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# GENERALIZED PRODUCT THEOREM FOR THE MELLIN TRANSFORM AND ITS APPLICATIONS

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Abstract. In this paper, we introduce the generalized product theorem for the Mellin transform and we solve certain classes of singular integral equations with kernels coincided with conditions of this theorem. Moreover, new inversion techniques for $n$-th iterate of the $\mathcal{L}_2$-transform are obtained. A very simple inversion formula for the Widder potential transform is also given.

## 1. Introduction and Preliminaries

One of the classical integral transform is the Mellin transform

$$\mathcal{M}\{f(x);p\} = F(p) = \int_0^\infty x^{p-1} f(x)dx, \quad c_1 < \Re p < c_2 \tag{1.1}$$

and its inversion formula is written in terms of the Bromwich's integral in the following form

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(p)x^{-p}dp, \qquad c_1 < c < c_2. \tag{1.2}$$

For convergence of the relation (1.2), function $F(p)$ must belong to the $\mathcal{K}$-class functions defined in Appendix 1.

This transform is used for expressing many problems in the applied sciences. An application of this transform may occur in problems leading to following singular integral equation

$$\int_0^\infty k(x,y)g(y)dy = f(x), \quad x > 0. \tag{1.3}$$

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It is of interest to have inversion techniques for formal solution of the above singular integral equation in terms of an improper integral as follows

$$g(y) = \int_0^\infty h(x,y)f(x)dx. \tag{1.4}$$

For this purpose in this paper, in operational calculus of the Mellin transform we state *the generalized product theorem* for the Mellin transform and consider the following contributions

- The study of the certain class of singular integral equation (1.3) which kernel is coincided with the conditions of the generalized product theorem.
- Finding new inversion techniques for the Wright and the Mittag-Leffler transforms arising in fractional calculus.
- Obtaining the kernels of $n$-th iterates of the Laplace transform and the $\mathcal{L}_2$-transform[1] and showing that the Mellin transform of kernels satisfies the conditions of the generalized product theorem.
- Finding new inversion technique for $n$-th iterate of the $\mathcal{L}_2$-transform in terms of the inverse Mellin Transform.

For organization of our motivations, at first step we write some main properties of the Mellin transform and state the generalized product theorem. In Section 2, we solve some singular integral equations with kernels in terms of the elementary functions. In Section 3, we introduce new approaches for finding inversion formulas for the Wright and the Mittag-Leffler integral transforms. These integral transforms play fundamental roles in fractional calculus and it is important to get inversion formulas for them.

In Section 4, we investigate on $n$-th iterates of the Laplace transform and the $\mathcal{L}_2$-transform and show that the kernels of $n$-th iterates of these transforms are coincided with conditions of the generalized product theorem. By this result, new inversion formula for $n$-th iterate of the $\mathcal{L}_2$-transform is also given. Finally, an Appendix is included for definition of $\mathcal{K}$-class functions and some $\mathcal{K}$-class functions used in this paper. Main conclusions are also given.

First, we recall some fundamental properties of the Mellin transform which can be easily written with respect to the definition (1.1). For more details and properties of this transform, see [6-8], [11], [13].

[1] Yurekli and Sadek [16] introduced the $\mathcal{L}_2$-transform

$$f(x) = \int_0^\infty y e^{-x^2y^2} g(y)dy$$

and showed the Parseval-Goldstein theorems involving the $\mathcal{L}_2$-transform and the Laplace transform can be used to obtain identities involving several well-known integral transforms and infinite integrals of elementary and special functions. Also, Aghili and Ansari applied this transform to solve some systems of ODEs, PFDEs and singular integral equations with the special kernels [1, 2].

