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ON CHAIN CONDITIONS AND FINITELY GENERATED MULTIPLICATION MODULES

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ABSTRACT. In this short paper, we study multiplication modules that satisfy ascending (respectively, descending) chain condition for multiplication submodules and we investigate some properties of finitely generated multiplication modules.

1. INTRODUCTION

Let R be a commutative ring with identity and Mbe a unitary R-module. According to [2], M is called a *multiplication module* if every submodule of M is of the form IM, for some ideal I of R. Barnard [2] showed that distributive modules are characterized as modules for which every finitely generated submodule is a multiplication module. Then many mathematicians worked on multiplication modules, for example see [1, 3–6, 8]. A survey about this subject is collected in [7]. For any submodule N of an R-module M, we define $(N : M) = \{r \in R \mid rM \subseteq N\}$ and denote $(0 : M)$ by $Ann_R(M)$.

2. MAIN RESULTS

Remark 2.1. Let M be a multiplication R-module and $\varphi \in End(M)$. Since $ker\varphi \cap MI = (ker\varphi)I$ for every ideal I of R, we conclude that $ker\varphi$ is a multiplication submodule of M. Also, according to Note 1.4 in [7], every homomorphic image of a multiplication module is a multiplication module.

Proposition 2.1. *Let M be a multiplication R-modules. If M satisfies ascending and descending chain conditions for multiplication submodules and $\varphi \in End(M)$, then there exists $n \geq 1$ such that $M = ker\varphi^n \oplus Im\varphi^n$.*

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Proof. By Remark 2.1, the proof is straightforward. □

We recall that an R-module M is indecomposable if it is non-zero and can not be written as a direct sum of two non-zero submodules.

Proposition 2.2. *Let M be an indecomposable multiplication R-modules. If M satisfies ascending and descending chain conditions for multiplication submodules and $\varphi \in End(M)$, then φ is bijective or nilpotent.*

Proof. By Proposition 2.1, the proof is straightforward. □

Lemma 2.1. [7] *Let M be a multiplication R-module. If N is a submodule of M such that $N \cap IM = IN$ for every ideal I of R, then N is a multiplication module.*

Theorem 2.1. *Let $M = Rm_1 + \ldots + Rm_j$ be a finitely generated multiplication R-module. Let I be an ideal of R and $Im_i = Rm_i \cap IM$, for every i. Then, M satisfies the ascending (respectively, descending) chain conditions on multiplication submodules if and only if for every i, Rm_i satisfies the ascending (respectively, descending) chain conditions of multiplication submodules.*

Proof. We prove the theorem for the ascending chain condition. The proof for the descending chain condition is analogous.

Since $Im_i = Rm_i \cap IM$, by Lemma 2.1, Im_i is a multiplication submodule of Rm_i. Suppose that

$$I_1 m_i \subseteq I_2 m_i \subseteq I_3 m_i \subseteq \ldots$$

is a chain of multiplication submodules of Rm_i. Then,

$$Ann_R\left(\frac{Rm_i}{I_1 m_i}\right) \subseteq Ann_R\left(\frac{Rm_i}{I_2 m_i}\right) \subseteq Ann_R\left(\frac{Rm_i}{I_3 m_i}\right) \subseteq \ldots$$

is a chain of ideals of R. Since $Ann_R\left(\frac{Rm_i}{I_k m_i}\right) = I_k + Ann_R(m_i)$,

$$(I_1 + Ann_R(m_i))\, M \subseteq (I_2 + Ann_R(m_i))\, M \subseteq (I_3 + Ann_R(m_i))\, M \subseteq \ldots$$

is a chain of multiplication submodules of M. Hence, there exists a positive integer n such that for every $k \geq n$,

$$(I_n + Ann_R(m_i))\, M = (I_k + Ann_R(m_i))\, M.$$

Thus,

$$\begin{aligned} I_k m_i &= Rm_i \cap I_k M \subseteq Rm_i \cap (I_k + Ann_R(m_i))M = Rm_i \cap (I_n + Ann_R(m_i))M \\ &= (I_n + Ann_R(m_i))m_i = I_n m_i. \end{aligned}$$

On the other hand, for every $k \geq n$, we have $I_n m_i \subseteq I_k m_i$. Therefore, Rm_i satisfies the ascending chain condition.

