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# GEOMETRY OF POSITION FUNCTION OF TOTALLY REAL SUBMANIFOLDS IN COMPLEX EUCLIDEAN SPACES

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*Dedicated to Professor Leopold Verstraelen for his 65th birthday*

ABSTRACT. A submanifold of a Euclidean space is said to be of constant-ratio if the ratio of the length of the tangential and normal components of its position vector function is constant. The notion of constant-ratio submanifolds was first introduced and studied by the author in [5, 8] during the early 2000s. Such submanifolds relate to a problem in physics concerning the motion in a central force field which obeys the inverse-cube law of Newton (cf. [1, 15]). Recently, it was pointed out in [13] that constant-ratio submanifolds also relate closely to D'Arcy Thomson's basic principal of natural growth in biology. In this paper, we provide a fundamental study of totally real submanifolds of $\mathbf{C}^m$ in terms of the positive function $x$ of the submanifolds and the complex structure $J$ of $\mathbf{C}^m$. In particular, we classify constant-ratio totally real submanifolds in $\mathbf{C}^m$. Some related results are also obtained.

## 1. INTRODUCTION

Let $x : M \to \mathbb{E}^q$ be an isometric immersion of a Riemannian $n$-manifold $M$ into a Euclidean $q$-space $\mathbb{E}^q$. We denote by $x$ the immersion of $M$ as well as the position vector function of $M$ in $\mathbb{E}^q$. The position vector function is the simplest and the most natural geometric object associated with submanifolds in Euclidean spaces.

There is a natural orthogonal decomposition of the position vector $x$ at each point on each submanifold $M$ in $\mathbb{E}^q$; namely,

$$x = x^T + x^N, \tag{1.1}$$

where $x^T$ and $x^N$ denote the tangential and normal components of $x$ at the point, respectively. Let $|x^T|$ and $|x^N|$ denote the length of $x^T$ and $x^N$, respectively.

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Using the decomposition (1.1), the author introduced and studied the notion of constant-ratio submanifolds in the early 2000s (see [5, 8]).

**Definition 1.1.** A submanifold $M$ of a Euclidean space is said to be *of constant-ratio* if the ratio $|x^N| : |x^T|$ is constant on $M$.

Clearly, a submanifold $M$ in a Euclidean space is of constant-ratio if and only if either $x^T = 0$ or the ratio $|x| : |x^T|$ is constant. Consequently, for each constant-ratio submanifold with $x^T \neq 0$, there is a positive number $\gamma$ such that

$$|x|^2 = \gamma\, |x^T|^2, \quad |x|^2 = \langle x, x \rangle\,,$$

where $\langle\ ,\ \rangle$ is the inner product on the Euclidean space. Moreover, a submanifold $M$ in a Euclidean space is of constant-ratio if and only if the angle between the position vector field and tangent spaces of $M$ is constant on $M$ [8]. For this reason, constant-ratio submanifolds are also known as *equiangular submanifolds* in [13].

The notion of constant-ratio submanifolds relates to the notion of convolution of Riemannian manifolds in the sense of [6, 7]. Convolution manifolds are defined as follows: Let $N_1, N_2$ be Riemannian manifolds equipped with metrics $g_1, g_2$, respectively. Consider a symmetric tensor ${}_hg_1 *_f g_2$ of type (0,2) on $N_1 \times N_2$ defined by

$${}_hg_1 *_f g_2 = h^2 g_1 + f^2 g_2 + 2fh df \otimes dh$$

for some positive functions $f$ and $h$ on $N_1$ and $N_2$, respectively. The symmetric tensor ${}_hg_1 *_f g_2$ is called a *convolution of* $g_1$ *and* $g_2$. The product manifold $N_1 \times N_2$ endowed with a convolution metric $g = {}_hg_1 *_f g_2$ is called a *convolution manifold* [6, 7].

Submanifolds of constant-ratio also relate closely to a problem in physics concerning the motion in a central force field which obeys the inverse-cube law. In fact, the trajectory of a particle subject to a central force of attraction located at the origin which obeys the inverse-cube law is a constant-ratio curve. The inverse-cube law was originated from I. Newton in his letter sent to R. Hooke on December 13, 1679. This letter is of great historical importance because it reveals the state of Newton's development of dynamics at that time (cf. [1, 14] and [15, §II, Proposition IX]).

Recently, it was pointed out by S. Haesen, A. I. Nistor and L. Verstraelen in [13] that constant-ratio submanifolds also relate closely to D'Arcy Thomson's basic principal of natural growth in biology (see also [12, 16–18]).

An isometric immersion $f : M \to \tilde{M}^m$ of a Riemannian $n$-manifold $M$ into a Kähler $m$-manifold $\tilde{M}^m$ is called *totally real* if the complex structure $J$ of $\tilde{M}$ carries each tangent vector of $M$ into a normal vector (cf. [11]). Totally real submanifolds form an important and natural class of submanifolds of Kähler manifolds, which includes Lagrangian submanifolds (cf. [4, pages 322–335]).