**i) The convolution theorem for the Mellin transform:**

$$F(p)G(p) = \mathcal{M}\{f*g;p\} = \mathcal{M}\left\{\int_0^\infty g(u)f(\frac{x}{u})\frac{du}{u};p\right\} \tag{1.5}$$

**ii) The Mellin transform of $\delta_x$-derivatives:**

$$\mathcal{M}\{\delta_x^n f(x);p\} = (-p)^n F(p), \quad \delta_x = x\frac{d}{dx} \tag{1.6}$$

**iii) Change of scale property of the Mellin transform :**

$$\mathcal{M}\{f(x^a);p\} = \frac{1}{a}F(\frac{p}{a}), \quad a > 0 \tag{1.7}$$

**iv) Translation property of the Mellin transform:**

$$\mathcal{M}\{x^a f(x);p\} = F(p+a), \quad a > 0 \tag{1.8}$$

**v) The Mellin transform of polar form of a function:**

$$\mathcal{M}\{\Im[f(re^{i\theta})];p\} = -F(p)\sin(p\theta) \tag{1.9}$$

$$\mathcal{M}\{\Re[f(re^{i\theta})];p\} = F(p)\cos(p\theta) \tag{1.10}$$

Now, we state the generalized product theorem for the Mellin transform.

**Theorem 1.1.** ***(The generalized product theorem)***
*Let $\mathcal{M}\{g(x);p\} = G(p) \in \mathcal{K}$ and assume that $\Psi_1(p) \in \mathcal{K}$ and $\Psi_2(p)$ is an analytic function such that, $\mathcal{M}\{k(x,y);p\} = \Psi_1(p)y^{\Psi_2(p)-1}$. Then the following relation holds for continuous function $k(x,y)$ on the rectangular region $a \le x \le b, c \le y \le d$, $[a,b]\times[c,d] \subset (0,\infty)\times(0,\infty)$*

$$\mathcal{M}\left\{\int_0^\infty k(x,y)g(y)dy;p\right\} = \Psi_1(p)G(\Psi_2(p)). \tag{1.11}$$

*Proof.* Using the definition of the Mellin transform and considering the condition of continuous function $k(x,y)$ in order to change the order of integration, we get

$$\begin{aligned}\mathcal{M}\left\{\int_0^\infty k(x,y)g(y)dy;p\right\} &= \int_0^\infty x^{p-1}\int_0^\infty k(x,y)g(y)dydx \\ &= \int_0^\infty g(y)\int_0^\infty x^{p-1}k(x,y)dxdy \\ &= \Psi_1(p)\int_0^\infty y^{\Psi_2(p)-1}g(y)dy = \Psi_1(p)G(\Psi_2(p)).\end{aligned}$$

□

With considering of the above theorem, in next section we find formal solutions of some singular integral equations with kernels in the $\mathcal{K}$-class functions.

## 2. Singular Integral Equations with Kernels of Elementary Functions

**Problem 2.1.** *Solve the singular integral equation with the following logarithmic kernel (see* [9]*)*

$$\int_0^\infty \ln(|x-y|)g(y)dy = f(x), \quad x > 0. \tag{2.1}$$

By showing the above equation in the following form

$$\int_0^\infty \ln(|\frac{x-y}{y}|)g(y)dy = f(x) - f(0), \quad x > 0 \tag{2.2}$$

and applying the Mellin transform on both sides of equation, we get

$$\mathcal{M}\left\{\int_0^\infty \ln\left(|\frac{x-y}{y}|\right) g(y)dy; p\right\} = \mathcal{M}\{\phi(x), p\},$$

where $\phi(x)$ is defined as $\phi(x) = f(x) - f(0)$.

