Abstract. In this paper we point out some inequalities of Hermite-Hadamard type for double integrals of functions whose partial derivatives of higher order are co-ordinated $s$-convex in the second sense. Our established results generalize the Hermite-Hadamard type inequalities established for co-ordinated $s$-convex functions and refine those results established for differentiable functions whose partial derivatives of higher order are co-ordinated convex proved in recent literature.

1. Introduction

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on $I$ if the inequality

$$f \left( \lambda x + (1 - \lambda) y \right) \leq \lambda f(x) + (1 - \lambda) f(y),$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. The inequality (1.1) holds in reverse direction if $f$ is concave.

The most famous inequality concerning the class of convex functions, is the Hermite-Hadamard’s inequality.

This double inequality is stated as

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}$$

where $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ a convex function, $a, b \in I$ with $a < b$. The inequalities in (1.2) are in reversed order if $f$ a concave function.

The inequalities (1.2) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a
variety of settings. Moreover, many inequalities of special means can be obtained for a particular choice of the function \( f \). Due to the rich geometrical significance of Hermite-Hadamard’s inequality (1.2), there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example [8, 14, 19, 29, 32, 33] and the references therein.

In the paper [15], Hudzik and Maligranda considered, among others, the class of functions which are \( s \)-convex in the second sense. This class is defined follows.

A function \( f : [0, \infty) \to \mathbb{R} \) is said to be \( s \)-convex in the second sense if
\[
f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)
\]
holds for all \( x, y \in [0, \infty) \), \( \lambda \in [0, 1] \) and for some fixed \( s \in (0, 1] \).

It can be easily seen that for \( s = 1 \), \( s \)-convexity reduces to ordinary convexity of functions defined on \([0, \infty)\).

In [9], Dragomir and Fitzpatrick proved a variant of Hadamard’s inequality which holds for \( s \)-convex functions in the second sense.

**Theorem 1.1.** [9] Suppose that \( f : [0, \infty) \to [0, \infty) \) is an \( s \)-convex function in the second sense, where \( s \in (0, 1) \) and \( a, b \in [0, \infty) \), \( a < b \). If \( f \in L^1[a,b] \), then the following inequalities hold
\[
2^{s-1} f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{s+1}.
\]
The constant \( k = \frac{1}{s+1} \) is the best possible in the second inequality in (1.3).

For more about properties and Hermite-Hadamard type inequalities of \( s \)-convex functions in the second sense we refer the interested readers to [7, 9, 12, 15, 20].

Let us consider now a bidimensional interval \( \Delta =: [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). A mapping \( f : \Delta \to \mathbb{R} \) is said to be convex on \( \Delta \) if the inequality
\[
f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda)f(z,w)
\]
holds for all \( (x,y), (z,w) \in \Delta \) and \( \lambda \in [0, 1] \).

A modification for convex functions on \( \Delta \), known as co-ordinated convex functions, was introduced by S. S. Dragomir [10] as follows.

A function \( f : \Delta \to \mathbb{R} \) is said to be convex on the co-ordinates on \( \Delta \) if the partial mappings \( f_y : [a,b] \to \mathbb{R}, f_y(u) = f(u,y) \) and \( f_x : [c,d] \to \mathbb{R}, f_x(v) = f(x,v) \) are convex where defined for all \( x \in [a,b], y \in [c,d] \).

A formal definition for co-ordinated convex functions may be stated as follow.

**Definition 1.1.** [21] A function \( f : \Delta \to \mathbb{R} \) is said to be convex on the co-ordinates on \( \Delta \) if the following inequality holds for all \( t, r \in [0, 1] \) and \( (x,u), (y,w) \in \Delta \)
\[
f(tx + (1-t)y, ru + (1-r)w) \leq tf(x,u) + t(1-r)f(x,w) + r(1-t)f(y,w) + (1-t)(1-r)f(y,u).
\]
Clearly, every convex mapping \( f : \Delta \to \mathbb{R} \) is convex on the co-ordinates but converse may not be true [10].

The following Hermite-Hadamard type inequalities for co-ordinated convex functions on the rectangle from the plane \( \mathbb{R}^2 \) were established in [10].

**Theorem 1.2.** [10] Suppose that \( f : \Delta \to \mathbb{R} \) is co-ordinated convex on \( \Delta \), then

\[
\begin{align*}
&f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \\
&\quad \leq \frac{1}{2} \left[ \frac{1}{b - a} \int_{a}^{b} f \left( \frac{x}{2}, \frac{c + d}{2} \right) \, dx + \frac{1}{d - c} \int_{c}^{d} f \left( \frac{a + b}{2}, y \right) \, dy \right] \\
&\quad \leq \frac{1}{(b - a)(d - c)} \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx \\
&\quad \leq \frac{1}{4} \left[ \frac{1}{b - a} \int_{a}^{b} \left[ f(x, c) + f(x, d) \right] \, dx \\
&\quad \quad + \frac{1}{d - c} \int_{c}^{d} \left[ f(a, y) + f(b, y) \right] \, dy \right] \\
&\quad \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\end{align*}
\]

The above inequalities are sharp.

The concept of s-convex functions on the co-ordinates in the second sense was introduced by Alomari and Darus in [3] as a generalization of the usual co-ordinated convexity.

**Definition 1.2.** [3] Consider the bidimensional interval \( \Delta = [a, b] \times [c, d] \) in \( [0, \infty)^2 \) with \( a < b \) and \( c < d \). The mapping \( f : \Delta \to \mathbb{R} \) is s-convex in the second sense on \( \Delta \) if \( f(\lambda x + (1 - \lambda) z, \lambda y + (1 - \lambda) w) \leq \lambda^s f(x, y) + (1 - \lambda)^s f(z, w) \), holds for all \( (x, y), (z, w) \in \Delta \), \( \lambda \in [0, 1] \) with some fixed \( s \in (0, 1] \).

A function \( f : \Delta \subseteq [0, \infty)^2 \to \mathbb{R} \) is called s-convex in the second sense on the co-ordinates on \( \Delta \) if the partial mappings \( f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y) \) and \( f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v) \), are s-convex in the second sense for all \( y \in [c, d], x \in [a, b] \) and \( s \in (0, 1] \), i.e., the partial mappings \( f_y \) and \( f_x \) are s-convex in the second sense with some fixed \( s \in (0, 1] \).

A formal definition of co-ordinated s-convex function in second sense may be stated as follows.

**Definition 1.3.** A function \( f : \Delta \subseteq [0, \infty)^2 \to \mathbb{R} \) is called s-convex in the second sense on the co-ordinates on \( \Delta \) if

\[
f(tx + (1 - t)y, ru + (1 - r)w) \leq t^s r^s f(x, u) + t^s (1 - r)^s f(x, w) \\
+ r^s (1 - t)^s f(y, u) + (1 - t)^s (1 - r)^s f(y, w)
\]

holds for all \( t, r \in [0, 1] \) and \( (x, u), (y, u), (x, w), (y, w) \in \Delta \), for some fixed \( s \in (0, 1] \). The mapping \( f \) is concave on the co-ordinates on \( \Delta \) if the inequality (1.5) holds in reversed direction for all \( t, r \in [0, 1] \) and \( (x, y), (u, w) \in \Delta \) with some fixed \( s \in (0, 1] \).
Furthermore, Alomari and Darus [5] introduced a new class of $s$-convex functions on the co-ordinates on the rectangle from the plane as follows.

