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POTENTIALLY GRAPHIC SEQUENCES OF SPLIT GRAPHS

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Abstract. A sequence $\pi = (d_1, d_2, \ldots, d_n)$ of non-negative integers is said to be graphic if it is the degree sequence of a simple G on n vertices, and such a graph G is referred to as a realization of π. The set of all non-increasing non-negative integer sequences $\pi = (d_1, d_2, \ldots, d_n)$ is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be graphic if it is the degree sequence of a graph G on n vertices, and such a graph G is called a realization of π. The set of all graphic sequences in NS_n is denoted by GS_n. A split graph $K_r + \overline{K_s}$ on $r + s$ vertices is denoted by $S_{r,s}$. A graphic sequence π is potentially H-graphic if there is a realizaton of π containing H as a subgraph. In this paper, we determine the graphic sequences of subgraphs H, where H is $S_{r_1,s_1} + S_{r_2,s_2} + S_{r_3,s_3} + \ldots + S_{r_m,s_m}$, $S_{r_1,s_1} \vee S_{r_2,s_2} \vee \ldots \vee S_{r_m,s_m}$ and $S_{r_1,s_1} \times S_{r_2,s_2} \times \ldots \times S_{r_m,s_m}$ and $+$, V and $\times$ denotes the standard join operation, the normal join operation and the cartesian product in these graphs respectively.

1. Introduction

Let G be an undirected simple graph (graph without multiple edges and loops) with n vertices and m edges having vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Any undefined notations follows that of Bondy and Murty [1]. Throughout the paper, we denote such a graph by $G(n, m)$. The set of all non-increasing non-negative integer sequences $\pi = (d_1, d_2, \ldots, d_n)$ is denoted by NS_n. There are several famous results, Havel and Hakimi [5, 6] and Erdös and Gallai [3] which give necessary and sufficient conditions for a sequence $\pi = (d_1, d_2, \ldots, d_n)$ to be the degree sequence of a simple graph G. Unfortunately, knowing that a sequence has a realization gives no information about the properties that such a graph might have. In this paper, we explore this question of properties of a graph which is related to work originally introduced by A. R. Rao [9]. A sequence $\pi \in NS_n$ is said to be graphic if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a realization of π. The sequence

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$\pi = (d_1, d_2, \ldots, d_n)$ is graphic if and only if the sequence π' obtained from π by laying off an element is graphic [7]. Also $d^{r_1 \times r_2}$ means d occurs $r_1 \times r_2$ times in π. The set of all graphic sequences in NS_n is denoted by GS_n. A graphic sequence π is potentially H-graphic if there is a realization of π containing H as a subgraph, while π is forcibly H graphic if every realization of π contains H as a subgraph. If π has a realization in which the $r+1$ vertices of largest degree induce a clique, then π is said to be potentially A_{r+1}-graphic. The graphic sequence π is potentially K_{k+1}-graphic if and only if π is potentially A_{k+1}-graphic [10]. Let $\sigma(\pi) = d_1 + d_2 + \ldots + d_n$. If G and G_1 are graphs, then $G \bigcup G_1$ is the disjoint union of G and G_1. If $G = G_1$, we abbreviate $G \bigcup G_1$ as $2G$. We denote $G + H$ as the graph with $V(G+H) = V(G) \bigcup V(H)$ and $E(G+H) = E(G) \bigcup E(H) \bigcup \{xy : x \in V(G), y \in V(H)\}$. Let K_k, C_k, T_k and P_k respectively denote a complete graph on k vertices, a cycle on k vertices, a tree on $k+1$ vertices and a path on $k+1$ vertices. Let F_k denote the friendship graph on $2k+1$ vertices, that is, the graph of k triangles intersecting in a single vertex. For $0 \leq r \leq t$, denote the generalized friendship graph on $kt - kr + r$ vertices by $F_{t,r,k}$, where $F_{t,r,k}$ is the graph of k copies of K_t meeting in a common r set.