Now, by using the generalized product theorem (1.11) and considering the relation A1 in Appendix 2 for the Mellin transform of $\ln\left(|\frac{x}{y} - 1|\right)$, we rewrite the above equation in the form

$$\frac{\pi}{p}\cot(\pi p)\int_0^\infty y^p g(y)dy = \Phi(p).$$

The above equation can be rewritten as the Mellin transform of function $g(x)$ as

$$G(p+1) = \frac{p}{\pi}\tan(\pi p)\Phi(p). \tag{2.3}$$

By implementation of the inverse Mellin transform and considering the translation property (1.8) and the convolution property (1.5), simultaneously, we obtain

$$yg(y) = \frac{1}{\pi}\int_0^\infty \mathcal{M}^{-1}\left\{p\tan(\pi p), \frac{y}{u}\right\}\phi(u)\frac{dy}{u}. \tag{2.4}$$

At this point, by reconsidering the following relation for $\mathcal{M}^{-1}\{p\tan(\pi p)\}$ in view of the relation A7 in Appendix 2

$$\begin{aligned}\mathcal{M}^{-1}\{p\tan(\pi p); y\} &= \mathcal{M}^{-1}\left\{p^2\frac{\tan(\pi p)}{p}; y\right\} = \left(y\frac{d}{dy}\right)^2 \mathcal{M}^{-1}\left\{\frac{\tan(\pi p)}{p}; y\right\} \\ &= \frac{1}{\pi}\left(y\frac{d}{dy}\right)^2 \ln\left(\frac{1+\sqrt{y}}{1-\sqrt{y}}\right)\end{aligned} \tag{2.5}$$

and substituting in the relation (2.3), the solution of the singular integral equation (2.1) is written as

$$g(y) = \frac{1}{\pi^2}\frac{d}{dy}y\frac{d}{dy}\int_0^\infty \ln\left(|\frac{\sqrt{y}+\sqrt{u}}{\sqrt{y}-\sqrt{u}}|\right)\phi(u)\frac{du}{u}. \tag{2.6}$$

**Problem 2.2.** *Solve the singular integral equation with the following inverse trigonometric kernel*

(2.7) $$\int_0^\infty \sin^{-1}(y\sqrt{y^2+x^2})g(y)dy = f(x), \quad x>0.$$

By applying the Mellin transform on both sides of equation

$$\mathcal{M}\left\{\int_0^\infty \sin^{-1}(y\sqrt{y^2+x^2})g(y)dy; p\right\} = \mathcal{M}\{f(x), p\}$$

and using the relation A2 in Appendix 2 for the Mellin transform of kernel $\sin^{-1}(y\sqrt{y^2+x^2})$, we get

$$\frac{1}{p}\sin\left(\frac{\pi p}{2}\right)\int_0^\infty y^p g(y)dy = F(p),$$

which implies that

(2.8) $$G(p+1) = p\csc\left(\frac{\pi p}{2}\right)F(p).$$

By applying the inverse Mellin transform and using the translation and the convolution properties, we obtain

(2.9) $$yg(y) = \int_0^\infty \mathcal{M}^{-1}\left\{p\csc\left(\frac{\pi p}{2}\right), \frac{y}{u}\right\} f(u)\frac{du}{u}.$$

According to the relation A8 in Appendix 2 and the Mellin transform of delta derivatives (1.6), we finally get the solution of (2.7) as follows

(2.10) $$g(y) = -\frac{2}{\pi}\frac{d}{dy}\int_0^\infty \frac{1}{\sqrt{u}(\sqrt{y}+\sqrt{u})}f(u)du.$$

**Problem 2.3.** *Solve the singular integral equation with the following exponential kernel*

(2.11) $$\int_0^\infty e^{-\frac{1}{y}x^\alpha} g(y)dy = f(x), \quad \alpha>0.$$

In the same procedure as in the previous problem, after applying the Mellin transform on equation and using the relation A3 in Appendix 2, we get

(2.12) $$\Gamma\left(\frac{p}{\alpha}\right)G\left(\frac{p}{\alpha}+1\right) = \alpha F(p),$$

or equivalently

(2.13) $$\Gamma(p)G(p+1) = \alpha F(\alpha p).$$

Also, by applying the inverse of Mellin transform and using the convolution property we obtain

(2.14) $$\int_0^\infty e^{-\frac{y}{u}}g(u)du = f(y^{\frac{1}{\alpha}}).$$

The above equation implies that the function $g$ can be obtained in terms of the inverse Laplace transform as follows

$$g(y) = \frac{1}{y^2}\mathfrak{L}^{-1}\left\{f(u^{\frac{1}{\alpha}}); \frac{1}{y}\right\}. \tag{2.15}$$

## 3. Inversion Techniques for Some Integral Transforms

Similar to the procedures in previous section for solving singular integral equations, in this section we find new inversion formulas for the Wright transform and Mittag-Leffler transform. These integral transforms have been recently arisen in fractional calculus [10], and it is necessary to have inversion techniques for them.