**Definition 1.4.** [5] Consider the bidimensional interval $\Delta =: [a, b] \times [c, d]$ in $[0, \infty)^2$ with $a < b$ and $c < d$. The mapping $f : \Delta \to \mathbb{R}$ is $s$-convex in the second sense on $\Delta$ if there exist $s_1, s_2 \in (0, 1]$ with $s = \frac{s_1 + s_2}{2}$ such that

$$f(\lambda x + (1 - \lambda) z, \lambda y + (1 - \lambda) w) \leq \lambda^{s_1} f(x, y) + (1 - \lambda)^{s_2} f(z, w)$$

holds for all $(x, y), (z, w) \in \Delta$, $\lambda \in [0, 1]$. This class of functions is denoted by $MWO^{s_1, s_2}_{s_1, s_2}$.

A function $f : \Delta \subseteq [0, \infty)^2 \to \mathbb{R}$ is called $s$-convex in the second sense on the co-ordinates on $\Delta$ if the partial mappings $f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v)$, are $s_1$-convex and $s_2$-convex in the second sense for all $y \in [c, d]$, $x \in [a, b]$ and $s_1, s_2 \in (0, 1]$ with $s = \frac{s_1 + s_2}{2}$, respectively, i.e., the partial mappings $f_y$ and $f_x$ are $s_1$-convex and $s_2$-convex in the second sense, $s_1, s_2 \in (0, 1]$ with $s = \frac{s_1 + s_2}{2}$.

The definition 1.3 can be generalized as follows.

**Definition 1.5.** A function $f : \Delta =: [a, b] \times [c, d] \subseteq [0, \infty)^2 \to \mathbb{R}$ is called $s$-convex in the second sense on the co-ordinates on $\Delta$ if

$$f(tx + (1 - t)y, ru + (1 - r)w) \leq t^{s_1} r^{s_2} f(x, u) + t^{s_1} (1 - r)^{s_2} f(x, w)$$

holds for all $t, r \in [0, 1]$ and $(x, u), (y, u), (x, w), (y, w) \in \Delta$, $s_1, s_2 \in (0, 1]$ with $s = \frac{s_1 + s_2}{2}$. The mapping $f$ is concave on the co-ordinates on $\Delta$ if the inequality (1.6) holds in reversed direction for all $t, r \in [0, 1]$ and $(x, y), (u, w) \in \Delta$, $s_1, s_2 \in (0, 1]$ with $s = \frac{s_1 + s_2}{2}$.

In [5], Alomari et al. also proved a variant of inequalities given above by (1.4) for $s$-convex functions in the second sense on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

**Theorem 1.3.** [5] Suppose $f : \Delta \subseteq [0, \infty)^2 \to [0, \infty)$ is $s$-convex function in the second sense on the co-ordinates on $\Delta$. Then one has the inequalities

$$\frac{4^{s_1-1} + 4^{s_2-1}}{2} - f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{2^{s_1-2}}{b - a} \int_a^b f \left( x, \frac{c + d}{2} \right) dx$$

$$+ \frac{2^{s_2-2}}{d - c} \int_c^d f \left( \frac{a + b}{2}, y \right) dy$$

$$\leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx$$
\[
\leq \frac{1}{2(s_1+1)} \left( \frac{1}{b-a} \int_a^b [f(x,c) + f(x,d)] \, dx + \frac{1}{d-c} \int_c^d [f(a,y) + f(b,y)] \, dy \right)
\]
\[
\leq \frac{1}{2} \left( \frac{1}{(s_1+1)^2} + \frac{1}{(s_2+1)^2} \right) [f(a,c) + f(b,c) + f(a,d) + f(b,d)].
\]

In recent years, many authors have proved several inequalities for co-ordinated convex functions. These studies include, among others, the works in [1, 3, 4, 5, 6], [10], [13], [21]-[24], [25]-[28] and [31]. Alomari et al. [1, 3, 4, 5, 6], proved several Hermite-Hadamard type inequalities for co-ordinated \( s \)-convex functions and co-ordinated log-convex functions. Dragomir [10], proved the Hermite-Hadamard type inequalities for co-ordinated convex functions. Hwang et. al [13], also proved some Hermite-Hadamard type inequalities for co-ordinated convex function of two variables by considering some mappings directly associated to the Hermite-Hadamard type inequality for co-ordinated convex mappings of two variables. Latif et. al [12]-[14], proved some inequalities of Hermite-Hadamard type for differentiable co-ordinated convex functions, differentiable functions whose higher order partial derivatives are co-ordinated convex, product of two co-ordinated convex mappings and for co-ordinated \( h \)-convex mappings. Özdemir et. al [25]-[28], proved Hadamard’s type inequalities for co-ordinated convex functions, co-ordinated \( s \)-convex functions and co-ordinated \( m \)-convex and \((\alpha, m)\)-convex functions.

The main aim of this paper is to establish some new Hermite-Hadamard type inequalities for differentiable functions whose partial derivatives of higher order are co-ordinated \( s \)-convex in the second sense on the rectangle from the plane \( \mathbb{R}^2 \) which generalize the Hermite-Hadamard type inequalities proved for co-ordinated \( s \)-convex functions in the second sense and refine those results established for differentiable functions whose partial derivatives of higher order are co-ordinated convex on the rectangle from the plane \( \mathbb{R}^2 \) (see [24]).

2. Main Results

In this section we establish new Hermite-Hadamard type inequalities for double integrals of functions whose partial derivatives of higher order are co-ordinated \( s \)-convex in the second sense.

To make the presentation easier and compact to understand, we make some symbolic representations as follows

\[
A' = \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b [f(x,c) + f(x,d)] \, dx + \frac{1}{d-c} \int_c^d [f(a,y) + f(b,y)] \, dy \right]
\]
\[
+ \frac{1}{2} \sum_{l=2}^{m-1} \frac{(l-1)(d-c)}{2(l+1)!} \left[ \frac{\partial^l f(a,c)}{\partial y^l} + \frac{\partial^l f(b,c)}{\partial y^l} \right]
\]
\[
+ \frac{1}{2} \sum_{k=2}^{n-1} \frac{(k-1)(b-a)}{2(k+1)!} \left[ \frac{\partial^k f(a,c)}{\partial x^k} + \frac{\partial^k f(a,d)}{\partial x^k} \right]
\]

Theorem 2.1. [18] Let \( f : \Delta \to \mathbb{R} \) be a continuous mapping such that the partial derivatives \( \frac{\partial^{k+l} f}{\partial x^k \partial y^l} \), \( k = 0, 1, \ldots, n-1 \), \( l = 0, 1, \ldots, m-1 \) exist on \( \Delta^\circ \) and are continuous on \( \Delta \), then

\[
\int_a^b \int_c^d f \left( t, r \right) dr dt = ( -1)^{m+n} \int_a^b \int_c^d K_n \left( x, t \right) S_m \left( y, r \right) \frac{\partial^{n+m} f \left( t, r \right)}{\partial t^n \partial r^m} dr dt \\
+ \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} X_k \left( x \right) Y_l \left( y \right) \frac{\partial^{k+l} f \left( x, y \right)}{\partial x^k \partial y^l} + \left( -1 \right)^m \sum_{k=0}^{n-1} X_k \left( x \right) \int_c^d S_m \left( y, r \right) \frac{\partial^{k+m} f \left( x, r \right)}{\partial x^k \partial r^m} dr \\
+ \left( -1 \right)^n \sum_{l=0}^{m-1} Y_l \left( y \right) \int_a^b K_n \left( x, t \right) \frac{\partial^{n+l} f \left( t, y \right)}{\partial t^n \partial y^l} dt,
\]

