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HOMOTHETIC FUNCTIONS WITH ALLEN'S PERSPECTIVE AND ITS GEOMETRIC APPLICATIONS

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Abstract. In this paper, we completely classify the homothetic functions of 2 variables by using their Allen's matrices. We give some applications of Allen's matrices to composite functions. Several geometric results are also obtained for graphs of homothetic functions.

1. introduction

In economics, a *production function* is a mathematical expression which denotes the physical relations between the output generated of a firm, an industry or an economy and inputs that have been used. Explicitly, a production function is a map which has non-vanishing first derivatives defined by

$$f : \mathbb{R}_{+}^{n} \longrightarrow \mathbb{R}_{+}, \quad f = f\left(x_1, x_2, \ldots, x_n\right),$$

where f is the quantity of output, n are the number of inputs and $x_1, x_2, \ldots, x_n$ are the inputs. For more details properties of production functions, see [15, 20, 21, 22, 23].

Almost all economic theories presuppose a production function, either on the firm level or the aggregate level. In this sense, the production function is one of the key concepts of mainstream neoclassical theories. By assuming that the maximum output technologically possible from a given set of inputs is achieved, economists using a production function in analysis are abstracting from the engineering and managerial problems inherently associated with a particular production process (cf. [4, 8]).

A production function $f\left(x_1, x_2, \ldots, x_n\right)$ is said to be *homogeneous of degree* p or $p-$*homogenous* if

$$f\left(tx_1, \ldots, tx_n\right) = t^p f\left(x_1, \ldots, x_n\right) \tag{1.1}$$

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holds for each $t \in \mathbb{R}_+$ for which (1.1) is defined. A homogeneous function of degree one is called *linearly homogeneous.* Many important properties of homogeneous production functions in economics was interpreted in terms of the geometry of their graphs by [4, 5, 11, 18, 19].

A. D. Vilcu and G. E. Vilcu [24] gave an exact classification for homogeneous production functions with proportional marginal rate of substitution and with constant elasticity of labor and capital.

A homothetic function is a production function of the form

$$f\left(x_1, \ldots, x_n\right) = F\left(h\left(x_1, \ldots, x_n\right)\right),$$

where $h\left(x_1, \ldots x_n\right)$ is homogeneous function of arbitrary given degree and F is a monotonically increasing function. Homothetic functions are production functions whose marginal technical rate of substitution is homogeneous of degree zero [9, 12, 16].

The most common quantitative indices of production factor substitutability are forms of the elasticity of substitution. R. G. D. Allen and J. R. Hicks [1] suggested two generalizations of Hicks' original two variables elasticity concept.

The first concept, called Hicks elasticity of substitution, is defined as follows. Let $f\left(x_1, \ldots, x_n\right)$ be a production function. Then *Hicks elasticity of substitution* of the $i-$th production variable with respect to the $j-$th production variable is given by

$$H_{ij}\left(\mathbf{x}\right) = -\frac{\dfrac{1}{x_i f_i} + \dfrac{1}{x_j f_j}}{\dfrac{f_{ii}}{\left(f_i\right)^2} - \dfrac{2f_{ij}}{f_i f_j} + \dfrac{f_{jj}}{\left(f_j\right)^2}} \quad \left(\mathbf{x} \in \mathbb{R}^n_+,\ i, j = 1, \ldots, n,\ i \neq j\right),$$

where $f_i = \partial f / \partial x_i$, $f_{ij} = \partial^2 f / \partial x_i \partial x_j$.

L. Losonczi [17] classified homogeneous production functions of 2 variables which have constant Hicks elasticiy of substitution. Then, the classification of L. Losonczi was extended to n variables by B.-Y. Chen [6].