### 3.1. The Wright Transform

For the following Wright transform [10]

$$\int_0^\infty W(-\gamma, 0; -\frac{1}{y}x^{-\gamma})g(y)dy = f(x), \quad x > 0,\ 0 < \gamma < 1 \tag{3.1}$$

where the Wright function is presented by the following relation

$$W(\alpha, \beta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\ \Gamma(\alpha k + \beta)}, \quad \alpha > -1,\ \beta \in \mathbb{C},\ z \in \mathbb{C}, \tag{3.2}$$

we show an inversion formula for the function $g(y)$. At first, by applying the Laplace transform on both sides of equation with respect to $x$

$$\mathfrak{L}\left\{\int_0^\infty W(-\gamma, 0; -\frac{1}{y}x^{-\gamma})g(y)dy; s\right\} = \mathfrak{L}\{f(x); s\}, \quad x > 0. \tag{3.3}$$

and using the fact that (see [2])

$$e^{-\frac{1}{y}s^\gamma} = \int_0^\infty e^{-sx}\frac{1}{x}W(-\gamma, 0; -\frac{1}{y}x^{-\gamma})dx \tag{3.4}$$

we get

$$\int_0^\infty e^{-\frac{1}{y}s^\gamma} g(y)dy = \Phi(s), \tag{3.5}$$

where $\Phi$ is the Laplace transform of the function

$$\phi(x) = \frac{f(x)}{x}.$$

Now, by considering the singular integral equation with exponential kernel (2.11) and its solution we obtain the following relation for the inverse function $g$

$$g(y) = \frac{1}{y^2}\mathfrak{L}^{-1}\left\{\Phi(s^{\frac{1}{\gamma}}); \frac{1}{y}\right\} = \frac{1}{y^2}\mathfrak{L}^{-1}\left\{\mathfrak{L}\left\{\frac{f(x)}{x}; s^{\frac{1}{\gamma}}\right\}; \frac{1}{y}\right\}, \tag{3.6}$$

provided that the integrals involved converge absolutely.

### 3.2. The Mittag-Leffler Transform

If we consider the Mittag-Leffler transform [10]

$$\int_0^\infty E_\alpha(-ys^\alpha)g(y)dy = f(s), \quad 0<\alpha<1. \tag{3.7}$$

where the Mittag-Leffler function is given by the following relation

$$E_\alpha(y) = \sum_{n=0}^{\infty}\frac{y^n}{\Gamma(\alpha n+1)}, \tag{3.8}$$

then by applying the inverse Laplace transform on both sides of equation with respect to $s$ and using the fact that [2]

$$\mathcal{L}^{-1}\{E_\alpha(-ys^\alpha);r\} = \frac{1}{\pi}\frac{yr^{\alpha-1}\sin(\alpha\pi)}{y^2+2yr^\alpha\cos(\alpha\pi)+r^{2\alpha}}, \tag{3.9}$$

we get a transformed equation in the following form

$$\frac{1}{\pi}\int_0^\infty \frac{yr^{\alpha-1}\sin(\alpha\pi)}{y^2+2yr^\alpha\cos(\alpha\pi)+r^{2\alpha}}g(y)dy = \phi(r), \quad 0<\alpha<1, \tag{3.10}$$

where $\phi(r)$ is the inverse Laplace transform of $f(s)$.