where, for \( (x, y) \in \Delta \), we have

\[
K_n \left( x, t \right) := \begin{cases} 
\frac{\left( t-a \right)^n}{n!}, t \in [a, x] \\
\frac{\left( b-t \right)^n}{n!}, t \in (x, b]
\end{cases}
\]

\[
S_m \left( y, r \right) := \begin{cases} 
\frac{\left( r-c \right)^m}{m!}, r \in [c, y] \\
\frac{\left( r-d \right)^m}{m!}, r \in (y, d]
\end{cases}
\]

and

\[
X_k \left( x \right) := \frac{\left( b-x \right)^{k+1} \left( -1 \right)^k \left( x-a \right)^{k+1}}{(k+1)!} \\
Y_l \left( y \right) := \frac{\left( d-y \right)^{l+1} \left( -1 \right)^l \left( y-c \right)^{l+1}}{(l+1)!}
\]
Theorem 2.2. Let $x$ and $y$ be co-ordinates on $\Delta$. Suppose $f: \Delta \to \mathbb{R}$, be a continuous mapping such that $\frac{\partial^{m+n} f}{\partial x^m \partial y^n}$ exists on $\Delta^o$ and $\frac{\partial^{m+n} f}{\partial x^m \partial y^n} \in L(\Delta)$ for $m, n \geq 1$, then

\begin{equation}
\frac{(b-a)^m (d-c)^n}{4n!m!} \int_0^1 \int_0^1 t^{n-1} r^{m-1} (n-2t) (m-2r) \frac{\partial^{m+n} f (ta + (1 - t)b, cr + (1 - r)d)}{\partial t^n \partial r^m} \, dt \, dr + A' = \frac{f (a, c) + f (a, d) + f (b, c) + f (b, d)}{4} + \frac{1}{(b-a) (d-c)} \int_a^b \int_c^d f (x, y) \, dy \, dx.
\end{equation}

Now we prove our main results.

Lemma 2.1. [24] Let $f: \Delta \to \mathbb{R}$, be a continuous mapping such that $\frac{\partial^{m+n} f}{\partial x^m \partial y^n}$ exists on $\Delta^o$ and $\frac{\partial^{m+n} f}{\partial x^m \partial y^n} \in L(\Delta)$. If $\frac{\partial^{m+n} f}{\partial x^m \partial y^n}$ is s-convex on the co-ordinates on $\Delta$ in the second sense, for $m, n \in \mathbb{N}$, $m, n \geq 2$, then we have the following inequality

\begin{equation}
\frac{f (a, c) + f (a, d) + f (b, c) + f (b, d)}{4} + \frac{1}{(b-a) (d-c)} \int_a^b \int_c^d f (x, y) \, dy \, dx - A' \leq \frac{(b-a)^n (d-c)^m}{4n!m!} \left[ LB_{(n,m)} + MC_{(n,m)} + ND_{(n,m)} + RE_{(n,m)} \right],
\end{equation}

where $s_1, s_2 \in (0, 1]$ with $s = \frac{s_1 + s_2}{2}$,

$L = \frac{n (n-1) + s_1 (n-2)}{(n + s_1) (n + s_1 + 1)} \left[ \frac{m (m-1) + s_2 (m-2)}{(m + s_2) (m + s_2 + 1)} \right],$

$M = \frac{n (n-1) + s_1 (n-2)}{(n + s_1) (n + s_1 + 1)} \left[ mb (m, s_2 + 1) - 2B (m + 1, s_2 + 1) \right],$

$N = \frac{m (m-1) + s_2 (m-2)}{(m + s_2) (m + s_2 + 1)} \left[ nB (n, s_1 + 1) - 2B (n + 1, s_1 + 1) \right],$

$R = \frac{nB (n, s_1 + 1) - 2B (n + 1, s_1 + 1)}{mB (m, s_2 + 1) - 2B (m + 1, s_2 + 1)} \left[ nB (n, s_1 + 1) - 2B (n + 1, s_1 + 1) \right],$

and $B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt$ is the Euler Beta function.

Proof. Suppose $m, n \geq 2$. By Lemma 2.1, we have

\begin{equation}
\frac{f (a, c) + f (a, d) + f (b, c) + f (b, d)}{4} + \frac{1}{(b-a) (d-c)} \int_a^b \int_c^d f (x, y) \, dy \, dx - A' \leq \frac{(b-a)^n (d-c)^m}{4n!m!} \int_0^1 \int_0^1 t^{n-1} r^{m-1} (n-2t) (m-2r) \frac{\partial^{m+n} f (ta + (1 - t)b, cr + (1 - r)d)}{\partial t^n \partial r^m} \, dt \, dr.
\end{equation}
By \( s\)-convexity of \( \left| \frac{\partial^{n+m}f}{\partial^m t \partial^n r} \right| \) on the co-ordinates on \( \Delta \), we get that
\[
\int_0^1 \int_0^1 t^{n-1}r^{m-1} (n-2t) (m-2r) \times \left| \frac{\partial^{n+m}f (ta + (1-t) b, cr + (1 - r) d)}{\partial^m t \partial^n r} \right| \, dt \, dr
\]
(2.4)
\[
\leq B_{(n,m)} \int_0^1 \int_0^1 t^{n+s_1-1}r^{m+s_2-1} (n-2t) (m-2r) \, dr \, dt
\]
\[
+ C_{(n,m)} \int_0^1 \int_0^1 t^{n+s_1-1}r^{m-1} (1 - r)^{s_2} (n-2t) (m-2r) \, dr \, dt
\]
\[
+ E_{(n,m)} \int_0^1 \int_0^1 t^{n-1} (1-t)^{s_1} (n-2t) r^{m-1} (1 - r)^{s_2} (m-2r) \, dr \, dt
\]
\[
+ D_{(n,m)} \int_0^1 \int_0^1 t^{n-1}r^{m+s_2-1} (1 - t)^{s_1} (n-2t) (m-2r) \, dr \, dt.
\]
Since
\[
\int_0^1 \int_0^1 t^{n+s_1-1}r^{m+s_2-1} (n-2t) (m-2r) \, dr \, dt
\]
(2.5)
\[
= \int_0^1 t^{n+s_1-1} (n-2t) \, dt \int_0^1 r^{m+s_2-1} (m-2r) \, dr
\]
\[
= \frac{[n(n-1)+s_1(n-2)]}{(n+s_1)(n+s_1+1)} [\frac{m(m-1)+s_2(m-2)}{(m+s_2)(m+s_2+1)}].
\]
Analogously,
\[
\int_0^1 \int_0^1 t^{n+s_1-1}r^{m-1} (1 - r)^{s_2} (n-2t) (m-2r) \, dr \, dt
\]
(2.6)
\[
= \frac{[n(n-1)+s_1(n-2)]}{(n+s_1)(n+s_1+1)} [mB(m,s_2+1) - 2B(m+1,s_2+1)],
\]
\[
\int_0^1 \int_0^1 t^{n-1}r^{m+s_2-1} (1 - t)^{s_1} (n-2t) (m-2r) \, dr \, dt
\]
(2.7)
\[
= \frac{[m(m-1)+s_2(m-2)]}{(m+s_2)(m+s_2+1)} [nB(n,s_1+1) - 2B(n+1,s_1+1)]
\]
and
\[
\int_0^1 \int_0^1 t^{n-1} (1-t)^{s_1} (n-2t) r^{m-1} (1 - r)^{s_2} (m-2r) \, dr \, dt
\]
(2.8)
\[
= [nB(n,s_1+1) - 2B(n+1,s_1+1)] [mB(m,s_2+1) - 2B(m+1,s_2+1)].
\]
From (2.4)-(2.8) in (2.3), we get the required inequality. This completes the proof of the theorem. \( \square \)
Theorem 2.3. Let \( f : \Delta \subset [0, \infty) \times [0, \infty) \rightarrow [0, \infty), \) \( a < b, c < d, \) be a continuous mapping such that \( \frac{\partial^{m+n} f}{\partial x^m \partial y^n} \) exists on \( \Delta^0 \) and \( \frac{\partial^{m+n} f}{\partial x^m \partial y^n} \in L(\Delta). \) If \( \left| \frac{\partial^{m+n} f}{\partial x^m \partial y^n} \right|^q, q \geq 1, \) is s-convex on the co-ordinates on \( \Delta, m, n \in \mathbb{N}, m, n \geq 2, \) then