In this paper, we provide a fundamental study of totally real submanifolds of the complex Euclidean $m$-space $\mathbf{C}^m$ in terms of the positive function $x$ of the totally real submanifolds and the complex structure $J$ of $\mathbf{C}^m$. Consequently, we classify constant-ratio totally real submanifolds in $\mathbf{C}^m$. Some related results are also presented.

## 2. Preliminaries

We follow the notations from [2, 3, 9].

2.1. **Basic formulas and definitions.** Let $f : M \to \mathbf{C}^m$ be an isometric immersion of a Riemannian $n$-manifold $M$ into the complex Euclidean $m$-space $\mathbf{C}^m$. We denote the Riemannian connections of $M$ and $\mathbf{C}^m$ by $\nabla$ and $\tilde{\nabla}$, respectively. Let $D$ denote the normal connection of the submanifold.

The formulas of Gauss and Weingarten are given respectively by

$$\begin{aligned} &\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \\ &\tilde{\nabla}_X \zeta = -A_\zeta X + D_X \zeta, \end{aligned} \tag{2.1}$$

for tangent vector fields $X$ and $Y$ and normal vector field $\zeta$. The second fundamental form $h$ is related to the shape operator $A$ by

$$\langle h(X, Y), \zeta \rangle = \langle A_\zeta X, Y \rangle. \tag{2.2}$$

A submanifold $M$ is called totally geodesic if its second fundamental form $h$ vanishes identically.

If we denote the Riemann curvature tensor of $\nabla$ by $R$, then the equations of Gauss and Codazzi are given respectively by

$$\begin{aligned} &\langle R(X, Y)Z, W \rangle = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle, \\ &(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z), \end{aligned}$$

where $X, Y, Z, W$ are vector fields tangent to $M$ and $\nabla h$ is defined by

$$(\nabla_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

When $M$ is a totally real submanifold, we also have (cf. [9, 11])

$$\langle h(X, Y), JZ \rangle = \langle h(Y, Z), JX \rangle = \langle h(Z, X), JY \rangle.$$

2.2. **Sasakian manifolds and anti-invariant submanifolds.** An odd-dimensional Riemannian manifold $(M, g)$ is called an *almost contact metric manifold* if there exist on $M$ a $(1, 1)$-tensor field $\phi$, a vector field $\xi$ and a 1-form $\eta$ such that

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \end{aligned}$$

for vector fields $X, Y$ on $M$. On an almost contact metric manifold, we also have

$$\phi\xi = 0, \quad \eta \circ \phi = 0.$$

The vector field $\xi$ is called the *structure vector field*.

By a *contact* $(2n+1)$*-manifold* we mean a $(2n+1)$-dimensional manifold $M$ together with a global 1-form $\eta$ satisfying

$$\eta \wedge (d\eta)^n \neq 0$$

on $M$. If $\eta$ of an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ is a contact form and if $\eta$ satisfies

$$d\eta(X, Y) = g(X, \phi Y)$$

for all vectors $X, Y$ tangent to $M$, then $M$ is called a *contact metric manifold.* On a contact metric $(2n+1)$-manifold $M$, $\eta = 0$ defines a $2n$-dimensional distribution in $TM$, which is called the *contact distribution.*

An almost contact metric structure of $M$ is called *normal* if the Nijenhuis torsion $[\phi, \phi]$ of $\phi$ equals to $-2\mathrm{d}\eta \otimes \xi$. A normal contact metric manifold is called a *Sasakian manifold.*

It can be proved that an almost contact metric manifold is Sasakian if and only if the Riemann curvature tensor $R$ satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

for any vector fields $X, Y$ on $M$.

A plane section of a Sasakian manifold $(M, \phi, \xi, g)$ is called a *$\phi$-section* if it is spanned by $v, \phi(v)$ for some tangent vector $v$. The section curvature of a $\phi$-section is called a *$\phi$-sectional curvature.*

Let $S^{2m-1}(1)$ denote the unit hypersphere of $\mathbf{C}^m$ centered at the origin, i.e.,

$$S^{2m-1}(1) = \{z \in \mathbf{C}^m : \langle z, z\rangle = 1\}.$$

Let $x$ denote the position function of $S^{2m-1}(1)$ in $\mathbf{C}^m$.

If we put $\xi = Jx$ and let $\phi(X)$ denote the tangential component of $JX$ for each $X \in T(S^{2m-1}(1))$, then $\xi$ is a unit tangent vector field of $S^{2m-1}(1)$. Let $g$ on $S^{2m-1}(1)$ be the metric induced from the Euclidean metric of $\mathbf{C}^m$ and let $\eta$ be the dual 1-form of $\xi$. Then $(S^{2m-1}(1), \phi, \xi, \eta, g)$ is a Sasakian manifold with constant $\phi$-sectional curvature one.

**Definition 2.1.** A submanifold $M$ of the Sasakian manifold is called *anti-invariant* if $\phi(T_pM) \subset T_p^{\perp}M$ holds at each point $p \in M$.

**Definition 2.2.** A submanifold $M$ of the Sasakian manifold is called *a $C$-totally real submanifold* if $\xi$ is a normal vector field of $M$.

*Remark* 2.1. A direct consequence of Definitions 2.1 and 2.2 is that each $C$-totally real submanifold of the Sasakian $S^{2m-1}(1)$ is an anti-invariant submanifold.