The second concept, investigated by R. G. D. Allen and H. Uzawa [21], is the following. Let f be a production function. Then *Allen elasticity of substitution* of the $i-$th production variable with respect to the $j-$th production variable is defined by

$$A_{ij}\left(\mathbf{x}\right) = -\frac{x_1 f_1 + x_2 f_2 + \cdots + x_n f_n}{x_i x_j} \frac{D_{ij}}{D} \quad \left(\mathbf{x} = \left(x_1, \ldots, x_n\right) \in \mathbb{R}^n_+\right)$$

where $i, j = 1, \ldots, n,\ i \neq j$ and D is the determinant of the matrix

$$M\left(f\right) = \begin{pmatrix} 0 & f_1 & \cdots & f_n \\ f_1 & f_{11} & \cdots & f_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ f_n & f_{n1} & \cdots & f_{nn} \end{pmatrix}$$

and D_{ij} is the co-factor of the element f_{ij} in the determinant D ($D \neq 0$ is assumed). $M\left(f\right)$ is called the *Allen's matrix* and we call $\det\left(M\left(f\right)\right)$ the *Allen determinant.*

It is a simple calculation to show that in case of two variables Hicks elasticity of substitution coincides with Allen elasticity of substitution.

In this paper, we classify the homothetic production functions of varibles 2 whose Allen's matrix is singular. We deduce that Allen's matrices of generalized Cobb-Douglas and ACMS production functions are always regular. Some geometric applications of Allen's matrices of the homothetic production functions are also given.

2. Homothetic production functions with Allen determinants

Let $h(\mathbf{x})$ be an $p-$homogeneous function, $\mathbf{x} = (x_1, \ldots x_n) \in \mathbb{R}^n_+$, and $f = F(h(\mathbf{x}))$ a homothetic production function of n variables. The following lemma provides a useful relation between Allen's matrices of f and h.

Lemma 2.1. *Let $f = F(h(x_1, \ldots, x_n))$ be a twice differentiable homothetic production function of n variables. Then we have the following*

$$\det(M(f)) = (F')^{n+1} \det(M(h))$$

for Allen's matrices $M(f)$ and $M(h)$ of f and h, where $F' = F'(u)$ and $u = h(\mathbf{x})$.

Proof. Let $f = F(h(x_1, \ldots, x_n))$ be a twice differentiable homothetic production function. The Allen's matrix for $f = F(h(x_1, \ldots, x_n))$ is given by

$$M(f) = \begin{pmatrix} 0 & f_1 & \cdots & f_n \\ f_1 & f_{11} & \cdots & f_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ f_n & f_{n1} & \cdots & f_{nn} \end{pmatrix},$$

where $f_i = \frac{\partial f}{\partial x_i}$, $f_{ij} = \frac{\partial^2 f}{\partial x_i x_j}$. Then, in our case,

$$f_i = h_i F', \; f_{ij} = h_{ij} F' + h_i h_j F'' \text{ and } h_i = \frac{\partial h}{\partial x_i}, \; h_{ij} = \frac{\partial^2 h}{\partial x_i x_j}.$$

So, we write the Allen determinant for the homothetic production function $f = F(h(x_1, \ldots, x_n))$ as follows

$$\det(M(f)) = \begin{vmatrix} 0 & h_1 F' & \cdots & h_n F' \\ h_1 F' & h_{11} F' + (h_1)^2 F'' & \cdots & h_{1n} F' + h_1 h_n F'' \\ \vdots & \vdots & \cdots & \vdots \\ h_n F' & h_{1n} F' + h_1 h_n F'' & \cdots & h_{nn} F' + (h_n)^2 F'' \end{vmatrix}. \tag{2.1}$$

Now we apply some elementary transformations for the determinant from the formula (2.1). We replace $j-$th column by $j-$th column minus $h_{j-1} F''/F'$ times $1-$st column for all $j \in \{2, \ldots, n+1\}$. Then we derive

$$\det(M(f)) = \begin{vmatrix} 0 & h_1 F' & \cdots & h_n F' \\ h_1 F' & h_{11} F' & \cdots & h_{1n} F' \\ \vdots & \vdots & \cdots & \vdots \\ h_n F' & h_{n1} F' & \cdots & h_{nn} F' \end{vmatrix} = (F')^{n+1} \det(M(h))$$

which completes proof. □

If F is monotonically increasing, then Lemma 2.1 immediately implies he following.

Corollary 2.1. *Let $f = F(h(x_1, \ldots, x_n))$ be a twice differentiable homothetic production function of n variables. Then the singularity of Allen's matrix $M(f)$ only depends on the function $h(x_1, \ldots, x_n)$.*

Next result completely classifies 2-input homothetic production functions whose Allen's matrices are singular.