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as subgraph? This number is denoted by $ex(n, H)$, and is known as the Turan number. In terms of graphic sequences, the number $2ex(n, H) + 2$ is the minimum even integer l such that every n-term graphic sequence π with $\sigma(\pi) \geq l$ is forcibly H-graphic. Erdös, Jacobson and Lehel [2] first considered the following variant: determine the minimum even integer l such that every n-term graphic sequence π with $\sigma(\pi) \geq l$ is potentially H-graphic. We denote this minimum l by $\sigma(H, n)$. A sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be potentially K_{r+1}-graphic if there is a realization G of π containing K_{r+1} as a subgraph. If π is a graphic sequence with a realization G containing H as a subgraph, then there is a realization $G^{\prime}$ of π containing H with the vertices of H having $|V(H)|$ largest degree of π [4]. Let $S_{r,s} = K_r + \overline{K_s}$ be split graph on $r+s$ vertices, where $\overline{K_s}$ is the complement of K_s and $+$ denotes the standard join operation. As seen in [11], $S_{r,1} = K_{r+1}$ and so the graph $S_{r,s}$ is an extension of the graph K_{r+1}. A sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be potentially $S_{r,s}$-graphic if there is a realization G of π containing $S_{r,s}$ as a subgraph. Yin Jain Hua and Haikou [11] obtained a Havel-Hakimi type procedure and a simple sufficient condition for π to be potentially $S_{r,s}$-graphic. We have the following definitions.

Definition 1.1. [8] For the graphs G_1, G_2 with disjoint vertex set $V(G_1)$, $V(G_2)$ the cartesian product is a graph $G = G_1 \times G_2$ with vertex set $V(G_1) \times V(G_2)$ and an edge $((u_1, v_1), (u_2, v_2))$ iff $u_1 = u_2$ and $v_1 v_2$ is an edge of G_2.

Definition 1.2. [11] The standard join of S_{r_1,s_1}, S_{r_2,s_2} is a graph $S = S_{r_1,s_1} \vee S_{r_2,s_2}$ with vertex set $V(S_{r_1,s_1}) \cup V(S_{r_2,s_2})$ and an edge set consisting of all edges of S_{r_1,s_1} and S_{r_2,s_2} together with the edges joining each vertex of K_{r_1} of S_{r_1,s_1} with every vertex of S_{r_2,s_2} and s_1 vertices of S_{r_1,s_1} are joined with only vertices of K_{r_2} in S_{r_2,s_2}.

Definition 1.3. [8] The join (complete product) of G_1 and G_2 is a graph $G = G_1 \vee G_2$ with vertex set $V(G_1) \cup V(G_2)$ and an edge set consisting of all edges of G_1 and G_2 together with the edges joining each vertex of G_1 with every vertex of G_2.

Definition 1.4. [9] The split graph $K_r + \overline{K_s}$ on $r+s$ vertices is denoted by $S_{r,s}$ where $+$ denotes the standard join operation and $\overline{K_s}$ is the complement of K_s. A non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of non-negative integers is said to be potentially $S_{r,s}$-graphic if there exists a realization G of π containing $S_{r,s}$ as a subgraph.

Definition 1.5. If π has a realization G containing K_{r+1} on those vertices having degree $d_1, d_2, \ldots, d_{r+1}$, then π is potentially A_{r+1}-graphic.

Definition 1.6. [10] The tensor product (conjuction), denoted by $G = G_1 \wedge G_2$, is the graph with vertex set $V = V_1 \times V_2$ and for any two vertices $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ in V; $u_1, u_2 \in V_1$ and $v_1, v_2 \in V_2$, there is an edge $w_1 w_2 \in E(G)$ if and only if $u_1 u_2 \in E_1$ and $v_1 v_2 \in E_2$.

2. Main Results

We start with the following result.