According to the relation A4 in Appendix 2, the integral equation (3.10) is coincided with the generalized product theorem. Therefore, by applying the Mellin transform on the above equation we get

$$\frac{1}{\alpha}G\left(\frac{p}{\alpha}+1\right)\csc\left(\pi\frac{p}{\alpha}\right)\sin(\pi p) = \mathcal{M}\left\{\frac{\phi(r)}{r};p\right\},$$

or equivalently

$$G(p+1)\csc(\pi p)\sin(\alpha\pi p) = \alpha\mathcal{M}\left\{\frac{\phi(r)}{r};\alpha p\right\}.$$

At this point, by using the inverse Mellin transform and using the convolution and the change of scale properties, we get the function $g$ as follows

$$g(y) = \frac{1}{y}\int_0^\infty \psi\left(\frac{y}{u}\right)\frac{\phi\left(u^{\frac{1}{\alpha}}\right)}{u^{\frac{1}{\alpha}+1}}du, \tag{3.11}$$

provided that the above integral converges absolutely. Also, the function $\psi$ is given by

$$\psi(x) = \mathcal{M}^{-1}\left\{\frac{\sin(\pi p)}{\sin(\alpha\pi p)};x\right\} = \sum_{n=1}^{\infty}(-1)^n x^{\frac{n}{\alpha\pi}}\sin\left(\frac{n}{\alpha}\right). \tag{3.12}$$

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[2] This relation can be obtained in view of the Titchmarsh theorem

$$f(r) = \frac{1}{\pi}\int_0^\infty e^{-sr}\Im\{F(se^{i\pi})\}ds$$

for inverses of the Laplace transform of functions which have branch cut on the real negative semiaxis, see [3].

## 4. Kernels of $n$-th Iterates of the Laplace transform and the $\mathcal{L}_2$-Transform

In this section, by using the generalized product theorem for the Mellin transform, we find kernels for $n$-th iterates of the Laplace transform and the $\mathcal{L}_2$-transform. Srivastava and Yurekli [12, 14] showed that the second iterates of the Laplace transform and the $\mathcal{L}_2$-transform are the Stieltjes and the Widder potential transforms respectively, and Brown et al. [4, 5] introduced that the third iterates of the Laplace transform and the $\mathcal{L}_2$-transform are as the exponential and the $\mathcal{E}_{2,1}$ -transforms respectively. These integral transforms have been shown in the second and third rows of Table 1 and Table 2 in terms of the exponential integral function $E_1(x)$ defined as

$$E_1(x) = -\mathrm{Ei}(-x) = \int_x^\infty \frac{e^{-u}}{u} du. \tag{4.1}$$

Now, for finding the $n$-th iterates, $(n \geq 4)$, of the Laplace transform and the $\mathcal{L}_2$-transform, we state the following theorem.

**Theorem 4.1.** *The Mellin transform of the kernel of $n$-th iterates of the Laplace transform and the $\mathcal{L}_2$-transform satisfies the following relation*

$$\mathcal{M}\{k_n(x,y);p\} = \psi_{n,1}(p) y^{\psi_{n,2}(p)-1} \tag{4.2}$$

*where $\psi_{n,1}(p)$ belongs to $\mathcal{K}$-class functions and $\psi_{n,2}(p)$ is an analytic function.*

*Proof.* By using the following relations which are easily obtained by definitions of the Laplace, Mellin and the $\mathcal{L}_2$-transform [15]

$$\mathcal{M}\{\mathcal{L}\{k_n(x,y);s\};p\} = \Gamma(p)\mathcal{M}\{k_n(x,y);1-p\}, \tag{4.3}$$

$$\mathcal{M}\{\mathcal{L}_2\{k_n(x,y);s\};p\} = \frac{1}{2}\Gamma(\frac{p}{2})\mathcal{M}\{k_n(x,y);2-p\}, \tag{4.4}$$

and considering the relations A5, A6 in Appendix 2 as first iterates of the Laplace transform and the $\mathcal{L}_2$-transform we see that in each iterate the Mellin transform of kernels satisfies the relation (4.2). □

Table 1 and Table 2 show the kernels of $n$-th iterates of the Laplace transform and the $\mathcal{L}_2$-transform and their Mellin transform. The relations A7- A10 in Appendix 2 have been used to complete the third columns of Table 1 and Table 2.