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx - A \right| \\
\leq \frac{(b-a)^n (d-c)^m}{4n!m!} \left( \frac{(n-1)(m-1)}{(n+1)(m+1)} \right)^{1-1/q} \times \sqrt{LB_{(n,m)}^q + MD_{(n,m)}^q + NC_{(n,m)}^q + RE_{(n,m)}^q},
\]

where \( s_1, s_2 \in (0, 1] \) with \( s = \frac{s_1 + s_2}{2} \) and \( L, M, N, R \) and \( B(x,y) \) are as defined in Theorem 2.2.

Proof. The case \( q = 1 \) is the Theorem 2.2. Suppose \( q > 1, \) then by Lemma 2.1 and the power mean inequality, we have

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx - A \right| \\
\leq \frac{(b-a)^n (d-c)^m}{4n!m!} \left\{ \int_0^1 \int_0^1 t^{n-1}r^{m-1}(n-2t)(m-2r) \, dr \, dt \right\}^{1-1/q} \times \left\{ \int_0^1 \int_0^1 t^{n-1}r^{m-1}(n-2t)(m-2r) \, dr \, dt \right\}^{1/q}
\]

By the similar arguments used to obtain (2.2) and the fact

\[
\int_0^1 \int_0^1 t^{n-1}r^{m-1}(n-2t)(m-2r) \, dr \, dt = \frac{(n-1)(m-1)}{(n+1)(m+1)},
\]

we get (2.9). This completes the proof of the theorem. \( \square \)

Theorem 2.4. Let \( f : \Delta \subset [0, \infty) \times [0, \infty) \rightarrow [0, \infty), \) \( a < b, c < d, \) be a continuous mapping such that \( \frac{\partial^{m+n} f}{\partial x^m \partial y^n} \) exist on \( \Delta^0 \) and \( \frac{\partial^{m+n} f}{\partial x^m \partial y^n} \in L(\Delta). \) If \( \left| \frac{\partial^{m+n} f}{\partial x^m \partial y^n} \right|^q, q \geq 1, \) is s-convex on the co-ordinates on \( \Delta, s_1, s_2 \in (0, 1] \) with \( s = \frac{s_1 + s_2}{2} \), \( m, n \in \mathbb{N}, m, n \geq 1. \)
Then
\[
\sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{1 + (-1)^k}{2^{k+l+2}} \frac{1 + (-1)^l}{(k+1)! (l+1)!} (b - a)^k (d - c)^l \frac{\partial^{k+l} f (\frac{a+b}{2}, \frac{c+d}{2})}{\partial x^k \partial y^l}
\]
\[
+ \frac{1}{(b - a) (d - c)} \int_a^b \int_c^d f (t, r) \, dr \, dt
\]
\[
+ \frac{(-1)^{m+1}}{(d - c) m!} \sum_{k=0}^{n-1} \frac{1 + (-1)^k}{2^{k+1} (k+1)!} \int_c^d Q(r) \frac{\partial^{k+m} f (\frac{a+b}{2}, r)}{\partial x^k \partial r^m} \, dr
\]
\[
+ \frac{(-1)^{n+1}}{(b - a) n!} \sum_{l=0}^{m-1} \frac{1 + (-1)^l}{2^{l+1} (l+1)!} \int_a^b P(t) \frac{\partial^{n+l} f (t, \frac{c+d}{2})}{\partial t^n \partial y^l} \, dt
\]
\[
(2.11)
\]
\[
\leq \frac{1}{4n!m!} \left( \frac{4}{(n+1) (m+1)} \right)^{1 - \frac{1}{q}} \left( \frac{b - a}{2} \right)^n \left( \frac{d - c}{2} \right)^m
\]
\[
\times \left[ \left( B^q_{(n,m)} + C^q_{(n,m)} + D^q_{(n,m)} + E^q_{(n,m)} \right) B (n+1, s_1 + 1) B (m+1, s_2 + 1) + 2 \left( G^q_{(n,m)} + J^q_{(n,m)} \right) B (n+1, s_1 + 1) + 2 \left( H^q_{(n,m)} + J^q_{(n,m)} \right) B (m+1, s_2 + 1)
\]
\[
+ \frac{4F^q_{(n,m)}}{(n + s_1 + 1) (m + s_2 + 1)} \right]^{\frac{1}{q}},
\]
\[
\text{where}
\]
\[
P(t) := \begin{cases} 
(t - a)^n, & t \in \left[ a, \frac{a+b}{2} \right] \\
(t - b)^n, & t \in \left[ \frac{a+b}{2}, b \right]
\end{cases}
\]
and
\[
Q(r) := \begin{cases} 
(r - c)^m, & r \in \left[ c, \frac{c+d}{2} \right] \\
(r - d)^m, & r \in \left[ \frac{c+d}{2}, d \right]
\end{cases}
\].

Proof. By letting \( x \mapsto \frac{a+b}{2} \) and \( y \mapsto \frac{c+d}{2} \) in Theorem 2.1 and using the properties of the absolute value, we obtain
\[
\sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{1 + (-1)^k}{2^{k+l+2}} \frac{1 + (-1)^l}{(k+1)! (l+1)!} (b - a)^k (d - c)^l \frac{\partial^{k+l} f (\frac{a+b}{2}, \frac{c+d}{2})}{\partial x^k \partial y^l}
\]
\[
+ \frac{1}{(b - a) (d - c)} \int_a^b \int_c^d f (t, r) \, dr \, dt
\]
\[
+ \frac{(-1)^{m+1}}{(d - c) m!} \sum_{k=0}^{n-1} \frac{1 + (-1)^k}{2^{k+1} (k+1)!} \int_c^d Q(r) \frac{\partial^{k+m} f (\frac{a+b}{2}, r)}{\partial x^k \partial r^m} \, dr
\]
\[
(2.12)
\]
By the power mean inequality for double integrals, we have

\[
\int_a^b \int_c^d |P(t)| |Q(r)| \left| \frac{\partial^{n+m} f (t, r)}{\partial t^n \partial r^m} \right| \, dr \, dt
\]