Theorem 2.1. *If $\pi_1 = (d_1^1, d_2^1, \ldots, d_m^1)$ is potentially K_{p_1}-graphic and $\pi_2 = (d_1^2, d_2^2, \ldots, d_n^2)$ is potentially K_{p_2}-graphic, $p_1 \leq m$ and $p_2 \leq n$, then the graphic sequence π of $G = G_1 \times G_2$ is potentially $p_1 + p_2 - 2$ regular graphic.*

Proof. Let $\pi_1 = (d_1^1, d_2^1, \ldots, d_m^1)$ and $\pi_2 = (d_1^2, d_2^2, \ldots, d_n^2)$ be respectively K_{p_1}-graphic and K_{p_2}-graphic. Then there exists graphs G_1 and G_2 respectively realizing π_1 and π_2 and respectively containing K_{p_1} and K_{p_2} as subgraphs. Let $G = G_1 \times G_2$ be the cartesian product of G_1 and G_2 and let $\pi_3 = (d_{11}, d_{12}, \ldots, d_{1m}, d_{21}, d_{22}, \ldots, d_{2m}, \ldots, d_{m1}, d_{m2}, \ldots, d_{mn})$ be the graphic sequence of $G_1 \times G_2$. Then $d_{ij} = d_i^1 + d_j^2$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ where d_{ij} is the degree of ij^{th} vertex in G. We have to show that the realization G of π contains $p_1 + p_2 - 2$ as a regular subgraph. To prove this, it is enough to show that sum of degrees of this subgraph is equal to $p_1 p_2 (p_1 + p_2 - 2)$. Clearly,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} d_i^1 + d_j^2 = (d_1^1 + d_1^2) + \ldots + (d_1^1 + d_n^2) + \ldots + (d_m^1 + d_n^2).$$

This is true for all m and n. In particular, it holds for $m = p_1$ and $n = p_2$. Therefore

$$\begin{aligned}\sum_{i=1}^{p_1} \sum_{j=1}^{p_2} d_{ij} &= (d_1^1 + d_1^2) + (d_1^1 + d_2^2) + \ldots + (d_1^1 + d_{p_2}^2) + \ldots + (d_{p_1}^1 + d_{p_2}^2) \\ &= (p_1 - 1 + p_2 - 1) + (p_1 - 1 + p_2 - 1) + \ldots + (p_1 - 1 + p_2 - 1) \\ &\quad + \ldots + (p_1 - 1 + p_2 - 1) \\ &= p_1 p_2 (p_1 + p_2 - 2).\end{aligned}$$

□

Theorem 2.1 can be generalized as follows.

Theorem 2.2. *If* $\pi_i = (d_1^i, d_2^i, \cdots, d_{n_j}^i)$ *is potentially* K_{p_i}*-graphic for* $i, j = 1, 2, \ldots, r$ *with* $p_i \leq n_j$*, then the graphic sequence* π *of* $G = G_1 \times G_2 \times \ldots \times G_r$ *is a potentially* $\sum_{i=1}^{r} p_i - r$ *regular graphic.*

Proof. The proof follows by induction on r. □

Theorem 2.3. *If* $\pi_1 = (d_1^1, d_2^1, \ldots, d_m^1)$ *is potentially* K_{p_1}*-graphic and* $\pi_2 = (d_1^2, d_2^2, \ldots, d_n^2)$ *is potentially* K_{p_2}*-graphic,* $p_1 \leq m$ *and* $p_2 \leq n$*, then the graphic sequence* π *of* $G = G_1 + G_2$ *is potentially* $K_{p_1+p_2}$*-graphic.*