*Remark* 2.2. Since $\phi$ is necessarily of rank $2m-2$ on the Sasakian $S^{2m-1}(1)$, we have $n \leq m$ for every $n$-dimensional anti-invariant submanifold of $S^{2m-1}(1)$.

*Remark* 2.3. On a Sasakian manifold of dimension $2n+1$, there exist $C$-totally real submanifolds of the contact distribution of dimension less than or equal to $n$, but of no higher dimension.

## 3. A characterization of totally real cones

By a *cone* in a complex Euclidean $m$-space $\mathbf{C}^m$, we mean a ruled submanifold generated by a family of lines passing through a fixed point in $\mathbf{C}^m$. A submanifold $M$ of $\mathbf{C}^m$ is called a *totally real cone* if $M$ is a cone which is a totally real submanifold (with respect to the complex structure $J$ of $\mathbf{C}^m$).

The following example shows that there exist ample examples of totally real cones in complex Euclidean spaces.

*Example* 3.1. Let $\psi : N^{n-1} \to S^{2m-1}(1) \subset \mathbf{C}^m$ be a $C$-totally real submanifold of $S^{2m-1}(1)$ and let $\mathbb{R} = \mathbb{R}^* - \{0\}$. Then the map

$$x : \mathbb{R}^* \times N^{n-1} \to \mathbf{C}^m,$$

defined by $x(t,p) \mapsto t\psi(p)$, for every $(t,p) \in \mathbb{R}^* \times N^{n-1}$, is a totally real cone in $\mathbf{C}^m$.

Totally real cones can be characterized as follows.

**Proposition 3.1.** *Let $x : M \to \mathbf{C}^m$ be a totally real immersion of a Riemannian $n$-manifold $M$ into $\mathbf{C}^m$. Then the following three statements are equivalent:*

(1) $x \in TM$.
(2) $Jx \in J(TM)$
(3) $x : M \to \mathbf{C}^m$ *is an open portion of a totally real cone with vertex at the origin.*

*Proof.* Under the hypothesis, the equivalence of (1) and (2) is trivial due to $J^2 = -I$.

(1) $\Rightarrow$ (3). If $x \in TM$, then we have $x = x^T$. Hence, $e_1 = x/|x|$ is a unit tangent vector field of $M$. Thus we have

$$\tilde{\nabla}_{e_1} x = e_1, \quad \tilde{\nabla}_{e_1} x = \tilde{\nabla}_{e_1}(\rho e_1) = (e_1\rho)e_1 + \rho\tilde{\nabla}_{e_1} e_1,$$

which implies $\tilde{\nabla}_{e_1} e_1 = 0$. Therefore, the integral curves of $e_1$ are open portions of straight lines in $\mathbf{C}^m$. Moreover, since the position vector is always tangent to $M$, the generating lines, given by the integral curves of $e_1$, always pass through the origin. Consequently, $M$ is an open portion of a totally real cone with vertex at the origin.

(3) $\Rightarrow$ (1). This is trivial. □

We also need the following.

**Lemma 3.1.** *Let $\phi : M \to S^{2m-1}(1)$ be an $n$-dimensional anti-invariant submanifold of $S^{2m-1}(1)$ and let $\iota$ denote the inclusion map of $S^{2m-1}(1)$ in $\mathbf{C}^m$. Then the composition $x = \iota \circ \phi : M \to S^{2m-1}(1) \subset \mathbf{C}^m$ is a totally real submanifold in $\mathbf{C}^m$.*

*Proof.* Assume that $\phi : M \to S^{2m-1}(1)$ is an $n$-dimensional anti-invariant submanifold. Then we have $\phi(TM) \subset T^\perp M$.

For each point $p \in M$, let $\mathcal{C}_p$ be the contact subspace of $T_pM$ defined by

$$\mathcal{C}_p = \{X \in T_pM : \langle X, \xi(p)\rangle = 0\}.$$

Then we have $\dim \mathcal{C}_p \geq n-1$ and

$$\phi(Y) = JY, \text{ for every } Y \in \mathcal{C}_p. \tag{3.1}$$

*Case* (i). $\xi(p) \in T_pM$. In this case, we have $T_pM = \mathcal{D}_p \oplus \operatorname{Span}\{\xi\}$. Since $J\xi = -x \in T_p^\perp M$, we obtain from (3.1) that $J(T_pM) \subset T_p^\perp M$.

*Case* (ii). $\xi(p) \in T_p^\perp M$. In this case, we have $T_pM = \mathcal{C}_p$. Therefore, (3.1) implies $J(T_pM) \subset T_p^\perp M$.

*Case* (iii). $\xi(p) \notin T_pM$ *and* $\xi(p) \notin T_p^\perp M$. If we put

$$\xi(p) = \xi^T(p) + \xi^\perp(p)$$

with $\xi^T(p) \in T_pM$ and $\xi^\perp(p) \in T_p^\perp M$, then $T_pM$ is spanned by $\xi^T(p)$ and $\mathcal{C}_p$.