(2.13)

\[
\leq \left( \int_a^b \int_c^d |P(t)| |Q(r)| \, dr \, dt \right)^{1 - \frac{1}{q}} \left( \int_a^b \int_c^d |P(t)| |Q(r)| \left| \frac{\partial^{n+m} f (t, r)^q}{\partial t^n \partial r^m} \right| \, dr \, dt \right)^{\frac{1}{q}}
\]

\[
= \left( \int_a^b \int_c^d |P(t)| |Q(r)| \, dr \, dt \right)^{1 - \frac{1}{q}} \left[ \int_a^{a+b} \int_c^{c+d} (t-a)^n (r-c)^m \left| \frac{\partial^{n+m} f (t, r)^q}{\partial t^n \partial r^m} \right| \, dr \, dt \right]^{\frac{1}{q}}
\]

Now we calculate each integral in (2.13). Since

[\text{expression]}

and

[\text{expression}],

By the co-ordinated s-convexity of \( \frac{\partial^{n+m} f (t, r)^q}{\partial t^n \partial s^m} \), we have

\[
\int_a^{a+b} \int_c^{c+d} (t-a)^n (r-c)^m \left| \frac{\partial^{n+m} f (t, r)^q}{\partial t^n \partial r^m} \right| \, dr \, dt \leq \left( \frac{2}{b-a} \right)^{s_1} \left( \frac{2}{d-c} \right)^{s_2}
\]

\[
\times \left[ B_{(n,m)}^{q} \int_a^{a+b} \int_c^{c+d} (t-a)^n (r-c)^m \left( \frac{a+b}{2} - t \right)^{s_1} \left( \frac{c+d}{2} - r \right)^{s_2} \, dr \, dt \right]
\]

\[
+ G_{(n,m)}^{q} \int_a^{a+b} \int_c^{c+d} (t-a)^n \left( \frac{a+b}{2} - t \right)^{s_1} (r-c)^{s_2+m} \, dr \, dt
\]

\[
+ H_{(n,m)}^{q} \int_a^{a+b} \int_c^{c+d} (t-a)^{s_1+n} \left( \frac{c+d}{2} - r \right)^{s_2} (r-c)^m \, dr \, dt
\]

\[
+ F_{(n,m)}^{q} \int_a^{a+b} \int_c^{c+d} (t-a)^{s_1+n} (r-c)^{s_2+m} \, dr \, dt \right].
\]
Now by the change of variables \( u = t - a, \) \( v = r - c \) and then by the change of variables \( x = \frac{2u}{b-a}, \) \( y = \frac{2v}{d-c}, \) we get that

\[
(2.15) \quad \left( \frac{2}{b-a} \right)^{s_1} \left( \frac{2}{d-c} \right)^{s_2} \times \int_a^b \int_c^{c+d} (t-a)^n (r-c)^m \left( \frac{a+b}{2} - t \right)^{s_1} \left( \frac{c+d}{2} - r \right)^{s_2} \, drdt
\]

\[
= \left( \frac{2}{b-a} \right)^{s_1} \left( \frac{2}{d-c} \right)^{s_2} \int_0^{\frac{b-a}{2}} u^n \left( \frac{b-a}{2} - u \right)^{s_1} \int_0^{\frac{d-c}{2}} v^m \left( \frac{d-c}{2} - v \right)^{s_2} \, dv\]

\[
= \int_0^{\frac{b-a}{2}} u^n \left( 1 - \frac{2u}{b-a} \right) du \int_0^{\frac{d-c}{2}} v^m \left( 1 - \frac{2v}{d-c} \right) dv
\]

\[
= \left( \frac{b-a}{2} \right)^{n+1} \left( \frac{d-c}{2} \right)^{m+1} \int_0^1 x^n (1-x)^s \, dx \int_0^1 y^m (1-y)^s \, dy
\]

\[
= \left( \frac{b-a}{2} \right)^{n+1} \left( \frac{d-c}{2} \right)^{m+1} B \left( n + 1, s_1 + 1 \right) B \left( m + 1, s_2 + 1 \right).
\]

Similarly,

\[
(2.16) \quad \left( \frac{2}{b-a} \right)^{s_1} \left( \frac{2}{d-c} \right)^{s_2} \int_a^b \int_c^{c+d} (t-a)^n \left( \frac{a+b}{2} - t \right)^{s_1} (r-c)^{s_2+m} \, drdt
\]

\[
= \left( \frac{b-a}{2} \right)^{n+1} \left( \frac{d-c}{2} \right)^{m+1} \frac{B \left( n + 1, s_1 + 1 \right)}{m + s_2 + 1},
\]

\[
(2.17) \quad \left( \frac{2}{b-a} \right)^{s_1} \left( \frac{2}{d-c} \right)^{s_2} \int_a^b \int_c^{c+d} (t-a)^{s_1+n} \left( \frac{c+d}{2} - r \right)^{s_2} (r-c)^m \, drdt
\]

\[
= \left( \frac{b-a}{2} \right)^{n+1} \left( \frac{d-c}{2} \right)^{m+1} \frac{B \left( m + 1, s_2 + 1 \right)}{n + s_1 + 1}.
\]

and

\[
(2.18) \quad \left( \frac{2}{b-a} \right)^{s_1} \left( \frac{2}{d-c} \right)^{s_2} \int_a^b \int_c^{c+d} (t-a)^{s_1+n} (r-c)^{s_2+m} \, drdt
\]

\[
= \left( \frac{b-a}{2} \right)^{n+1} \left( \frac{d-c}{2} \right)^{m+1} \frac{B \left( n + s_1 + 1 \right) \left( m + s_2 + 1 \right)}{n + s_1 + 1}.
\]

Using (2.15)-(2.18) in (2.14), we obtain

\[
(2.19) \quad \int_a^b \int_c^{c+d} (t-a)^n (r-c)^m \left| \frac{\partial^{n+m} f (t, r)}{\partial t^n \partial r^m} \right|^q \, drdt \leq \left( \frac{b-a}{2} \right)^{n+1} \left( \frac{d-c}{2} \right)^{m+1}
\]

\[
\times \left[ B^q_{(n,m)} B \left( n + 1, s_1 + 1 \right) B \left( m + 1, s_2 + 1 \right) + \frac{G^q_{(n,m)} B \left( n + 1, s_1 + 1 \right)}{m + s_2 + 1} \right]
\]
Analogously,\[\begin{align*}
\int_{a+b}^{b} \int_{c}^{d} (b-t)^n (r-c)^m \left| \frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m} \right|^q \, dr \, dt & \leq \left( \frac{b-a}{2} \right)^{n+1} \left( \frac{d-c}{2} \right)^{m+1} \\
& \times \left[ \frac{H^q_{(n,m)} B(m+1,s_2+1)}{n+s_1+1} + D^q_{(n,m)} B(n+1,s_1+1) B(m+1,s_2+1) \\
& + \frac{F^q_{(n,m)}}{(n+s_1+1)(m+s_2+1)} + \frac{I^q_{(n,m)} B(n+1,s_1+1)}{m+s_2+1} \right],
\end{align*}\]

and
\[\begin{align*}
\int_{a+b}^{b} \int_{c}^{d} (b-t)^n (d-r)^m \left| \frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m} \right|^q \, dr \, dt & \leq \left( \frac{b-a}{2} \right)^{n+1} \left( \frac{d-c}{2} \right)^{m+1} \left[ \frac{F^q_{(n,m)}}{(n+s_1+1)(m+s_2+1)} + \frac{I^q_{(n,m)} B(n+1,s_1+1)}{m+s_2+1} \\
& + \frac{J^q_{(n,m)} B(m+1,s_2+1)}{n+s_1+1} + E^q_{(n,m)} B(n+1,s_1+1) B(m+1,s_2+1) \right].
\end{align*}\]