Proof. Let $\pi_1 = (d_1^1, d_2^1, \ldots, d_m^1)$ be potentially K_{p_1}-graphic. Then there exists a graph G_1 which realizes π_1 and will contain K_{p_1} as a subgraph. Let $\pi_2 = (d_1^2, d_2^2, \ldots, d_n^2)$ be potentially $K_{p_2}-$ graphic, so there exists a graph G_2 which realizes π_2 and will contain K_{p_2} as a subgraph. Let $G = G_1 + G_2$ be the join of G_1 and G_2 and let $\pi = (d_1, d_2, \ldots, d_{m+n})$ be the graphic sequence of $G = G_1 + G_2$. Then we have

$$\begin{aligned} d_i &= d_i^1 + n \text{ for } i = 1, 2, \ldots, m \\ d_{m+j} &= d_j^2 + m \text{ for } j = 1, 2, \ldots, n. \end{aligned} \tag{2.1}$$

We have to show that the realization of π contains $K_{p_1+p_2}$ as a subgraph. To prove this it is enough to show that

$$\sum_{i=1}^{p_1} d_i + \sum_{j=1}^{p_2} d_{m+j} = (p_1 + p_2)(p_1 + p_2 - 1).$$

We take the summation to the equations in (2.1) respectively from $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ and get $\sum_{i=1}^{m} d_i = \sum_{i=1}^{m} d_i^1 + \sum_{i=1}^{m} n$ and $\sum_{j=1}^{m} d_{m+j} = \sum_{j=1}^{n} d_j^2 + \sum_{j=1}^{n} m$. These two equations imply

$$\sum_{i=1}^{m} d_i = \sum_{i=1}^{m} d_i^1 + mn \tag{2.2}$$

and

$$\sum_{j=1}^{m} d_{m+j} = \sum_{j=1}^{n} d_j^2 + nm. \tag{2.3}$$

As (2.2) and (2.3) is true for all m and n, therefore, in particular it is true for $m = p_1$ and $n = p_2$. So,

$$\begin{aligned}\sum_{i=1}^{p_1} d_i + \sum_{i=1}^{p_2} d_{m+j} &= \sum_{i=1}^{p_1} d_i^1 + \sum_{i=1}^{p_2} d_{m+j}^2 + 2p_1p_2 \\ &= d_1^1 + d_2^1 + \ldots + d_{p_1}^1 + d_1^2 + d_2^2 + \ldots + d_{p_2}^2 + 2p_1p_2 \\ &= (p_1 - 1) + \ldots + (p_1 - 1) + (p_2 - 1) + \ldots + (p_2 - 1) + 2p_1p_2 \\ &= p_1(p_1 - 1) + p_2(p_2 - 1) + p_1p_2 + p_1p_2 \\ &= p_1(p_1 + p_2 - 1) + p_2(p_2 + p_1 - 1) \\ &= (p_1 + p_2)(p_1 + p_2 - 1).\end{aligned}$$

□

Theorem 2.3 can be generalized as follows.

Theorem 2.4. *If $\pi_i = (d_1^i, d_2^i, \ldots, d_{n_j}^i)$ is potentially K_{p_i}-graphic for $i = 1, 2, \ldots, r$ with $p_i \leq n_j$. Then the graphic sequence π of $G = G_1 + G_2 + \ldots + G_r$ is potentially $K_{\sum_{i=1}^{r} p_i}$-graphic.*

Proof. This can be proved by induction on r. □

Theorem 2.5. *If π_i is potentially S_{r_i,s_i}-graphic for $i = 1, 2, \ldots, m$, then*

(1) *The graphic sequence π of $G = G_1 + G_2 + \ldots + G_m$ is potentially $S_{\sum_{i=1}^{m} r_i, \sum_{i=1}^{m} s_i}$-graphic, where $+$ denotes the standard join operation in S_{r_i,s_i}.*

(2) *The graphic sequence of $S_{\sum_{i=1}^{m} r_i, \sum_{i=1}^{m} s_i}$ for $j = 1, 2, \ldots, m$ is*

$$\pi' = \left(\left(\sum_{i=1}^{m} (r_i + s_i - 1) \right)^{r_j}, \left(\sum_{i=1}^{m} r_i \right)^{s_j} \right),$$

