence $(ii) \iff (iv)$ and the chain of implications $(iii) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii)$.

For the part $(ii) \Rightarrow (iv)$, we denote

$$t = \frac{p_0}{p_0 + p_1 + \cdots + p_n}$$

and

$$y = \frac{p_1 x_1 + \cdots + p_n x_n}{p_1 + \cdots + p_n}.$$

Then $t \in (0,1)$ and $y \in I$. From assertion (ii), the function $s : I \to \mathbb{R}$ defined by

$$s(x) = tf(x) + (1-t)f(y) - f(tx - (1-t)y)$$

for all $x \in I$, is nonincreasing on $\underline{I}(y)$ and is nondecreasing on $\overline{I}(y)$. We obtain that the function l is satisfying the property from the assertion (iv).

For the part $(iv) \Rightarrow (ii)$, let be $n = 1, p_0 = t \in (0,1), p_1 = 1-t$ and $x_1 = y$. Then we have $l = g$ and the conclusion follows if we are using the relation $x_0 = y$.

In same mode we obtain the implication $(iii) \Rightarrow (ii)$.

For the part $(ii) \Rightarrow (i)$,we choose $t \in (0,1)$, because the case $t \in \{0,1\}$ is trivial. Let be $x, y \in I$. Then, from the assertion (ii) we obtain $g(x) \geq g(y)$. But $g(y) = 0$ and this implies that $g(x) \geq 0$ and

$$tf(x) + (1-t)f(y) - f(tx + (1-t)y) \geq 0,$$

so the function f is convex.

In the last part of the proof we will prove the implication $(i) \Rightarrow (iii)$. Let be $u, v \in \underline{I}(x_0)$ with $u < v$.Then

$$h(u) - h(v) =$$

$$= p_0 \left[f(u) - f(v)\right] - \left(\sum_{i=0}^{n} p_i\right) \left[f\left(\frac{p_0 u + \sum_{i=1}^{n} p_i x_n}{p_0 + \sum_{i=1}^{n} p_i}\right) - f\left(\frac{p_0 v + \sum_{i=1}^{n} p_i x_n}{p_0 + \sum_{i=1}^{n} p_i}\right)\right]$$

$$= p_0(u-v)\left[\frac{f(u) - f(v)}{u - v} - \frac{f\left(\frac{p_0 u + p_1 x_1 + \cdots + p_n x_n}{p_0 + p_1 + \cdots + p_n}\right) - f\left(\frac{p_0 v + p_1 x_1 + \cdots + p_n x_n}{p_0 + p_1 + \cdots + p_n}\right)}{\frac{p_0 u + p_1 x_1 + \cdots + p_n x_n}{p_0 + p_1 + \cdots + p_n} - \frac{p_0 v + p_1 x_1 + \cdots + p_n x_n}{p_0 + p_1 + \cdots + p_n}}\right].$$

But we have $u < v$,

$$u \leq \frac{p_0 u + p_1 x_1 + \cdots + p_n x_n}{p_0 + p_1 + \cdots + p_n}$$

and

$$v \leq \frac{p_0 v + p_1 x_1 + \cdots + p_n x_n}{p_0 + p_1 + \cdots + p_n}.$$

Last inequality is true from the definition of x_0. From Corollary 1.1, we obtain $h(u) - h(v) \geq 0$ and the function h is nonincreasing on $\underline{I}(x_0)$. In same mode we obtain that the function h is nondecreasing on $\overline{I}(x_0)$. □

A classical results of the mathematical analysis said that the inequalities Hermite-Hadamard characterizes the convex functions when the functions are continuous. For example, we can see the Theorem 3.7.3. from [1]. The next proposition offers a similar characterization.

Proposition 2.2. *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is a convex function if and only if for all $a_1, a_2, b_1, b_2, c_1, c_2 \in I$ with $a_1 < a_2 \leq b_1 < b_2 \leq c_1 < c_2$ the next inequality is true*

$$(c-a) \cdot \frac{\int_{b_1}^{b_2} f(x)dx}{b_2 - b_1} \leq (c-b) \cdot \frac{\int_{a_1}^{a_2} f(x)dx}{a_2 - a_1} + (b-a) \cdot \frac{\int_{c_1}^{c_2} f(x)dx}{c_2 - c_1}$$

where $a = a_1 + a_2, b = b_1 + b_2, c = c_1 + c_2$.