Also, by using the generalized product theorem we can obtain new inversion techniques for $n$-th iterate of the $\mathcal{L}_2$-transform. Since, the Mellin transforms of kernels $k_{2n}(x,y)$ and $k_{2n+1}(x,y)$ satisfy the conditions of generalized product theorem

$$\mathcal{M}\{k_{2n}(x,y);p\} = \pi^n \csc^n(\pi p) y^{p-1}, \tag{4.5}$$

$$\mathcal{M}\{k_{2n+1}(x,y);p\} = \pi^n \csc^n(\pi p)\Gamma(p) y^{-p}, \tag{4.6}$$

we can easily see that by implementation of relation (1.9) the inverse function $g(y)$ is written as $n$-th iterate of imaginary part of the inverse Mellin transform. These results have been shown in Table 3.

| The $n$-th iterate | The integral Transform | The Mellin Transform of kernel |
|---|---|---|
| The first iterate | $\int_0^\infty e^{-xy} g(y)dy$ | $\Gamma(p)y^{-p}$ |
| The second iterate | $\int_0^\infty \frac{1}{y+x} g(y)dy$ | $\pi\csc(\pi p)y^{p-1}$ |
| The third iterate | $\int_0^\infty e^{xy} E_1(xy)g(y)dy$ | $\pi\Gamma(p)\csc(\pi p)y^{-p}$ |
| The fourth iterate | $\int_0^\infty \frac{\ln(\frac{x}{y})}{x-y} g(y)dy$ | $\pi^2\csc^2(\pi p)y^{p-1}$ |
| The fifth iterate | $\int_0^\infty k_5(x,y)g(y)dy$ | $\pi^2\csc^2(\pi p)\Gamma(p)y^{-p}$ |
| The sixth iterate | $\int_0^\infty k_6(x,y)g(y)dy$ | $\pi^3\csc^3(\pi p)y^{p-1}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| The $2n$-th iterate | $\int_0^\infty k_{2n}(x,y)g(y)dy$ | $\pi^n\csc^n(\pi p)y^{p-1}$ |
| The $2n+1$-th iterate | $\int_0^\infty k_{2n+1}(x,y)g(y)dy$ | $\pi^n\csc^n(\pi p)\Gamma(p)y^{-p}$ |

TABLE 1. The $n$-th iterate of the Laplace transform

| The $n$-th iterate | The integral Transform | The Mellin Transform of kernel |
|---|---|---|
| The first iterate | $\int_0^\infty ye^{-x^2y^2} g(y)dy$ | $\frac{1}{2}y^{1-p}\Gamma(\frac{p}{2})$ |
| The second iterate | $\frac{1}{2}\int_0^\infty \frac{y}{y^2+x^2} g(y)dy$ | $\frac{\pi}{4}y^{p-1}\csc(\frac{\pi p}{2})$ |
| The third iterate | $\frac{1}{4}\int_0^\infty ye^{x^2y^2} E_1(x^2y^2)g(y)dy$ | $\frac{\pi}{8}\Gamma(\frac{p}{2})y^{1-p}\csc(\frac{\pi p}{2})$ |
| The fourth iterate | $\frac{1}{8}\int_0^\infty \frac{y\ln(\frac{x^2}{y^2})}{x^2-y^2} g(y)dy$ | $\frac{\pi^2}{16}\csc^2(\frac{\pi p}{2})y^{p-1}$ |
| The fifth iterate | $\int_0^\infty k_5(x,y)g(y)dy$ | $\frac{\pi^2}{32}\Gamma(\frac{p}{2})\csc^2(\frac{\pi p}{2})y^{1-p}$ |
| The sixth iterate | $\int_0^\infty k_6(x,y)g(y)dy$ | $\frac{\pi^3}{64}\csc^3(\frac{\pi p}{2})y^{p-1}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| The $2n$-th iterate | $\int_0^\infty k_{2n}(x,y)g(y)dy$ | $(\frac{\pi}{4})^n\csc^n(\frac{\pi p}{2})y^{p-1}$ |
| The $2n+1$-th iterate | $\int_0^\infty k_{2n+1}(x,y)g(y)dy$ | $\frac{1}{2}(\frac{\pi}{4})^n\Gamma(\frac{p}{2})\csc^n(\frac{\pi p}{2})y^{1-p}$ |