(3) *Also,* $\sigma(\pi') = \left(\sum_{i=1}^{m} r_i \right)^2 + 2 \left(\sum_{i=1}^{m} r_i \right) \left(\sum_{i=1}^{m} s_i \right) - \left(\sum_{i=1}^{m} r_i \right).$

Proof. Let π be potentially S_{r_i,s_i}-graphic for $i = 1, 2, \ldots, m$. Then there exists a graph G_i which realizes π_i and will contain S_{r_i,s_i} as a subgraph. Let $G = G_1 + G_2 + \ldots + G_m$ be the graph obtained from $G_1, G_2, \ldots, G_m$ by using join operation. Therefore, clearly the graphic sequence π of G is potentially $S_{\sum_{i=1}^{m} r_i, \sum_{i=1}^{m} s_i}$-graphic follows from Theorem 2.4. To prove part (2), we use induction on m. For $k = 1$, the result is obvious. For $k = 2$, we have $G = G_1 + G_2$. Therefore, in particular $S_{r_1+r_2, s_1+s_2} = S_{r_1,s_1} + S_{r_2,s_2}$. Now by Theorem 2.4, we have for every $i = 1, 2, \ldots, r_1$ and $i = 1, 2, 3, \ldots, r_2$ and $j = 1, 2, 3, \ldots, s_1$ and $j = 1, 2, 3, \ldots, s_2$

$$\overline{d_i} = d_i + r_2 + s_1 + s_2 \tag{2.4}$$

and

$$\overline{d_j} = r_1 + r_2, \tag{2.5}$$

where $\overline{d_i}$ and $\overline{d_j}$ are respectively the degree of $\overline{v_i^{th}}$ and $\overline{v_j^{th}}$ vertex in $S_{r_1+r_2,s_1+s_2}$ and d_i is the degree of i^{th} vertex in K_{r_1}. Equations (2.4) and (2.5) hold for every i, j. Thus

$$\pi^2 = \left(\left(r_1 + r_2 + s_1 + s_2 - 1\right)^{r_1}, \left(r_1 + r_2 + s_1 + s_2 - 1\right)^{r_2}, \left(r_1 + r_2\right)^{s_1}, \left(r_1 + r_2\right)^{s_2}\right)$$
$$= \left(\left(\sum_{i=1}^{2}(r_i + s_i)\right)^{r_j}, \left(\sum_{i=1}^{2} r_i\right)^{s_j}\right).$$

This shows that the result is true for $k = 2$. Assume that the result holds for $k = m-1$, therefore $\pi^{m-1} = \left(\left(\sum_{i=1}^{m-1}(r_i + s_i)\right)^{r_j}, \left(\sum_{i=1}^{m-1} r_i\right)^{s_j}\right)$, for $j = 1, 2, \ldots, m-1$. Now for $k = m$ we have that $G = S_{r_1,s_1} + S_{r_2,s_2} + \ldots + S_{r_{m-1},s_{m-1}} + S_{r_m,s_m} = A + S_{r_m,s_m}$, where $A = S_{r_1,s_1} + S_{r_2,s_2} + \ldots + S_{r_{m-1},s_{m-1}}$.

Since the result is proved for every $k = m-1$ and using the fact that the result is proved for each pair and since the result is already proved for $k = 2$, it follows by induction hypothesis that the result holds for $k = m$ also. That is,

$$\pi = \pi^m = \left(\left(\sum_{i=1}^{m}(r_i + s_i - 1)\right)^{r_j}, \left(\sum_{i=1}^{m} r_i\right)^{s_j}\right).$$