Proof. For the only if part we use the Hadamard-Hermite inequalities. Then

$$(c-a) \cdot \frac{\int_{b_1}^{b_2} f(x)dx}{b_2 - b_1} \leq (c-a) \cdot \frac{f(b_2) + f(b_1)}{2} \tag{2.1}$$

By using the Corollary 1.1 for $x \in [a_1, a_2]$ we obtain

$$\frac{f(x) - f(b_1)}{x - b_1} \leq \frac{f(b_2) - f(b_1)}{b_2 - b_1}.$$

This relation is equivalent with the next

$$f(x) \geq f(b_1) + \frac{f(b_2) - f(b_1)}{b_2 - b_1}(x - b_1),$$

for all $x \in [a_1, a_2]$. This is true because $x - b_1 < 0$ for all $x \in [a_1, a_2]$.

Then we obtain

$$\begin{aligned}\int_{a_1}^{a_2} f(x)dx &\geq f(b_1)(a_2 - a_1) + \frac{f(b_2) - f(b_1)}{b_2 - b_1} \cdot \frac{(a_2 - b_1)^2 - (a_1 - b_1)^2}{2} \\ &= f(b_1)(a_2 - a_1) + \frac{f(b_2) - f(b_1)}{b_2 - b_1} \cdot \frac{(a_2 - a_1)(a_2 + a_1 - 2b_1)}{2}.\end{aligned}$$

After we divide with $a_2 - a_1$, we have

$$\frac{\int_{a_1}^{a_2} f(x)dx}{a_2 - a_1} \geq f(b_1) + \frac{1}{2}(a - 2b_1)\frac{f(b_2) - f(b_1)}{b_2 - b_1}. \tag{2.2}$$

Similar, for $x \in [c_1, c_2]$, we obtain

$$\frac{f(x) - f(b_1)}{x - b_1} \geq \frac{f(b_2) - f(b_1)}{b_2 - b_1},$$

which is equivalent with

$$f(x) \geq f(b_1) + \frac{f(b_2) - f(b_1)}{b_2 - b_1}(x - b_1),$$

for all $x \in [c_1, c_2]$. After we integrate on the interval $[c_1, c_2]$ and we made the calculus in same mode, we have

$$\frac{\int_{c_1}^{c_2} f(x)dx}{c_2 - c_1} \geq f(b_1) + \frac{1}{2}(c - 2b_1)\frac{f(b_2) - f(b_1)}{b_2 - b_1}. \tag{2.3}$$

By using the relations 2.2 and 2.3 we have

$$\begin{aligned}
&(c-b)\cdot\frac{\int_{a_1}^{a_2} f(x)dx}{a_2-a_1}+(b-a)\cdot\frac{\int_{c_1}^{c_2} f(x)dx}{c_2-c_1}\geq\\
&\geq f(b_1)(c-a)+\frac{1}{2}\cdot\frac{f(b_2)-f(b_1)}{b_2-b_1}\left[(a-2b_1)(c-b)+(c-2b_1)(b-a)\right]\\
&=f(b_1)(c-a)+\frac{1}{2}\cdot\frac{f(b_2)-f(b_1)}{b_2-b_1}(c-a)(b_2-b_1)=(c-a)\cdot\frac{f(b_2)+f(b_1)}{2}.
\end{aligned}$$

Now , we use 2.1 and the conclusion follows.