From (3.1) we get

$$\left\langle J\xi^T(p), Y \right\rangle = -\left\langle \xi^T(p), JY \right\rangle = -\left\langle \xi^T(p), \phi(Y) \right\rangle = 0, \text{ for every } Y \in \mathcal{C}_p.$$

Obviously, we also have $\left\langle J\xi^T(p), \xi^T(p) \right\rangle = 0$. Therefore, we have $J(T_pM) \subset T_p^\perp M$.

Hence, in all of the three cases given above, $J(T_pM) \subset T_p^\perp M$ holds. Consequently, $x = \iota \circ \phi : M \to S^{2m-1}(1) \subset \mathbf{C}^m$ is a totally real submanifold in $\mathbf{C}^m$. □

## 4. Totally real submanifolds in $\mathbf{C}^m$ with $Jx \in TM$

The following result characterizes and classifies totally real submanifolds of $\mathbf{C}^m$ satisfying $\xi = Jx \in TM$.

**Theorem 4.1.** *Let $x : M \to \mathbf{C}^m$ be a totally real immersion of a Riemannian $n$-manifold $M$ into $\mathbf{C}^m$. Then the following three statements are equivalent:*

(1) *$Jx \in TM$.*
(2) *Up to a dilation of $\mathbf{C}^m$, $M$ is an anti-invariant submanifold of the Sasakian $S^{2m-1}(1) \subset \mathbf{C}^m$ with $\xi \in TM$.*
(3) *$M$ is an open part of the Riemannian product of a circle $S^1$ and a Riemannian $(n-1)$-manifold $N^{n-1}$. Moreover, up to a suitable dilation of $\mathbf{C}^m$, the immersion is given by*

$$x(t, u_2, \ldots, u_n) = e^{\mathrm{i}t}\phi(u_2, \ldots, u_n),$$

*where $\phi$ is a $C$-totally real immersion of $N^{n-1}$ into the Sasakian $S^{2m-1}(1)$.*

*Proof.* Let $x : M \to \mathbf{C}^m$ be a Lagrangian immersion of $M$ into $\mathbf{C}^m$.

(1) ⇒ (2). Suppose that $\xi = Jx$ is a tangent vector field of $M$. If we put

$$\mathcal{D}_1 = \operatorname{Span}\{\xi\}, \quad \mathcal{D}_2 = (\mathcal{D}_1)^\perp,$$

then we have $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$. Since $\mathbf{C}^m$ is Kählerian, we find

$$\tilde{\nabla}_Z \xi = J\tilde{\nabla}_Z x = JZ, \text{ for every } Z \in TM. \tag{4.1}$$

From (4.1) we find

$$\nabla_Z \xi = 0, \tag{4.2}$$

$$h(Z,\xi) = JZ, \tag{4.3}$$

$$h(\xi,\xi) = J\xi = -x, \tag{4.4}$$

for any $Z \in TM$. It follows from (4.2) that

$$Z\langle x,x\rangle = Z\langle \xi,\xi\rangle = 2\,\langle \nabla_Z\xi,\xi\rangle = 0, \text{ for every } Z \in TM.$$

Therefore, $\langle x,x\rangle$ is a constant. Hence, $x(M)$ is contained in a hypersphere of $\mathbf{C}^m$ centered at the origin. Consequently, after applying suitable dilation, $x(M)$ lies in the unit hypersphere $S^{2m-1}(1)$.

Now, because $x : M \to \mathbf{C}^m$ is a totally real immersion, we have $J(TM) \subset T^{\perp}M$, which implies $\phi(TM) \subset T^{\perp}M$. Consequently, $M$ is anti-invariant in $S^{2m-1}(1)$ with $\xi \in TM$. This gives statement (2).

(2) $\Rightarrow$ (3). It follows from (4.2) that $\mathcal{D}_1$ is a totally geodesic distribution, i.e., $\mathcal{D}_1$ is an integrable distribution whose leaves are totally geodesic in $M$. Moreover, it also follows from (4.2) that each integral curve of $\xi$ is an open portion of a great circle of $S^{2m-1}(1)$.

Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are orthogonal complementary distributions, we obtain from (4.2) that $\langle [X,Y],\xi\rangle = 0$. Thus, $\mathcal{D}_2$ is an integral distribution.

Since $\xi$ is orthogonal to $\mathcal{D}_2$ and $M$ is totally real, we have

$$\langle \tilde\nabla_X Y,\xi\,\rangle = -\,\langle Y,\tilde\nabla_X\xi\,\rangle = -\,\langle Y,JX\,\rangle = 0. \tag{4.5}$$

On the other hand, it follows from formula (2.1) of Gauss that

$$\langle \tilde\nabla_X Y,\xi\,\rangle = \langle \nabla_X Y,\xi\,\rangle\,. \tag{4.6}$$

Therefore, by combining (4.5) and (4.6), we conclude that $\mathcal{D}_2$ is also a totally geodesic distribution. Consequently, $M$ is locally the Riemannian product $S^1 \times N^{n-1}$ of a circle $S^1$ and a Riemannian $(n-1)$-manifold $N^{n-1}$, according to the well-known de Rham decomposition theorem. Hence, there exists a local coordinate system $\{t,u_2,\ldots,u_n\}$ on $M$ such that $\xi = \frac{\partial}{\partial t}$ and $\frac{\partial}{\partial u_2},\ldots,\frac{\partial}{\partial u_n} \in \mathcal{D}_2$.