TABLE 2. The $n$-th iterate of the $\mathcal{L}_2$-transform

| The $n$-th iterate | The integral Transform | The inverse function |
|---|---|---|
| The first iterate | $f(x)=\int_0^\infty ye^{-x^2y^2}g(y)dy$ | $g(y)=\mathcal{M}^{-1}\{\frac{F(2-p)}{\Gamma(1-\frac{p}{2})};y\}$ |
| The second iterate | $f(x)=\frac{1}{2}\int_0^\infty \frac{y}{y^2+x^2}g(y)dy$ | $g(y)=-\frac{4}{\pi}\Im[f(iy)]$ |
| The third iterate | $f(x)=\frac{1}{4}\int_0^\infty ye^{x^2y^2}E_1(x^2y^2)g(y)dy$ | $g(y)=-\frac{8}{\pi}\Im\mathcal{M}^{-1}\{\frac{F(2-p)}{\Gamma(1-\frac{p}{2})};iy\}$ |
| The fourth iterate | $f(x)=\frac{1}{8}\int_0^\infty \frac{y\ln(\frac{x^2}{y^2})}{x^2-y^2}g(y)dy$ | $g(y)=\frac{16}{\pi^2}\Im\mathcal{M}^{-1}\{\Im\mathcal{M}^{-1}\{F(p);ir\};iy\}$ |
| The fifth iterate | $f(x)=\int_0^\infty k_5(x,y)g(y)dy$ | $g(y)=\frac{32}{\pi^2}\Im\mathcal{M}^{-1}\{\Im\mathcal{M}^{-1}\{\frac{F(2-p)}{\Gamma(1-\frac{p}{2})};ir\};iy\}$ |
| The sixth iterate | $f(x)=\int_0^\infty k_6(x,y)g(y)dy$ | $g(y)=-\frac{64}{\pi^3}\Im\mathcal{M}^{-1}\{\Im\mathcal{M}^{-1}\{\Im\mathcal{M}^{-1}\{F(p);ir\};iu\};iy\}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| The $2n$-th iterate | $f(x)=\int_0^\infty k_{2n}(x,y)g(y)dy$ | $g(y)=\frac{(-4)^n}{\pi^n}(\Im\mathcal{M}^{-1})^n\{F(p);iy\}$ |
| The $2n+1$-th iterate | $f(x)=\int_0^\infty k_{2n+1}(x,y)g(y)dy$ | $g(y)=\frac{(-2)^{2n+1}}{\pi^n}(\Im\mathcal{M}^{-1})^n\{\frac{F(2-p)}{\Gamma(1-\frac{p}{2})};iy\}$ |

TABLE 3. The inverse function of $n$-th iterate of the $\mathcal{L}_2$-transform

*Remark* 4.1. The second row of the Table 3 implies that for the Widder potential transform

$$\mathcal{P}\{g(y); p\} = f(x) = \int_0^\infty \frac{y}{y^2+x^2} g(y)dy \tag{4.7}$$

a very simple inversion formula can be written as the imaginary part of function $-\frac{2}{\pi} f(iy)$.