Conversely, suppose that for every i, Rm_i satisfies the ascending chain condition. By Lemma 2.1, for every i, I_iM is multiplication. Now, let

$$I_1M \subseteq I_2M \subseteq I_3M \subseteq \ldots$$

be a chain of multiplication submodules of M. Then, we have

$$Rm_i \cap I_1M \subseteq Rm_i \cap I_2M \subseteq Rm_i \cap I_3M \subseteq \ldots \quad (i = 1, \ldots, j).$$

Since $Rm_i \cap I_nM = I_nm_i$, for every $1 \leq i \leq j$ and $n \geq 1$ and Rm_i satisfies ascending chain condition, then there exists r_i such that $I_nm_i = I_{r_i}m_i$, for every $n \geq r_i$. Now, we take $r = \max\{r_1, r_2, \ldots, r_j\}$. Then, $I_nm_i = I_rm_i$, for every $n \geq r$. Thus,

$$I_nM = I_nm_1 + I_nm_2 + \ldots + I_nm_j = I_rm_1 + I_rm_2 + \ldots + I_rm_j = I_rM,$$

for every $n \geq r$. Therefore, M satisfies ascending chain condition. □

Theorem 2.2. *Let $M = Rm_1 + \ldots + Rm_j$ be a finitely generated multiplication R-module. Let for every ideal I of R, $Im_i = Rm_i \cap IM$ and M satisfy ascending (respectively, descending) chain condition of multiplication submodules. Then,*

(1) *M is a Noetherian (respectively, Artinian) R-module.*
(2) *$\frac{R}{Ann_R(M)}$ is a Noetherian (respectively, Artinian) ring.*

Proof. (1) Suppose that M satisfies ascending chain condition of multiplication submodules. Then, by previous theorem, for every i, Rm_i satisfies ascending chain condition of multiplication submodules. Every submodule of Rm_i is the form I_km_i, where I_k is an ideal of R. By Lemma 2.1, I_km_i is a multiplication submodule of Rm_i. Thus, Rm_i is Noetherian. Therefore, we conclude that $Rm_1 \oplus \ldots \oplus Rm_j$ is also Noetherian. The map $\varphi : Rm_1 \oplus \ldots \oplus Rm_j \to Rm_1 + \ldots + Rm_j$ by $\varphi(r_1m_1, \ldots, r_jm_j) = r_1m_1 + \ldots + r_jm_j$ is an epimorphism. Since the sequence

$$0 \to ker\psi \to Rm_1 \oplus \ldots \oplus Rm_j \to 0$$

is exact, we conclude that M is Noetherian.

(2) We consider the map $\psi : R \to Rm_1 \oplus \ldots \oplus Rm_j$ by $\psi(r) = (rm_1, \ldots, rm_j)$ such that $r \in R$. Then, ψ is an R-homomorphism and $ker\psi = Ann_R(M)$. Therefore, $\frac{R}{Ann_R(M)}$ is isomorphic to a submodule of $Rm_1 \oplus \ldots \oplus Rm_j$. Since $Rm_1 \oplus \ldots \oplus Rm_j$ is Noetherian and every submodule of a Noetherian module is Noetherian, we conclude that $\frac{R}{Ann_R(M)}$ is Noetherian. □

Definition 2.1. An element $m \in M$ is called *devisable* if for every $r \in R \setminus Z(R)$, there exists $m' \in M$ such that $m = rm'$. If every element of M is devisable, then M is a *devisable module*. In other words, M is devisable if $M = rM$ for every $r \in R \setminus Z(R)$.

Proposition 2.3. *Let R be an integral domain and $M = Rm_1 + \ldots + Rm_j$ be a finitely generated multiplication R-module. If $M \neq 0$ is a divisible module, then M is faithful. Moreover, if M is a faithful simple R-module, then M is a divisible R-module.*

Proof. Suppose that $r \in Ann_R(M)$. Hence, for every $m \in M$, $rm = 0$. Since M is divisible, for every $m \in M$, $r' \in R \setminus Z(R)$, there exists $m' \in M$ such that $m = r'm'$. Thus, $rr'm = 0$. Since $M \neq 0$ and R is integral domain, we obtain $r = 0$ and so M is faithful.

If M is faithful simple R-module, then $M = RM$ and so M is divisible. □

References

[1] M. M. Ali, *Idempotent and nilpotent submodules of multiplication modules*, Comm. Algebra **36** (2008), 4620–4642.

[2] A. Barnard, *Multiplication modules*, J. Algebra **71** (1981), 174–178.

[3] Y. H. Cho, *On multiplication modules. IV*, Korean Ann. Math. **23** (2006), 77–83.

[4] S. C. Lee, C. Sang and R. Varmazyar, *Semiprime submodules of graded multiplication modules*, J. Korean Math. Soc. **49** (2012), 435–447.

[5] S. C. Lee, *Multiplication modules whose endomorphism rings are integral domains*, Bull. Korean Math. Soc. **47** (2010), 1053–1066.

[6] A. Parkash, A.K. Maloo, *Distributive and multiplication modules*, Beitr. Algebra Geom. **52** (2011), 405–412.

[7] A. A. Tuganbaev, *Multiplication modules*, Journal of Mathematical Sciences **132** (2004), 3839–3905.

[8] A. A. Tuganbaev, *Multiplication modules and ideals*, Algebra. J. Math. Sci. (N. Y.) **136** (2006), 4116–4130.

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