This proves part (2). To prove part (3), we have for $j = 1, 2, \ldots, m$ that

$$\sigma(\pi') = r_j\left(\sum_{i=1}^{m}(r_i + s_i - 1)\right) + s_j\left(\sum_{i=1}^{m} r_i\right)$$
$$= r_j\left(\sum_{i=1}^{m}(r_i + s_i)\right) - r_j + s_j\left(\sum_{i=1}^{m} r_i\right)$$
$$= \sum_{j=1}^{m} r_j\left(\sum_{i=1}^{m}(r_i + s_i)\right) - \sum_{j=1}^{m} r_j + \sum_{j=1}^{m} s_j \sum_{i=1}^{m} r_i$$
$$= \left(\sum_{i=1}^{m} r_i\right)^2 + 2\left(\sum_{i=1}^{m} r_i\right)\left(\sum_{i=1}^{m} s_i\right) - \left(\sum_{i=1}^{m} r_i\right).$$

□

Theorem 2.6. *If* $\pi_1 = (d_1^1, d_2^1, \ldots, d_m^1)$ *is potentially* S_{r_1,s_1}*-graphic and* $\pi_2 = (d_1^2, d_2^2, \ldots, d_n^2)$ *is potentially* S_{r_2,s_2}*-graphic. Then*

(1) $\pi_{s_1 \times s_2}$ *of* $S_1 \times S_2$ *is graphic,*

(2) *the graphic sequence of* $S_1 \times S_2$ *is* $\pi_{s_1 \times s_2} = (d_{ij}^{r_1 \times r_2}, d_{ij}^{r_1 \times s_2}, d_{ij}^{s_1 \times r_2}, d_{ij}^{s_1 \times s_2})$*, where* d_{ij} *is the degree of ijth vertex in* $S_1 \times S_2$*.*

Proof. Let $\pi_1 = (d_1^1, d_2^1, \ldots, d_m^1)$ be potentially S_{r_1,s_1}-graphic. Then there exists a graph G_1 which realizes π_1 and will contain S_{r_1,s_1} as a subgraph. Let $\pi_2 = (d_1^2, d_2^2, \ldots, d_n^2)$ be potentially S_{r_2,s_2}-graphic so that there exists a graph G_2 which realizes π_2 and will contain S_{r_2,s_2} as a subgraph. Let $G = G_1 \times G_2$ be the cartesian product of G_1 and G_2. Then we have $d_{ij} = d_i + d_j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. This relation is true for every vertex of the graph G, therefore it also holds for the graph $S = S_1 \times S_2$. Thus we can write $S = S_{r_1,s_1} \times S_{r_2,s_2}$. We have

$$d_{ij} = d_i + d_j \quad \text{for} \quad 1 \leq i \leq r_1 + s_1 \quad \text{and} \quad 1 \leq j \leq r_2 + s_2. \tag{2.6}$$

for $1 \leq i \leq r_1 + s_1$ and $1 \leq j \leq r_2 + s_2$.

If d_i is the degree of ith vertex of r_1 in S_{r_1,s_1} and d_j is the degree of jth vertex of r_2 in S_{r_2,s_2}, it can be seen by construction that degree of $ijth$ vertex of $r_1 \times r_2$ in S is d_{ij}, where d_{ij} is defined above and this term occurs in $r_1 \times r_2$ in $\pi_{S_1 \times S_2}$. Similarly other degree terms of the sequence occurs in $r_1 \times s_2$, $s_1 \times r_2$, $s_1 \times s_2$ by using definition of cartesian product of graphs. Thus $\pi_{s_1 \times s_2} = (d_{ij}^{r_1 \times r_2}, d_{ij}^{r_1 \times s_2}, d_{ij}^{s_1 \times r_2}, d_{ij}^{s_1 \times s_2})$. This completes the proof of the theorem. □

The following result is a generalization of Theorem 2.6 whose proof follows simply by induction.