It is not difficult to observe that
\[\int_{a}^{b} \int_{c}^{d} |P(t)||Q(r)| \, dr \, dt = \frac{4}{(n+1)(m+1)} \left( \frac{b-a}{2} \right)^{n+1} \left( \frac{d-c}{2} \right)^{m+1} .\]

From (2.12)-(2.23), we get the desired inequality. The proof of the Theorem for \(q = 1\) is the same. This completes the proof. \(\square\)

Some results can be deduced from the inequalities (2.9) and (2.12) as follows. Letting \(s_1 = s_2 = 1\) in Theorem 2.3 gives the following corollary.
Corollary 2.1. Let \( f : \Delta \subset [0, \infty) \times [0, \infty) \to [0, \infty) \), \( a < b, \ c < d \), be a continuous mapping such that \( \frac{\partial^{m+n+f}}{\partial x^m \partial r^f} \) exists on \( \Delta^o \) and \( \frac{\partial^{m+n+f}}{\partial x^m \partial r^f} \in L(\Delta) \). If \( \left| \frac{\partial^{m+n+f}}{\partial x^m \partial r^f} \right| ^q, \ q \geq 1, \) is convex on the co-ordinates on \( \Delta, m,n \in \mathbb{N}, m,n \geq 2, \) then

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx - A^\prime \right| \\
(2.24)
\]

\[
\leq \frac{(b-a)^n (d-c)m (n-1)^{1-1/q} (m-1)^{1-1/q}}{4(n+1)!(m+1)!(n+2)^{1/q}(m+2)^{1/q}} \left[ (m^2 - 2) (n^2 - 2) B_{(n,m)}^q + m (n^2 - 2) C_{(n,m)}^q + n (m^2 - 2) D_{(n,m)}^q + nm E_{(n,m)}^q \right]^{\frac{1}{q}}.
\]

Corollary 2.2. Under the assumptions of Corollary 2.1 with \( m = n = 2 \), we have

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx - A^\prime \right| \\
\leq \frac{(b-a)^2 (d-c)^2}{9 \cdot 2^{\frac{7}{4} + 1}} \sqrt{\left| \frac{\partial^4 f(a,c)}{\partial t^2 \partial r^2} \right|^q + \left| \frac{\partial^4 f(b,c)}{\partial t^2 \partial r^2} \right|^q + \left| \frac{\partial^4 f(a,d)}{\partial t^2 \partial r^2} \right|^q + \left| \frac{\partial^4 f(b,d)}{\partial t^2 \partial r^2} \right|^q}.
\]

The following corollary is a special case of Theorem 2.4 for \( s_1 = s_2 = 1 \).

Corollary 2.3. Let \( f : \Delta \subset [0, \infty) \times [0, \infty) \to [0, \infty) \), \( a < b, \ c < d \), be a continuous mapping such that \( \frac{\partial^{m+n+f}}{\partial x^m \partial r^f} \) exist on \( \Delta^o \) and \( \frac{\partial^{m+n+f}}{\partial x^m \partial r^f} \in L(\Delta) \). If \( \left| \frac{\partial^{m+n+f}}{\partial x^m \partial r^f} \right| ^q, \ q \geq 1, \) is convex on the co-ordinates on \( \Delta, m,n \in \mathbb{N}, m,n \geq 1, \) Then

\[
(2.25) \quad \left| - \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \left[ 1 + (-1)^k \right] \left[ 1 + (-1)^l \right] \frac{(b-a)^k (d-c)^l}{2^{k+l+2}} \frac{\partial^{k+l} f \left( \frac{a+b}{2}, \frac{c+d}{2} \right)}{(k+1)!(l+1)!} \right| \\
+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,r) \, dr \, dt \\
+ \frac{(-1)^m+1}{(d-c)m!} \sum_{k=0}^{n-1} \left[ 1 + (-1)^k \right] \frac{(b-a)^k}{2^{k+1}(k+1)!} \int_c^d Q(r) \frac{\partial^{k+m} f \left( \frac{a+b}{2}, r \right)}{\partial x^k \partial r^m} \, dr \\
+ \frac{(-1)^{m+1}}{(b-a)n!} \sum_{l=0}^{m-1} \left[ 1 + (-1)^l \right] \frac{(d-c)^l}{2^{l+1}(l+1)!} \int_a^b P(t) \frac{\partial^{m+l} f \left( t, \frac{c+d}{2} \right)}{\partial t^m \partial y^l} \, dt \right|
\]
\[
\leq \frac{(b - a)^n (d - c)^m}{2^{m+n+\frac{2}{q}} (n+1)! (m+1)!} \left[ \frac{B^q_{(n,m)} + C^q_{(n,m)} + D^q_{(n,m)} + E^q_{(n,m)}}{(n+2) (m+2)} + \frac{2 (m+1) \left( G^q_{(n,m)} + I^q_{(n,m)} \right)}{(n+2) (m+2)} + \frac{2 (n+1) \left( H^q_{(n,m)} + J^q_{(n,m)} \right)}{(n+2) (m+2)} + \frac{4 (n+1) (m+1) F^q_{(n,m)}}{(n+2) (m+2)} \right]^{\frac{1}{q}},
\]

where \( P(t) \) and \( Q(r) \) are as defined in Theorem 2.4.

The following corollary is a special case of Theorem 2.4 for \( s_1 = s_2 = 1 \) and \( m = n = 1 \), which gives tighter estimate than those from [23, Theorem 4, page 8].

**Corollary 2.4.** Under the assumptions of Corollary 2.3 with \( m = n = 1 \), we have

\[
\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,r) \, dr \, dt + f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \right|
\leq \frac{(b-a)(d-c)}{2^{4+\frac{2}{q}}} \left[ \frac{B^q_{(1,1)} + C^q_{(1,1)} + D^q_{(1,1)} + E^q_{(1,1)}}{9} + \frac{4 \left( C^q_{(1,1)} + I^q_{(1,1)} \right)}{9} + \frac{4 \left( H^q_{(1,1)} + J^q_{(1,1)} \right)}{9} + \frac{8 F^q_{(1,1)}}{9} \right]^{\frac{1}{q}},
\]

where \( P(t) \) and \( Q(r) \) are as defined in Theorem 2.4.

It is easy to see that, when \( \frac{\partial^{m+n}}{\partial t^m \partial r^n} f \), \( q \geq 1 \), is convex on the co-ordinates on \( \Delta \), \( m, n \in \mathbb{N}, m,n \geq 1 \), then

\[
2 \left( G^q_{(n,m)} + I^q_{(n,m)} \right) \leq B^q_{(n,m)} + C^q_{(n,m)} + D^q_{(n,m)} + E^q_{(n,m)},
\]

\[
2 \left( H^q_{(n,m)} + J^q_{(n,m)} \right) \leq B^q_{(n,m)} + C^q_{(n,m)} + D^q_{(n,m)} + E^q_{(n,m)}
\]

and

\[
4 F^q_{(n,m)} \leq B^q_{(n,m)} + C^q_{(n,m)} + D^q_{(n,m)} + E^q_{(n,m)}.
\]

Substituting these inequalities in Corollary 2.3, we get the following corollary which is [24, Theorem 2.3, page 12].