For the if part, let be $x, y \in I$ with $x < y$ and $t \in (0,1)$. We denote $z = tx+(1-t)y$. Then $x < z < y$. Let be $\varepsilon > 0$ so that

$$x < x+\varepsilon < z-\varepsilon < z < y-\varepsilon < y.$$

We choose $a_1 = x, a_2 = x+\varepsilon, b_1 = z-\varepsilon, b_2 = z, c_1 = y-\varepsilon, c_2 = y$ and we have

$$(2y-2x-2\varepsilon)\cdot\frac{\int_{z-\varepsilon}^{z} f(t)dt}{\varepsilon}\leq(2y-2z)\cdot\frac{\int_{x}^{x+\varepsilon} f(t)dt}{\varepsilon}+(2z-2x-2\varepsilon)\cdot\frac{\int_{y-\varepsilon}^{y} f(t)dt}{\varepsilon}.$$

If we put $\varepsilon \to 0$, the previous inequalities becames

$$(y-x)\cdot f(z)\leq(y-z)\cdot f(x)+(z-x)\cdot f(y),$$

equivalent with

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y).$$

These concludes that the function f is convex. □

Remark 2.1. It is obviously that the only if part of Proposition 2.2 is true without the continuity of the function f.

3. Consequences

In this paragraph we will present and prove some consequences of the results from the Proposition 2.1 and 2.2.

Corollary 3.1. [11] *If $f : I \to \mathbb{R}$ is a convex function then for any $n \in \mathbb{N}, n \geq 3$, for any $a_1, a_2, \ldots, a_n \in I$ with $a_1 \leq a_2 \leq \ldots \leq a_n$ and for all $p_1, p_2, \ldots, p_n \in (0,\infty)$ we have*

$$\begin{aligned}
&\frac{p_1f(a_1)+p_2f(a_2)+\cdots+p_nf(a_n)}{p_1+p_2+\cdots+p_n}-f\left(\frac{p_1a_1+p_2a_2\cdots+p_na_n}{p_1+p_2+\cdots+p_n}\right)\geq\\
&\geq\frac{(p_1+p_2)f(a_2)+\cdots+p_nf(a_n)}{p_1+p_2+\cdots+p_n}-f\left(\frac{(p_1+p_2)a_2+\cdots+p_na_n}{p_1+\cdots+p_n}\right)\geq\cdots\geq\\
&\geq\frac{(p_1+\cdots+p_{n-1})f(a_{n-1})+p_nf(a_n)}{p_1+p_2+\cdots+p_n}-f\left(\frac{(p_1+\cdots+p_{n-1})a_{n-1}+p_na_n}{p_1+p_2+\cdots+p_n}\right)\geq 0.
\end{aligned}$$

Proof. We will prove only the first inequality. The others will be obtains by using the mathematical induction. It easy to see that we need to study only the case $a_1 < a_2 < ... < a_n$, because the equality case is trivial. From the assertion (iii) of the Proposition 2.1, the function $g : I \to \mathbb{R}$ defined by

$$g(x) = p_1 f(x) + p_2 f(x) + \cdots + p_n f(a_n) - (p_1 + \cdots + p_n) f\left(\frac{p_1 x + p_2 a_2 + \cdots + p_n a_n}{p_1 + \cdots + p_n}\right),$$

for all $x \in I$, is nonincreasing on $\underline{I}(x_0)$, where $x_0 = \frac{p_2 a_2 + \cdots + p_n a_n}{p_2 + \cdots + p_n}$.

But $a_1 < a_2 < x_0$ and we have $g(a_1) \geq g(a_2)$ and conclusion follows now. □

Corollary 3.2. [3, 8, 14] *Let a, b be two real numbers with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a convex function. For some $n \in \mathbb{N}, n \geq 2$, let be $x_1, x_2, \ldots, x_n \in [a, b]$ and $p_1, p_2, \ldots, p_n \in (0, \infty)$ so that $p_1 + \cdots + p_n = 1$. Then*

$$p_1 f(x_1) + \cdots + p_n f(x_n) - f(p_1 x_1 + \cdots + p_n x_n) \leq$$
$$\leq \max\{p f(a) + (1-p) f(b) - f(pa + (1-p)b) : \ p \in]0, 1[\}.$$