Now, it follows from (4.2), (4.3) and (4.4) that the immersion $x : M \to \mathbf{C}^m$ satisfies the following system of partial differential equations:

$$\frac{\partial^2 x}{\partial t^2} = \mathrm{i}\frac{\partial x}{\partial t}, \tag{4.7}$$

$$\frac{\partial^2 x}{\partial t\partial u_j} = \mathrm{i}\frac{\partial x}{\partial u_j}, \tag{4.8}$$

$$\frac{\partial^2 x}{\partial u_i\partial u_j} = \sum_{k=2}^{n}\Gamma_{ij}^k\frac{\partial x}{\partial u_k} + h\left(\frac{\partial}{\partial u_j},\frac{\partial}{\partial u_j}\right) - \left\langle \frac{\partial}{\partial u_j},\frac{\partial}{\partial u_j}\right\rangle x,$$

for $i,j = 2,\ldots,n$, where $\Gamma_{ij}^k$ are the Christoffel symbols.

After solving (4.7) we get

$$x = A(u_2, \dots, u_n) + e^{\mathrm{i}t}\phi(u_2, \dots, u_n) \tag{4.9}$$

for some $\mathbf{C}^m$-valued functions $A$ and $\phi$.

By substituting (4.9) into (4.8) we find

$$\frac{\partial A}{\partial u_j} = 0, \quad j = 2, \dots, n.$$

Thus, $A$ is a constant vector, say $c_0$. Consequently, (4.9) becomes

$$x = c_0 + e^{\mathrm{i}t}\phi(u_2, \dots, u_n). \tag{4.10}$$

From (4.10) we get

$$\frac{\partial x}{\partial t} = \mathrm{i}e^{\mathrm{i}t}\phi,$$

which implies that $\langle \phi, \phi \rangle = 1$, since $\xi$ is a unit vector field. Therefore, it follows from (4.9) and $\langle L, L \rangle = 1$ that

$$0 = \langle c_0, c_0 \rangle + 2\left\langle c_0, e^{\mathrm{i}t}\phi \right\rangle. \tag{4.11}$$

Now, after taking the differentiation of (4.11) twice with resect to $t$, we obtain $\left\langle c_0, e^{\mathrm{i}t}\phi \right\rangle = 0$. By combining this with (4.11) gives $c_0 = 0$. Consequently, (4.10) reduces to

$$x = e^{\mathrm{i}t}\phi(u_2, \dots, u_n). \tag{4.12}$$

From (4.12) we have

$$\frac{\partial x}{\partial u_j} = e^{\mathrm{i}t}\frac{\partial \phi}{\partial u_j}, \quad j = 2, \dots, n,$$

which gives

$$\left\langle \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} \right\rangle = \left\langle \frac{\partial \phi}{\partial u_i}, \frac{\partial \phi}{\partial u_j} \right\rangle, \quad i, j = 2, \dots, n.$$

Therefore, $\phi$ is an isometric immersion. From (4.12) we get

$$\frac{\partial x}{\partial t} = Jx, \quad \frac{\partial x}{\partial u_j} = e^{\mathrm{i}t}\frac{\partial \phi}{\partial u_j}, \quad k = 2, \dots, n. \tag{4.13}$$

Since $x : M \to \mathbf{C}^m$ is a totally real immersion and the structure vector field $\xi$ is orthogonal to $\mathcal{D}_2$, we have

$$\left\langle \frac{\partial x}{\partial t}, \mathrm{i}\frac{\partial x}{\partial u_j} \right\rangle = \left\langle \frac{\partial x}{\partial t}, \frac{\partial x}{\partial u_j} \right\rangle = 0, \quad j = 2, \dots, n. \tag{4.14}$$

Now, we derive from (4.13) and (4.14) that $\left\langle Jx, \mathrm{i}\frac{\partial \phi}{\partial u_i} \right\rangle = 0$ holds for $j = 2, \dots, n$. Consequently, $\phi$ is a $C$-totally real immersion of $N^{n-1}$ into $S^{2m-1}(1)$. Hence, we obtain the statement (3).

(3) $\Rightarrow$ (1). If $x : M \to \mathbf{C}^m$ is given by

$$x(t, u_2, \ldots, u_n) = e^{\mathrm{i}t}\phi(u_2, \ldots, u_n),$$

such that $\phi$ is a $C$-totally real immersion of $N^{n-1}$ into $S^{2m-1}(1)$, then we obtain $Jx = \frac{\partial}{\partial t} \in TM$. This implies statement (1). $\square$

*Remark* 4.1. When $M$ is a Lagrangian submanifold of $\mathbf{C}^n$, Condition (1) of Theorem 4.1 holds automatically. In this case, Theorem 4.1 reduces to Theorem 1.1 of [10].

## 5. Totally real submanifolds in $\mathbf{C}^m$ with $Jx \in \nu$

Let $x : M \to \mathbf{C}^m$ be a totally real immersion. Then the normal bundle $T^{\perp}M$ of $M$ admits a canonical decomposition:

$$T^{\perp}M = J(TM) \oplus \nu,$$

where $\nu$ is a subbundle of the normal bundle satisfying $J\nu = \nu$.