**Corollary 2.5.** Let \( f : \Delta \subset [0, \infty) \times [0, \infty) \to [0, \infty) \), \( a < b \), \( c < d \), be a continuous mapping such that \( \frac{\partial^{m+n}}{\partial t^m \partial r^n} f \) exist on \( \Delta^o \) and \( \frac{\partial^{m+n}}{\partial t^m \partial r^n} f \in L(\Delta) \). If \( \frac{\partial^{m+n}}{\partial t^m \partial r^n} f \), \( q \geq 1 \), is
convex on the co-ordinates on $\Delta$, $m, n \in \mathbb{N}$, $m, n \geq 1$. Then

$$
(2.27)
- \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{1 + (-1)^k}{2^{k+l+2}} \frac{1 + (-1)^l}{(k+1)! (l+1)!} (b - a)^k (d - c)^l \frac{\partial^{k+l} f \left( \frac{a+b}{2}, \frac{c+d}{2} \right)}{\partial x^k \partial y^l}
$$

$$
+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,r) \, dr \, dt
$$

$$
+ \frac{(-1)^{m+1}}{(d-c)m!} \sum_{k=0}^{n-1} \frac{1 + (-1)^k}{2^{k+1} (k+1)!} \int_c^d Q(r) \frac{\partial^{k+m} f \left( \frac{a+b}{2}, r \right)}{\partial x^k \partial r^m} \, dr
$$

$$
+ \frac{(-1)^{n+1}}{(b-a)n!} \sum_{l=0}^{m-1} \frac{1 + (-1)^l}{2^{l+1} (l+1)!} \int_a^b P(t) \frac{\partial^{n+l} f \left( t, \frac{c+d}{2} \right)}{\partial t^n \partial y^l} \, dt
$$

$$
\leq \frac{(b-a)^n (d-c)^m}{2^{m+n+\frac{q}{2}} (n+1)! (m+1)!} \sqrt{B_{q(m,n)}^q + C_{q(n,m)}^q + D_{q(n,m)}^q + E_{q(n,m)}^q},
$$

where $P(t)$ and $Q(r)$ are as defined in Theorem 2.4.

A different approach leads us to the following result.

**Theorem 2.5.** Let $f : \Delta \subset [0, \infty) \times [0, \infty) \to [0, \infty)$, $a < b$, $c < d$, be a continuous mapping such that $\frac{\partial^{m+n} f}{\partial x^m \partial y^n}$ exist on $\Delta^c$ and $\frac{\partial^{m+n} f}{\partial x^m \partial y^n} \in L(\Delta)$. If $\left| \frac{\partial^{m+n} f}{\partial x^m \partial y^n} \right|^q$, $q \geq 1$, is $s$-convex on the co-ordinates on $\Delta$, $s_1, s_2 \in (0, 1]$ with $s = \frac{s_1 + s_2}{2}$, $m, n \in \mathbb{N}$, $m, n \geq 1$. Then

$$
(2.28)
- \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{1 + (-1)^k}{2^{k+l+2}} \frac{1 + (-1)^l}{(k+1)! (l+1)!} (b - a)^k (d - c)^l \frac{\partial^{k+l} f \left( \frac{a+b}{2}, \frac{c+d}{2} \right)}{\partial x^k \partial y^l}
$$

$$
+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,r) \, dr \, dt
$$

$$
+ \frac{(-1)^{m+1}}{(d-c)m!} \sum_{k=0}^{n-1} \frac{1 + (-1)^k}{2^{k+1} (k+1)!} \int_c^d Q(r) \frac{\partial^{k+m} f \left( \frac{a+b}{2}, r \right)}{\partial x^k \partial r^m} \, dr
$$

$$
+ \frac{(-1)^{n+1}}{(b-a)n!} \sum_{l=0}^{m-1} \frac{1 + (-1)^l}{2^{l+1} (l+1)!} \int_a^b P(t) \frac{\partial^{n+l} f \left( t, \frac{c+d}{2} \right)}{\partial t^n \partial y^l} \, dt
$$

$$
\leq \frac{1}{4n!m!} \left( \frac{1}{(n+1)(m+1)} \right)^{1-\frac{q}{2}} \left( \frac{b-a}{2} \right)^n \left( \frac{d-c}{2} \right)^m \times
$$
\[
\times \left\{ B^q_{(n,m)}B (n + 1, s_1 + 1) B (m + 1, s_2 + 1) + \frac{G^q_{(n,m)}B (n + 1, s_1 + 1)}{m + s_2 + 1} \right.
\]
\[
\quad + \frac{H^q_{(n,m)}B (m + 1, s_2 + 1)}{n + s_1 + 1} + \frac{F^q_{(n,m)}}{(n + s_1 + 1)(m + s_2 + 1)} \right] ^{\frac{1}{2}}
\]
\[
\quad + \left[ \frac{H^q_{(n,m)}B (m + 1, s_2 + 1)}{n + s_1 + 1} + D^q_{(n,m)}B (n + 1, s_1 + 1) B (m + 1, s_2 + 1) \right.
\]
\[
\quad + \left[ \frac{F^q_{(n,m)}}{(n + s_1 + 1)(m + s_2 + 1)} + \frac{I^q_{(n,m)}B (n + 1, s_1 + 1)}{m + s_2 + 1} \right)^{\frac{1}{2}}
\]
\[
\quad + \left[ \frac{C^q_{(n,m)}B (n + 1, s_1 + 1)}{m + s_2 + 1} + C^q_{(n,m)}B (n + 1, s_1 + 1) B (m + 1, s_2 + 1) \right.
\]
\[
\quad + \left[ \frac{F^q_{(n,m)}}{(n + s_1 + 1)(m + s_2 + 1)} + \frac{I^q_{(n,m)}B (n + 1, s_1 + 1)}{m + s_2 + 1} + \frac{J^q_{(n,m)}B (m + 1, s_2 + 1)}{n + s_1 + 1} \right.
\]
\[
\quad + F^q_{(n,m)}B (n + 1, s_1 + 1) B (m + 1, s_2 + 1) \right]\} ^{\frac{1}{2}},
\]

where \( P(t) \) and \( Q(r) \) are as defined in Theorem 2.4.

**Proof.** By letting \( x \mapsto \frac{a+b}{2} \) and \( y \mapsto \frac{c+d}{2} \) in Theorem 2.1, using the properties of the absolute value, we obtain

\[
\left| \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \left[ 1 + (-1)^k \right] \left[ 1 + (-1)^l \right] \frac{2^{k+l+2}}{2^{k+1} (k + 1)! (l + 1)!} \frac{(b-a)^k (d-c)^l}{(k+1)! (l+1)!} \frac{\partial f (a+b, c+d)}{\partial x^k \partial y^l} \right|
\]
\[
\quad + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f (t, r) \, dr \, dt
\]
\[
\quad + \frac{(-1)^{m+1}}{(d-c) \cdot m!} \sum_{k=0}^{n-1} \left[ 1 + (-1)^k \right] \cdot \frac{2^{k+1}}{2^{k+1} (k+1)!} \int_c^d Q(r) \frac{\partial f (a+b, r)}{\partial x^k \partial y^m} \, dr
\]
\[
\quad + \frac{(-1)^{n+1}}{(b-a) \cdot n!} \sum_{l=0}^{m-1} \left[ 1 + (-1)^l \right] \cdot \frac{2^{l+1}}{2^{l+1} (l+1)!} \int_a^b P(t) \frac{\partial f (t, c+d)}{\partial t^n \partial y^l} \, dt
\]
(2.29) \[
\leq \frac{1}{(b-a) (d-c) m! n!} \left[ \int_a^{a+b} \int_c^{c+d} (t-a)^n (r-c)^m \left| \frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m} \right| \, dr \, dt \\
+ \int_{a+b}^b \int_c^{c+d} (b-t)^n (r-c)^m \left| \frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m} \right| \, dr \, dt \\
+ \int_a^a \int_{c+d}^d (t-a)^n (d-r)^m \left| \frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m} \right| \, dr \, dt \\
+ \int_{a+b}^b \int_{c+d}^d (b-t)^n (d-r)^m \left| \frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m} \right| \, dr \, dt \right].
\]

Using the power-mean inequality for each integral on the right-side of (2.29) and by the similar arguments as in proving Theorem 2.4, we get (2.28).