Theorem 2.7. *If $\pi_i = (d_1^i, d_2^i, \ldots, d_{n_j}^i)$ is potentially S_{r_i,s_i}-graphic, then*

(1) *the sequence π of $G = S_{r_1,s_1} \times S_{r_2,s_2} \times \ldots \times S_{r_m,s_m}$ is graphic,*
(2) *the graphic sequence of π is $\pi_{s_1 \times s_2 \times \ldots \times s_m} = (d_{ijk\ldots m}^{r_1 \times r_2 \times \ldots \times r_m}, d_{ijk\ldots m}^{r_1 \times r_2 \times \ldots \times r_{m-1} \times s_m}, \ldots, d_{ijk\ldots m}^{s_1 \times s_2 \times s_3 \times \ldots \times s_{m-1} \times r_m}, \ldots, d_{ijk\ldots m}^{s_1 \times s_2 \times \ldots \times s_m})$, where $d_{ijk\ldots m} = d_i + d_j + d_k + \ldots + d_m$.*

Proof. This can be proved by induction on r. □

Theorem 2.8. *If π_i is potentially S_{r_i,s_i}-graphic for $i = 1, 2, \ldots, m$, then*

(1) *the graphic sequence π of $G = G_1 \vee G_2 \vee \ldots \vee G_m$ is potentially $S_{\sum_{i=1}^m r_i, \sum_{i=1}^m s_i}$-graphic, where $\vee$ denotes the join operation in $G_1, G_2, \ldots, G_n$,*
(2) *the graphic sequence of $S_{\sum_{i=1}^m r_i, \sum_{i=1}^m s_i}$ is*

$$\pi' = \left(\left(\sum_{i=1}^{m} (r_i + s_i - 1) \right)^{r_j}, \left(\sum_{i=1}^{m} r_i + \sum_{i=1, i \neq j}^{m} s_i \right)^{s_j} \right), \ for \ j = 1, 2, \ldots, m,$$

(3) *and* $\sigma(\pi') = \left(\sum_{i=1}^{m} r_i \right)^2 + 2 \sum_{i=1}^{m} r_i \sum_{j=1}^{m} s_j + \left(\sum_{i=1}^{m} s_i \right)^2 + \sum_{j=1}^{m} s_j \left(\sum_{i=1, i \neq j}^{m} s_i \right) - \sum_{i=1}^{m} r_i.$

Proof. Let π be potentially S_{r_i,s_i}-graphic for $i = 1, 2, \ldots, m$. Then there exists a graph G_i which realizes π_i and will contain S_{r_i,s_i} as a subgraph. Let $G = G_1 \vee G_2 \vee \ldots \vee G_m$ be the graph obtained from $G_1, G_2, \ldots, G_m$ by using join operation. Therefore, clearly the graphic sequence π of G is potentially $S_{\sum_{i=1}^m r_i, \sum_{i=1}^m s_i}$-graphic.

To prove part (2), we use induction on m. For $k = 1$, the result is obvious. For $k = 2$, we have $G = G_1 \vee G_2$, therefore, in particular we take the normal join operation between graphs S_{r_1,s_1} and S_{r_2,s_2}. Thus we have $S_{1,2} = S_{r_1,s_1} \vee S_{r_2,s_2}$. Now by Theorem 2.6, we have for every $i = 1, 2, \ldots, r_1$ and $i = 1, 2, 3, \ldots, r_2$ and $j = 1, 2, 3, \ldots, s_1$ and $j = 1, 2, 3, \ldots, s_2$

$$\overline{d_i} = d_i + r_2 + s_1 + s_2 \tag{2.7}$$

and

$$\overline{d_j} = r_1 + r_2 + s_2, \tag{2.8}$$

where $\overline{d_i}$ and $\overline{d_j}$ are respectively the degree of $\overline{v_i^{th}}$ and $\overline{v_j^{th}}$ vertex in $S_{r_1+r_2,s_1+s_2}$ and d_i is the degree of i^{th} vertex in K_{r_1}. Equations (2.7) and (2.8) hold for every i, j. Thus for $j = 1, 2$

$$\begin{aligned}\pi^2 &= \Bigg(\Big(r_1 + r_2 + s_1 + s_2 - 1\Big)^{r_1}, \Big(r_1 + r_2 + s_1 + s_2 - 1\Big)^{r_2}, \\ &\quad \Big(r_1 + r_2 + s_2\Big)^{s_1}, \Big(r_1 + r_2 + s_1\Big)^{s_2}\Bigg) \\ &= \Bigg(\Big(\sum_{i=1}^{2}(r_i + s_i - 1)\Big)^{r_j}, \Big(\sum_{i=1}^{2} r_i + \sum_{i=1, i\neq j}^{2} s_i\Big)^{r_j}\Bigg).\end{aligned}$$