The following theorem provides a very simple characterization of $C$-totally real submanifolds of $S^{2m-1}(1)$ in term of the position function $x$ and the complex structure $J$ of $\mathbf{C}^m$.

**Theorem 5.1.** *Let $x : M \to \mathbf{C}^m$ be a totally real immersion of a Riemannian $n$-manifold $M$ into the complex Euclidean $m$-space $\mathbf{C}^m$. Then the following two statements are equivalent:*

(1) *$Jx \in \nu$.*

(2) *Up to a dilation of $\mathbf{C}^m$, $M$ is a $C$-totally real submanifold of the Sasakian $S^{2m-1}(1) \subset \mathbf{C}^m$.*

*Proof.* Let $x : M \to \mathbf{C}^m$ be a Lagrangian isometric immersion of a Riemannian $n$-manifold $M$ into the complex Euclidean $m$-space $\mathbf{C}^m$. Assume that $\xi = Jx \in \nu$. Then we have

$$-A_{\xi}Z + D_Z\xi = \tilde{\nabla}_Z\xi = J\tilde{\nabla}_Z x = JZ, \text{ for every } Z \in TM,$$

which yields

$$\nabla_Z\xi = JZ, \text{ for every } Z \in TM. \tag{5.1}$$

It follows from (5.1) that

$$Z\langle x, x\rangle = Z\langle \xi, \xi\rangle = 2\,\langle D_Z\xi, \xi\rangle = 0$$

for any vector $Z$ tangent to $M$. Thus $\langle x, x\rangle$ is a constant. Hence $x(M)$ is contained in a hypersphere of $\mathbf{C}^m$ centered at the origin. Consequently, after applying a suitable dilation of $\mathbf{C}^m$, $x(M)$ lies in the unit hypersphere $S^{2m-1}(1)$.

Since $\xi$ lies in $\nu$, $\xi$ is a normal vector field of $M$. Therefore, by Definition 2.2, $M$ is a $C$-totally real submanifold of the Sasakian $S^{2m-1}(1)$.

Conversely, if $M$ is a $C$-totally real submanifold of the Sasakian $S^{2m-1}(1) \subset \mathbf{C}^m$, then we have

$$\langle \xi, JZ \rangle = - \langle J\xi, Z \rangle = \langle x, Z \rangle = 0, \quad \text{for all } Z \in TM.$$

Hence we obtain $Jx = \xi \in \nu$. □

## 6. Classification of constant-ratio totally real submanifolds

In this section, we classify constant-ratio totally real submanifolds in the complex Euclidean space $\mathbf{C}^m$.

**Theorem 6.1.** *Let $x : M \to \mathbf{C}^m$ be a totally real immersion of a Riemannian $n$-manifold $M$ into $\mathbf{C}^m$. Then $M$ is of constant ratio if and only if one of the following four statements holds:*

(1) *$M$ is an open portion of a totally real cone with vertex at the origin.*

(2) *Up to a suitable dilation, $x : M \to \mathbf{C}^m$ is given by*

$$x(t, u_2, \ldots, u_n) = e^{\mathrm{i}t} \phi(u_2, \ldots, u_n),$$

*where $\phi$ is an $(n-1)$-dimensional $C$-totally real submanifold of the Sasakian $S^{2m-1}(1)$.*

(3) *Up a suitable dilation, $M$ is an anti-invariant submanifold of the Sasakian $S^{2m-1}(1)$ with $\xi \notin TM$.*

(4) *Up to a suitable dilation, $x : M \to \mathbf{C}^m$ is given by*

$$x(s, u_2, \ldots, u_n) = bs\psi(s, u_2, \ldots, u_n), \quad s \neq 0,$$

*where $b$ is a positive number $< 1$ and $\psi : M \to S^{2m-1}(1)$ is an immersion satisfying*

(4.a) $\langle \psi_s, \psi_s \rangle = (1 - b^2)/(b^2 s^2)$,

(4.b) $\langle \psi, \mathrm{i}\psi_{u_i} \rangle = -s \langle \psi_s, \mathrm{i}\psi_{u_i} \rangle$, *and*

(4.c) $\langle \psi_{u_i}, \mathrm{i}\psi_{u_j} \rangle = \langle \psi_s, \psi_{u_j} \rangle = 0$,

*for $i, j = 2, \ldots, n$.*

*Proof.* Suppose that $x : M \to \mathbf{C}^m$ is a constant-ratio totally real immersion from a Riemannian $n$-manifold into $\mathbf{C}^m$. Then exactly one of the following three cases occurs:

(a) $x^N = 0$;

(b) $x^T = 0$;

(c) the ratio $|x^N|/|x^T|$ is a positive number.

*Case* (a): $x^N = 0$. In this case, Proposition 3.1 implies that $M$ is an open part of a totally real cone with vertex at the origin. This gives case (1) of the theorem.