Corollary 2.6. If the conditions of Theorem 2.5 are satisfied and if \( m = n = 1 \) and \( s_1 = s_2 = 1 \), then we have the inequality

\[
\left| \frac{1}{(b-a) (d-c)} \int_a^b \int_c^d f(t,r) \, dr \, dt + \frac{1}{2} \left( \frac{a+b}{2} , \frac{c+d}{2} \right) \right| \\
\leq \left( \frac{1}{4} \right)^{\frac{1}{q}} \left( \frac{b-a}{2} \right) \left( \frac{d-c}{2} \right) \left\{ \left[ \frac{1}{36} D_{(1,1)}^q + \frac{1}{18} G_{(1,1)}^q + \frac{1}{9} I_{(1,1)}^q \right] + \frac{1}{18} H_{(1,1)}^q \right\}^{\frac{1}{q}}
\]

\[
+ \left[ \frac{1}{18} H_{(1,1)}^q + \frac{1}{36} D_{(1,1)}^q + \frac{1}{9} I_{(1,1)}^q \right] \}^{\frac{1}{q}}
\]

\[
+ \left[ \frac{1}{18} G_{(1,1)}^q + \frac{1}{36} C_{(1,1)}^q + \frac{1}{18} J_{(1,1)}^q + \frac{1}{9} F_{(1,1)}^q \right]^{\frac{1}{q}}
\]

\[
+ \left[ \frac{1}{9} F_{(1,1)}^q + \frac{1}{18} I_{(1,1)}^q + \frac{1}{18} J_{(1,1)}^q + \frac{1}{36} E_{(1,1)}^q \right]^{\frac{1}{q}}
\].

If we use the Hölder’s inequality instead of the power-mean inequality we get the following result.

Theorem 2.6. Let \( f : \Delta \subset [0, \infty) \times [0, \infty) \to [0, \infty) \), \( a < b \), \( c < d \), be a continuous mapping such that \( \frac{\partial^{m+n} f}{\partial t^m \partial r^n} \) exist on \( \Delta^o \) and \( \frac{\partial^{m+n} f}{\partial t^m \partial r^n} \in L(\Delta) \). If \( \frac{\partial^{m+n} f}{\partial t^m \partial r^n} \in L(\Delta) \), is \( s \)-convex on the co-ordinates on \( \Delta \), \( s_1, s_2 \in (0, 1] \) with \( s = \frac{s_1 + s_2}{2} \), \( m, n \in \mathbb{N} \), \( m, n \geq 1 \).
Then for $P(t)$ and $Q(r)$ defined as in Theorem 2.4 and $\frac{1}{p} + \frac{1}{q} = 1$ we have

\[ (2.30) \quad \left| \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(t, r) \, dr dt - \int_{c}^{d} \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \right| \]

Then for $\Delta$ concave on the co-ordinates on $s$, we have the inequality

\[
\left| \int_{c}^{d} \left( \frac{a+b}{2} \right) - \int_{c}^{d} \left( \frac{a+b}{2} \right) \right|
\]

The inequality (2.30) follows from the Hölder’s inequality and (1.7).

**Proof.** The inequality (2.30) follows from the Hölder’s inequality and (1.7). □

**Corollary 2.7.** Under the assumptions of Theorem 2.6, if $m = n = 1$ and $s_1 = s_2 = 1$, then for $\frac{1}{p} + \frac{1}{q} = 1$ we have the inequality

\[
\left| \int_{c}^{d} \left( \frac{a+b}{2} \right) - \int_{c}^{d} \left( \frac{a+b}{2} \right) \right|
\]

Our last result is for the s-concave functions can be stated as follows.

**Theorem 2.7.** Let $f : \Delta \subset [0, \infty) \times [0, \infty) \to [0, \infty)$, $a < b$, $c < d$, be a continuous mapping such that $\frac{\partial^{m+n} f}{\partial t^m \partial r^m}$ exist on $\Delta^c$ and $\frac{\partial^{m+n} f}{\partial t^m \partial r^m} \in L(\Delta)$. If $\frac{\partial^{m+n} f}{\partial t^m \partial r^m}$, $p > 1$, is s-concave on the co-ordinates on $\Delta$, $s_1, s_2 \in (0, 1]$ with $s = \frac{s_1 + s_2}{2}$, $m, n \in \mathbb{N}$, $m, n \geq 1$. Then for $P(t)$ and $Q(r)$ defined as in Theorem 2.4 and $\frac{1}{p} + \frac{1}{q} = 1$ we have

\[
\left| \int_{c}^{d} \left( \frac{a+b}{2} \right) - \int_{c}^{d} \left( \frac{a+b}{2} \right) \right|
\]
\[
\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,r) \, dr \, dt \right|
- \frac{n-1}{(d-c)\, m!} \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \left[ 1 + (-1)^k \right] \frac{1}{2^{k+1} (k+1)!} \int_c^d Q(r) \frac{\partial^{k+m} f \left( \frac{a+b}{2}, \frac{c+d}{2} \right)}{\partial x^k \partial y^l} \, dr
+ \frac{(-1)^{m+1}}{(d-c)\, m!} \sum_{k=0}^{m-1} \left[ 1 + (-1)^k \right] \frac{(b-a)^k}{2^{k+1} (k+1)!} \int_c^d P(t) \frac{\partial^{n+m} f \left( t, \frac{c+d}{2} \right)}{\partial t^m \partial r^m} \, dt
\] (2.31)

\[
\leq \frac{(b-a)^n (d-c)^m}{2^{n+m+1} n! \left( (np+1)(mp+1) \right)^{\frac{1}{2}}} \left[ \frac{4^{s_1+1} + 4^{s_2+1}}{2} \right]^{\frac{1}{2}} \left| \frac{\partial^{n+m} f \left( \frac{a+b}{2}, \frac{c+d}{2} \right)}{\partial t^m \partial r^m} \right|.
\]

**Proof.** The inequality (2.31) follows from the Hölder's inequality and the inequality (1.7) with inequalities in reversed direction. \(\square\)

**Corollary 2.8.** If the conditions of Theorem 2.7 are satisfied and if \( m = n = 1 \) and \( s_1 = s_2 = 1 \), then for \( \frac{1}{p} + \frac{1}{q} = 1 \) we have the inequality

\[
\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,r) \, dr \, dt + f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \right|
- \frac{1}{2(d-c)} \int_c^d f \left( \frac{a+b}{2}, r \right) \, dr
- \frac{1}{2(b-a)} \int_a^b f \left( t, \frac{c+d}{2} \right) \, dt
\] \[
\leq \frac{(b-a)(d-c)}{2^{p-\frac{2}{q}}(p+1)^{\frac{q}{2}}} \left| \frac{\partial^2 f \left( \frac{a+b}{2}, \frac{c+d}{2} \right)}{\partial t \partial r} \right|.
\]

**References**


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