ON CHAIN CONDITIONS AND FINITELY GENERATED MULTIPLICATION MODULES

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Abstract. In this short paper, we study multiplication modules that satisfy ascending (respectively, descending) chain condition for multiplication submodules and we investigate some properties of finitely generated multiplication modules.

1. Introduction

Let $R$ be a commutative ring with identity and $M$ be a unitary $R$-module. According to [2], $M$ is called a multiplication module if every submodule of $M$ is of the form $IM$, for some ideal $I$ of $R$. Barnard [2] showed that distributive modules are characterized as modules for which every finitely generated submodule is a multiplication module. Then many mathematicians worked on multiplication modules, for example see [1, 3–6, 8]. A survey about this subject is collected in [7]. For any submodule $N$ of an $R$-module $M$, we define $(N : M) = \{ r \in R \mid rM \subseteq N \}$ and denote $(0 : M)$ by $\text{Ann}_R(M)$.

2. Main results

Remark 2.1. Let $M$ be a multiplication $R$-module and $\varphi \in \text{End}(M)$. Since $\ker \varphi \cap MI = (\ker \varphi)I$ for every ideal $I$ of $R$, we conclude that $\ker \varphi$ is a multiplication submodule of $M$. Also, according to Note 1.4 in [7], every homomorphic image of a multiplication module is a multiplication module.

Proposition 2.1. Let $M$ be a multiplication $R$-module. If $M$ satisfies ascending and descending chain conditions for multiplication submodules and $\varphi \in \text{End}(M)$, then there exists $n \geq 1$ such that $M = \ker \varphi^n \oplus \text{Im} \varphi^n$.

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Proof. By Remark 2.1, the proof is straightforward.

We recall that an $R$-module $M$ is indecomposable if it is non-zero and cannot be written as a direct sum of two non-zero submodules.

**Proposition 2.2.** Let $M$ be an indecomposable multiplication $R$-module. If $M$ satisfies ascending and descending chain conditions for multiplication submodules and $\varphi \in \text{End}(M)$, then $\varphi$ is bijective or nilpotent.

Proof. By Proposition 2.1, the proof is straightforward. □

**Lemma 2.1.** [7] Let $M$ be a multiplication $R$-module. If $N$ is a submodule of $M$ such that $N \cap IM = IN$ for every ideal $I$ of $R$, then $N$ is a multiplication module.

**Theorem 2.1.** Let $M = Rm_1 + \ldots + Rm_j$ be a finitely generated multiplication $R$-module. Let $I$ be an ideal of $R$ and $Im_i = Rm_i \cap IM$, for every $i$. Then, $M$ satisfies the ascending (respectively, descending) chain conditions on multiplication submodules if and only if for every $i$, $Rm_i$ satisfies the ascending (respectively, descending) chain conditions of multiplication submodules.

Proof. We prove the theorem for the ascending chain condition. The proof for the descending chain condition is analogous.

Since $Im_i = Rm_i \cap IM$, by Lemma 2.1, $Im_i$ is a multiplication submodule of $Rm_i$. Suppose that

$$I_1m_i \subseteq I_2m_i \subseteq I_3m_i \subseteq \ldots$$

is a chain of multiplication submodules of $Rm_i$. Then,

$$\text{Ann}_R \left( \frac{Rm_i}{I_1m_i} \right) \subseteq \text{Ann}_R \left( \frac{Rm_i}{I_2m_i} \right) \subseteq \text{Ann}_R \left( \frac{Rm_i}{I_3m_i} \right) \subseteq \ldots$$

is a chain of ideals of $R$. Since $\text{Ann}_R \left( \frac{Rm_i}{I_km_i} \right) = I_k + \text{Ann}_R(m_i)$,

$$(I_1 + \text{Ann}_R(m_i)) M \subseteq (I_2 + \text{Ann}_R(m_i)) M \subseteq (I_3 + \text{Ann}_R(m_i)) M \subseteq \ldots$$

is a chain of multiplication submodules of $M$. Hence, there exists a positive integer $n$ such that for every $k \geq n$,

$$(I_n + \text{Ann}_R(m_i)) M = (I_k + \text{Ann}_R(m_i)) M.$$ 

Thus,

$$I_km_i = Rm_i \cap I_kM \subseteq Rm_i \cap (I_k + \text{Ann}_R(m_i))M = Rm_i \cap (I_n + \text{Ann}_R(m_i))M = (I_n + \text{Ann}_R(m_i))m_i = I_nm_i.$$ 