*Case* (b): $x^T = 0$. In this case, the position vector field $x$ is normal to $M$. Thus, we have $Z\langle x, x \rangle = 2 \langle x, Z \rangle = 0$ for any $Z \in TM$. Hence, $\langle x, x \rangle$ is a positive constant. Therefore, $x(M)$ lies in a hypersphere of $\mathbf{C}^m$ centered at the origin.

If $\xi = Jx \in TM$, then Theorem 4.1 implies that case (2) of the theorem occurs.

If $\xi \notin TM$, then, after applying a suitable dilation on $\mathbf{C}^m$, $M$ becomes an anti-invariant submanifold of the Sasakian $S^{2m-1}(1)$. Thus, we get case (3) of the theorem.

*Case* (c): $|x^N|/|x^T|$ *is a positive number.* Let us put

$$\left\langle x^N, x^N \right\rangle = \alpha \left\langle x^T, x^T \right\rangle, \quad x^T = \beta e_1. \tag{6.1}$$

then $\alpha \in \mathbb{R}^+$ and $\beta = \langle x, e_1 \rangle = |x^T|$. We may extend $e_1$ to a local orthonormal frame $e_1, \ldots, e_n$ on $M$.

From (6.1), we get

$$\langle x, x \rangle = (1+\alpha)|x^T|^2. \tag{6.2}$$

Clearly, we must have $\alpha \neq -1$. By applying (6.2), we have

$$\langle x, e_1 \rangle^2 = c \, \langle x, x \rangle \tag{6.3}$$

with $c = (1+\alpha)^{-1}$. After taking the derivative of (6.3) with respect to an arbitrary tangent vector $X$ of $M$, we find

$$\langle e_1, x \rangle \left( \langle \, \tilde{\nabla}_X e_1, x \rangle + \langle e_1, X \rangle \, \right) = c \, \langle x, X \rangle \, . \tag{6.4}$$

In particular, for $X = e_1$, (6.4) gives

$$\langle x, h(e_1, e_1) \rangle = c - 1. \tag{6.5}$$

If $c = 1$, (6.3) reduces to $\langle x, e_1 \rangle^2 = \langle x, x \rangle$, which implies $x = x^T$. Thus, we have $x^N = 0$, which is a contradiction. Hence, we must have $c \neq 1$. Consequently, we may put

$$x^N = \mu e_{n+1}, \quad \mu \neq 0 \tag{6.6}$$

for some unit normal vector field $e_{n+1}$ of $M$. If we put $\kappa = \left\langle A_{e_{n+1}} e_1, e_1 \right\rangle$, then by applying (2.2), (6.5) and (6.6) we find

$$\mu\kappa = c - 1. \tag{6.7}$$

Since $c \neq 1$, we also have $\kappa \neq 0$. Since $x = |x| e_1$, (6.4) gives

$$\langle \, x, \tilde{\nabla}_{e_j} e_1 \, \rangle = 0, \;\; j = 2, \ldots, n.$$

From $\langle \, e_1, \tilde{\nabla}_{e_j} e_1 \, \rangle = \langle \, x, \tilde{\nabla}_{e_j} e_1 \, \rangle = 0$, we find

$$0 = \left\langle x^N, h(e_1, e_j) \right\rangle = \mu \left\langle A_{e_{n+1}} e_1, e_j \right\rangle.$$

Therefore, $e_1$ is an eigenvector of $A_{e_{n+1}}$ with eigenvalue $\kappa$, *i.e.,*

$$A_{e_{n+1}} e_1 = \kappa e_1. \tag{6.8}$$

On the other hand, by taking the derivative of $\langle x, e_j \rangle = 0$ with respect to $e_k$ for $j, k = 2, \ldots, n$, we find

$$0 = \delta_{jk} + \beta \, \langle e_1, \nabla_{e_k} e_j \rangle + \mu \, \langle e_{n+1}, h(e_k, e_j) \rangle \, . \tag{6.9}$$

Let us put

$$\nabla_{e_k} e_j = \sum_{s=1}^{n} \omega_j^s(e_k) e_s.$$

Then we find from (6.9) that

$$\beta \omega_j^1(e_k) = -\delta_{jk} - \mu \left\langle A_{e_{n+1}} e_j, e_k \right\rangle . \tag{6.10}$$

Since $\beta \neq 0$ and $A_{e_{n+1}}$ is self-adjoint, (6.10) yields

$$\omega_j^1(e_k) = \omega_k^1(e_j), \quad j,k = 2,\dots,n. \tag{6.11}$$

Let $\mathcal{F}$ denote the distribution spanned by $e_1$ and $\mathcal{F}^\perp$ denote the orthogonal complementary distribution of $\mathcal{F}$ in $TM$. Then (6.11) implies that $\mathcal{F}^\perp$ is a completely integrable distribution. Moreover, since $\mathcal{F}$ is of rank one, the distribution $\mathcal{F}$ is also completely integrable. Therefore, there exist local coordinate systems $\{s, u_2, \dots, u_n\}$ on $M$ such that $e_1 = \partial/\partial s$ and $\{\partial/\partial u_2, \dots, \partial/\partial u_n\}$ spans $\mathcal{F}^\perp$.