## 5. Conclusions

In this paper we provided new results in operational calculus for the Mellin transform. These results are derived from the generalized product theorem. New inversion formulas for the Wright and the Mittag-Leffler transforms were obtained. These formulas may be considered as promising approaches in expressing the Wright and the Mittag-Leffler functions in fractional calculus.
Also, new inversion technique for $n$-th iterate of the $\mathcal{L}_2$-transform was written and a simple inversion formula was derived for the Widder potential transform as second iterate of the $\mathcal{L}_2$-transform.

## Appendix.

### Appendix 1. The definition of the $\mathcal{K}$-class functions

A complex variable function $F(p)$ is said to belong to the $\mathcal{K}$-class functions if it is regular in the infinite strip $\mathcal{S} = \{s = \sigma + i\tau : c_1 < \sigma < c_2\}$ and for any arbitrary $\epsilon > 0$, $F(p)$ tends to zero uniformly as $| \tau | \to \infty$ in the strip $c_1 + \epsilon \le \sigma \le c_2 + \epsilon$. Also, the integral $\int_{-\infty}^{\infty} F(\sigma + i\tau)d\tau$ is absolutely convergent for each value of $\sigma$ in the open internal $(c_1, c_2)$.

### Appendix 2.

The Mellin transform of the $\mathcal{K}$-class functions coincides with the generalized product theorem [15]

**A1):** $\mathcal{M}\left\{\ln|\frac{x}{y} - 1|; p\right\} = \frac{\pi}{p} y^p \cot(\pi p), \quad -1 < \Re p < 0,$

**A2):** $\mathcal{M}\left\{\sin^{-1}(y\sqrt{y^2+x^2}); p\right\} = \frac{1}{p} y^p \sin\left(\frac{\pi p}{2}\right), \quad 0 < \Re p < 1,$

**A3):** $\mathcal{M}\left\{e^{-\frac{1}{y}x^\alpha}; p\right\} = \frac{1}{\alpha} y^{\frac{p}{\alpha}} \Gamma\left(\frac{p}{\alpha}\right), \quad \Re p > 0,$

**A4):** $\mathcal{M}\left\{\frac{yx\sin(\alpha\pi)}{y^2 + 2yx\cos(\alpha\pi) + x^2}; p\right\} = \pi y^p \csc(\pi p)\sin(\alpha\pi p), \quad -1 < \Re p < 1,$

**A5):** $\mathcal{M}\{e^{-xy}; p\} = \Gamma(p) y^{-p}, \quad 0 < \Re p,$

**A6):** $\mathcal{M}\{y e^{-y^2x^2}; p\} = \frac{1}{2} y^{1-p} \Gamma\left(\frac{p}{2}\right), \quad \Re p > 0,$

**A7):** $\mathcal{M}\left\{\frac{1}{y+x}; p\right\} = \pi y^{p-1}\csc(\pi p), \quad 0 < \Re p < 1,$

**A8):** $\mathcal{M}\left\{\frac{y}{y^2+x^2}; p\right\} = \frac{\pi}{2} y^{p-1} \csc\left(\frac{\pi p}{2}\right), \quad 0 < \Re p < 2,$

**A9):** $\mathcal{M}\{e^{xy} E_1(xy); p\} = \pi\Gamma(p) y^{-p}\csc(\pi p), \quad 0 < \Re p < 1,$

**A10):** $\mathcal{M}\{ye^{x^2y^2}E_1(x^2y^2);p\} = \frac{\pi}{2}\Gamma\left(\frac{p}{2}\right)y^{1-p}\csc\left(\frac{\pi p}{2}\right), \quad 0 < \Re p < 2.$

The Mellin transform of other functions used in this note

**A11):** $\mathcal{M}\left\{\ln(|\frac{1+\sqrt{x}}{1-\sqrt{x}}|);p\right\} = \pi\frac{\tan(\pi p)}{p}, \quad -1 < \Re p < 1,$

**A12):** $\mathcal{M}\left\{\frac{2}{\pi}\frac{1}{1+\sqrt{x}};p\right\} = \csc(\frac{\pi p}{2}), \quad 0 < \Re p < 1.$

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