On the other hand, for every $k \geq n$, we have $I_nm_i \subseteq I_km_i$. Therefore, $Rm_i$ satisfies the ascending chain condition.
Conversely, suppose that for every \(i\), \(Rm_i\) satisfies the ascending chain condition. By Lemma 2.1, for every \(i\), \(I_iM\) is multiplication. Now, let
\[
I_1M \subseteq I_2M \subseteq I_3M \subseteq \ldots
\]
be a chain of multiplication submodules of \(M\). Then, we have
\[
Rm_i \cap I_1M \subseteq Rm_i \cap I_2M \subseteq Rm_i \cap I_3M \subseteq \ldots \quad (i = 1, \ldots, j).
\]
Since \(Rm_i \cap I_nM = I_nm_i\), for every \(1 \leq i \leq j\) and \(n \geq 1\) and \(Rm_i\) satisfies ascending chain condition, then there exists \(r_i\) such that \(I_nm_i = I_r m_i\), for every \(n \geq r_i\). Now, we take \(r = \max\{r_1, r_2, \ldots, r_j\}\). Then, \(I_nm_i = I_r m_i\), for every \(n \geq r\). Thus,
\[
I_nM = I_nm_1 + I_n m_2 + \ldots + I_n m_j = I_r m_1 + I_r m_2 + \ldots + I_r m_j = I_r M,
\]
for every \(n \geq r\). Therefore, \(M\) satisfies ascending chain condition. \(\square\)

**Theorem 2.2.** Let \(M = Rm_1 + \ldots + Rm_j\) be a finitely generated multiplication \(R\)-module. Let for every ideal \(I\) of \(R\), \(I m_i = Rm_i \cap IM\) and \(M\) satisfy ascending (respectively, descending) chain condition of multiplication submodules. Then,

1. \(M\) is a Noetherian (respectively, Artinian) \(R\)-module.
2. \(\frac{R}{Ann_R(M)}\) is a Noetherian (respectively, Artinian) ring.

**Proof.**

(1) Suppose that \(M\) satisfies ascending chain condition of multiplication submodules. Then, by previous theorem, for every \(i\), \(Rm_i\) satisfies ascending chain condition of multiplication submodules. Every submodule of \(Rm_i\) is the form \(I_km_i\), where \(I_k\) is an ideal of \(R\). By Lemma 2.1, \(I_k m_i\) is a multiplication submodule of \(Rm_i\). Thus, \(Rm_i\) is Noetherian. Therefore, we conclude that \(Rm_1 \oplus \ldots \oplus Rm_j\) is also Noetherian. The map \(\varphi : Rm_1 \oplus \ldots \oplus Rm_j \to Rm_1 + \ldots + Rm_j\) by \(\varphi(r_1m_1, \ldots, r_j m_j) = r_1 m_1 + \ldots + r_j m_j\) is an epimorphism. Since the sequence
\[
0 \to \ker \psi \to Rm_1 \oplus \ldots \oplus Rm_j \to 0
\]
is exact, we conclude that \(M\) is Noetherian.

(2) We consider the map \(\psi : R \to Rm_1 \oplus \ldots \oplus Rm_j\) by \(\psi(r) = (rm_1, \ldots, rm_j)\) such that \(r \in R\). Then, \(\psi\) is an \(R\)-homomorphism and \(\ker \psi = Ann_R(M)\). Therefore, \(\frac{R}{Ann_R(M)}\) is isomorphic to a submodule of \(Rm_1 \oplus \ldots \oplus Rm_j\). Since \(Rm_1 \oplus \ldots \oplus Rm_j\) is Noetherian and every submodule of a Noetherian module is Noetherian, we conclude that \(\frac{R}{Ann_R(M)}\) is Noetherian. \(\square\)

**Definition 2.1.** An element \(m \in M\) is called devisable if for every \(r \in R \setminus Z(R)\), there exists \(m' \in M\) such that \(m = rm'\). If every element of \(M\) is devisable, then \(M\) is a devisable module. In other words, \(M\) is devisable if \(M = rM\) for every \(r \in R \setminus Z(R)\).

**Proposition 2.3.** Let \(R\) be an integral domain and \(M = Rm_1 + \ldots + Rm_j\) be a finitely generated multiplication \(R\)-module. If \(M \neq 0\) is a divisible module, then \(M\) is faithful. Moreover, if \(M\) is a faithful simple \(R\)-module, then \(M\) is a divisible \(R\)-module.
Proof. Suppose that $r \in \text{Ann}_R(M)$. Hence, for every $m \in M$, $rm = 0$. Since $M$ is divisible, for every $m \in M$, $r' \in R \setminus Z(R)$, there exists $m' \in M$ such that $m = r'm'$. Thus, $rr'm = 0$. Since $M \neq 0$ and $R$ is integral domain, we obtain $r = 0$ and so $M$ is faithful.

If $M$ is faithful simple $R$-module, then $M = RM$ and so $M$ is divisible. □

References


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