Proof. We apply the assertion (iii) of the Proposition 2.1for the function $g : [a, b] \to \mathbb{R}$ defined by

$$g(x) = p_1 f(x) + \cdots + p_n f(x_n) - (p_1 + \cdots + p_n) \cdot f\left(\frac{p_1 x + \cdots + p_n x_n}{p_1 + \cdots + p_n}\right),$$

for all $x \in [a, b]$. Then we have

$$g(x) \leq \max\{g(a), g(b)\}$$

and we obtain the conclusion. □

Corollary 3.3. [6] *Let be $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a convex function. Let s, t be two nonnegative real numbers so that $s + t = 1$. Then the next statements hold*

(i) *The function $\varphi_a : [a, b] \to \mathbb{R}$ defined by*

$$\varphi_a(x) = s f(a) + t f(x) - f(sa + tx)$$

for all $x \in [a, b]$, is nondecreasing;

(ii) *The function $\varphi_b : [a, b] \to \mathbb{R}$ defined by*

$$\varphi_b(x) = s f(x) + t f(b) - f(sx + tb)$$

for all $x \in [a, b]$, is nonincreasing;

(iii) *We have $s f(x) + t f(y) - f(sx + ty) \leq s f(a) + t f(b) - f(sa + tb)$ for all $x, y \in [a, b]$.*

Proof. The cases $s = 0$ or $t = 0$ are trivial , so that we admits $s, t > 0$. The points (i) and (ii) are consequences of the Proposition 2.1 if we choose $y = a$, respectively b, $t = s$ and $1 - t = v$. For the assertion (iii) we apply (i) and (ii) from this corollary. □

The following result is representing a particular case of the Theorem 1 from [10].

Corollary 3.4. [10] *Let* $f : I \to \mathbb{R}$ *be a convex function and* $a_1, a_2, b_1, b_2, c_1, c_2 \in I$ *so that* $a_1 < a_2 \le b_1 < b_2 \le c_1 < c_2$, $b_2 - b_1 = a_2 - a_1 + c_2 - c_1$ *and* $b_2^2 - b_1^2 = a_2^2 - a_1^2 + c_2^2 - c_1^2$. *Then*

$$\int_{b_1}^{b_2} f(x)dx \le \int_{a_1}^{a_2} f(x)dx + \int_{c_1}^{c_2} f(x)dx.$$

Proof. Using the Proposition 2.2 we obtain

$$(c-a) \cdot \frac{\int_{b_1}^{b_2} f(x)dx}{b_2 - b_1} \le (c-b) \cdot \frac{\int_{a_1}^{a_2} f(x)dx}{a_2 - a_1} + (b-a) \cdot \frac{\int_{c_1}^{c_2} f(x)dx}{c_2 - c_1},$$

where $a = a_1 + a_2,\ b = b_1 + b_2,\ c = c_1 + c_2$.

From the hypotesis, we have $b_2 - b_1 = a_2 - a_1 + c_2 - c_1$ and $b(b_2 - b_1) = c(c_2 - c_1) + a(a_2 - a_1)$, so

$$\frac{c-b}{a_2 - a_1} = \frac{b-a}{c_2 - c_1} = \frac{c-a}{b_2 - b_1}$$

which concludes our proof. ☐

Corollary 3.5. [5] *Let* $f : I \to \mathbb{R}$ *be a convex function. Let* $a_1, a_2, \ldots, a_{2n}, a_{2n+1} \in I$ *some real numbers that is forming an arithmetic progression with a a positive ration. Then*

$$\int_{a_n}^{a_{n+2}} f(x)dx \le \int_{a_1}^{a_2} f(x)dx + \int_{a_{2n}}^{a_{2n+1}} f(x)dx.$$

Proof. From the Proposition 2.2 and Remark 2.1 we obtain

$$(a_{2n+1} + a_{2n} - a_1 - a_2) \cdot \frac{\int_{a_n}^{a_{n+2}} f(x)dx}{a_{n+2} - a_n} \le$$

$$\le (a_{2n+1} + a_{2n} - a_{n+2} - a_n) \cdot \frac{\int_{a_1}^{a_2} f(x)dx}{a_2 - a_1} + (a_{2n+2} + a_n - a_2 - a_1) \cdot \frac{\int_{a_{2n}}^{a_{2n+1}} f(x)dx}{a_{2n+1} - a_{2n}},$$

and the conclusion follows after we make the simplification. ☐

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