Because $e_1$ is perpendicular to $\mathcal{F}^\perp$, we have

$$\langle x_s, x_{u_2} \rangle = \cdots = \langle x_s, x_{u_n} \rangle = 0, \quad x_s = \frac{\partial x}{\partial s}, \quad x_{u_j} = \frac{\partial x}{\partial u_j}. \tag{6.12}$$

It follows from (6.6) and (6.7) that

$$\langle x, e_{n+1} \rangle \kappa = c - 1.$$

Thus

$$(c-1) e_j \left( \frac{1}{\kappa} \right) = - \left\langle x^T, A_{e_{n+1}} e_j \right\rangle + \left\langle x^N, D_{e_j} e_{n+1} \right\rangle , \quad j = 2, \dots, n. \tag{6.13}$$

Since $e_{n+1}$ is a unit normal vector field and $x^N = \mu e_{n+1}$, we get

$$\left\langle x^N, D_{e_j} e_{n+1} \right\rangle = 0.$$

Consequently, by applying (6.8) and (6.13), we find $e_2\kappa = \cdots = e_n \kappa = 0$. So, $\kappa$ depends only on $s$, *i.e.,* $\kappa = \kappa(s)$. Therefore, after taking the derivative of (6.12) with respect to $s$, we find

$$0 = \kappa'(s) \langle x, e_{n+1} \rangle - \kappa^2(s) \langle x, e_1 \rangle , \tag{6.14}$$

in virtue of (6.8). By combining (6.7) and (6.14) we have

$$\langle x, e_1 \rangle = (c-1) \frac{\kappa'}{\kappa^3}. \tag{6.15}$$

Thus, after taking the derivative of (6.15) with respect to $s$, we get

$$1 + \langle\, x, \tilde{\nabla}_{e_1} e_1 \,\rangle = (c-1) \left( \frac{\kappa'}{\kappa^3} \right)'. \tag{6.16}$$

Combining (6.5) and (6.16) yields

$$\left( \frac{\kappa'}{\kappa^3} \right)' = \frac{c}{c-1}. \tag{6.17}$$

After solving (6.17) we obtain

$$\frac{1}{\kappa^2} = \left(\frac{c}{1-c}\right) s^2 + as + b,$$

where $a$ and $b$ are integrating constants. Hence, after applying a suitable translation on $s$, we have

$$\frac{1}{\kappa^2} = \left(\frac{c}{1-c}\right) s^2 + \delta \tag{6.18}$$

for some suitable constant $\delta$.

Now, by applying (6.1), (6.15) and (6.18), we obtain

$$|x^T| = \beta = (c-1)\frac{\kappa'}{\kappa^3} = cs. \tag{6.19}$$

By combining (6.1) and (6.19), we find

$$\langle x, x\rangle = cs^2. \tag{6.20}$$

For simplicity, let us put $c = b^2$ with $b > 0, b \neq 1$. Then we get from (6.19) and (6.20) that

$$|x| = bs, \quad |x^T| = b^2 s. \tag{6.21}$$

In view of (6.21), we may put

$$x(s, u_2, \ldots, u_n) = bs\psi(s, u_2, \ldots, u_n), \tag{6.22}$$

for some $\mathbf{C}^m$-valued function $\psi = \psi(s, u_2, \ldots, u_n)$ satisfying $\langle \psi, \psi\rangle = 1$.

It follows from $\langle \psi, \psi\rangle = 1$, (6.12) and (6.22) that

$$\left\langle \psi_s, \psi_{u_j} \right\rangle = 0, \quad j = 2, \ldots, n. \tag{6.23}$$

Moreover, since $x : M \to \mathbf{C}^m$ is a totally real immersion, we obtain conditions (4.b) and (4.c) from (6.21). Finally, it is easy to obtain condition (4.a) from (6.21), (6.22), (6.23) and $\langle x_s, x_s\rangle = 1$.

The converse follows from Lemma 3.1 and straightforward computation. □

Obviously, there exist many constant-ratio totally real submanifolds of types (1), (2) and (3) given in Theorem 6.1. The next example shows that there also exist many constant-ratio totally real submanifolds of type (4).

*Example* 6.1. Let $b$ be a positive number $< 1$ and let $\eta : N^{n-1} \to S^{2m-3}(1) \subset \mathbf{C}^{m-1}$ be a $C$-totally real submanifold of $S^{2m-3}(1)$.

Consider the map $x : \mathbb{R} \times N^{n-1} \to \mathbf{C}^m$ given by

$$x(s, u_2, \ldots, u_n) = bs\left(\sin\left(\frac{\sqrt{1-b^2}}{b}\ln s\right), \cos\left(\frac{\sqrt{1-b^2}}{b}\ln s\right)\eta(u_2, \ldots, u_n)\right).$$

If we put

$$\psi(s, u_2, \ldots, u_n) = \left(\sin\left(\frac{\sqrt{1-b^2}}{b}\ln s\right), \cos\left(\frac{\sqrt{1-b^2}}{b}\ln s\right)\eta(u_2, \ldots, u_n)\right),$$

then it is straight-forward to verify that $\psi$ satisfies conditions (4.a), (4.b) and (4.c); and $x : M \to \mathbf{C}^m$ defines a constant-ratio totally real submanifold of type (4) with ratio $|x^N| : |x^T| = 1 : b$.

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