This shows that the result is true for $k = 2$. Assume that the result holds for $k = m - 1$, therefore $\pi^{m-1} = \Bigg(\Big(\sum_{i=1}^{m-1}(r_i + s_i - 1)\Big)^{r_j}, \Big(\sum_{i=1}^{m-1} r_i + \sum_{i=1, i\neq j}^{m-1} s_i\Big)^{r_j}\Bigg)$, for all $j = 1, 2, ..., m - 1$. Now for $k = m$ we have that $G = S_{r_1,s_1} \vee S_{r_2,s_2} \vee \ldots \vee S_{r_{m-1},s_{m-1}} \vee S_{r_m,s_m} = A \vee S_{r_m,s_m}$, where $A = S_{r_1,s_1} \vee S_{r_2,s_2} \vee \ldots \vee S_{r_{m-1},s_{m-1}}$.

Since the result is proved for all $k = m - 1$ and using the fact that the result is proved for each pair and since the result is already proved for $k = 2$, it follows by induction hypothesis that result holds for $k = m$ also. That is,

$$\pi = \pi^m = \Bigg(\Big(\sum_{i=1}^{m}(r_i + s_i - 1)\Big)^{r_j}, \Big(\sum_{i=1}^{m} r_i + \sum_{i=1, i\neq j}^{m} s_i\Big)^{s_j}\Bigg).$$

This proves part (2). To prove part (3), we have for all $j = 1, 2, \ldots, m$

$$\begin{aligned}\sigma(\pi') &= r_j\Big(\sum_{i=1}^{m}(r_i + s_i - 1)\Big) + s_j\Big(\sum{}_{i=1}^{m} r_i + \sum_{i=1, i\neq j}^{m} s_i\Big) \\ &= r_j\Big(\sum_{i=1}^{m}(r_i + s_i)\Big) - r_j + s_j\Big(\sum_{i=1}^{m} r_i + \sum_{i=1, i\neq j}^{m} s_j\Big)\end{aligned}$$

$$= \left(\sum_{j=1}^{m} r_i\right)^2 + 2\sum_{i=1}^{m} r_i \sum_{i=1}^{m} s_i + \sum_{i=1}^{m} s_j \left(\sum_{i=1, i\neq j}^{m} s_i\right) - \sum_{i=1}^{m} r_i.$$

□

Remark 2.1. Let $\pi_1 = (d_1^1, d_2^1 \ldots, d_m^1)$ be potentially K_{p_1}-graphic $\pi_2 = (d_1^2, d_2^2 \cdots, d_m^2)$ be potentially K_{p_2} graphic. Then the graphic sequence π of $G = G_1 \wedge G_2$ is potentially H_p-graphic, where H_p is a p- regular graph and p depends upon p_1 and p_2. If $p_1 = 3$ and $p_2 = 2$, then π of $G = G_1 \wedge G_2$ is potentially H_2-graphic. If $p_1 = 3$ and $p_2 = 3$, then π of $G = G_1 \wedge G_2$ is potentially H_4-graphic. If $p_1 = 4$ and $p_2 = 4$, then π of $G = G_1 \wedge G_2$ is potentially H_9 graphic. If $p_1 = 3$ and $p_2 = 4$, then π of $G = G_1 \wedge G_2$ is potentially H_6-graphic. From this we conclude that p depends upon p_1 and p_2.

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