Theorem 2.1. *Let $f = F(h(u, v))$ be a homothetic production function of 2 varibles. Then the Allen's matrix of f is singular if and only if*

$$h(u, v) = (c_1 u + c_2 v)^p,$$

where c_1, c_2 are some nonzero constants and p is homogeneity degree of $h(u, v)$.

Proof. Let $f = F(h(u, v))$ be a homothetic production function. By Lemma 2.1, we get

$$\det(M(f)) = (F')^3 \left\{-h_{11}(h_2)^2 + 2h_1h_2h_{12} - (h_1)^2 h_{22}\right\}, \tag{2.2}$$

where $h_1 = h_u$, $h_2 = h_v$, $h_{uv} = h_{12}$, $h_{11} = h_{uu}$ and $h_{22} = h_{vv}$. If $M(f)$ is singular, then, because of $F' \neq 0$, the equality (2.2) reduces to

$$h_{11}(h_2)^2 - 2h_1h_2h_{12} + (h_1)^2 h_{22} = 0. \tag{2.3}$$

On the other hand, since h is a homogeneous of degree p, we write by using Euler Homogeneous Function Theorem

$$uh_u + vh_v = ph. \tag{2.4}$$

Taking partial derivatives with respect to u and v in equality (2.4), we obtain

$$uh_{11} + vh_{12} = (p-1)h_1, \tag{2.5}$$

$$uh_{12} + vh_{22} = (p-1)h_2. \tag{2.6}$$

For (2.5) and (2.6) we have two cases:

Case 1. $p = 1$. From (2.5) and (2.6), we have

$$h_{11} = -\left(\frac{v}{u}\right)h_{12} \text{ and } h_{22} = -\left(\frac{u}{v}\right)h_{12}. \tag{2.7}$$

After substituting (2.7) into (2.3), we derive

$$\left[(h_1)^2\left(\frac{u}{v}\right) + 2h_1h_2 + (h_2)^2\left(\frac{v}{u}\right)\right]h_{12} = 0$$

or

$$\left[(h_1)^2 u^2 + 2h_1h_2uv + (h_2)^2 v^2\right]h_{12} = 0. \tag{2.8}$$

From (2.4) and (2.8), we conclude that $h_{12} = 0 \Rightarrow h = \varphi(u) + \psi(v)$.

Since it is assumed that $p = 1$, that is, h is linearly homogeneous, we have to put $h = c_1u + c_2v$. This completes case 1.

Case 2. If $p \neq 1$, the let us choose the functions $\hat{F}(t) = F(t^p)$, and $\hat{h}(u,v) = (h(u,v))^{\frac{1}{p}}$. Thus $\hat{h}$ is a linear homogeneous function such that $f(u,v) = F(h(u,v)) = \hat{F}\left(\hat{h}(u,v)\right)$. Since the Allen's matrix of f is singular and $\deg \hat{h} = 1$ we may apply the some argument given in case 1 to deduce that $\hat{h}(u,v) = c_1 u + c_2 v$ for some constants c_1, c_2. Therefore we derive that $h(u,v) = (c_1 u + c_2 v)^p$.

The converse can be verify by direct calculation. □

3. Applications of allen determinants to composite functions

Theorem 3.1. *Let $F(u)$ be twice differentiable function with $F'(u) \neq 0$ and let*

$$f(\mathbf{x}) = F(c_1^p x_1^p + \cdots + c_n^p x_n^p)$$

be the composite of F and $r(\mathbf{x}) = c_1^p x_1^p + \cdots + c_n^p x_n^p$, where $p, c_1, \ldots, c_n$ are nonzero constants, $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ and $r(\mathbf{x}) \neq 0$. Then the Allen's matrix of f is singular if and only if r is linearly homogeneous.

Proof. Let assume that the Allen's matrix of the composite function $f = F \circ r$ is singular. Since a homothetic function is a composite function, we can apply Lemma 2.1, obtained for the homothetic functions, to the composite function $f(\mathbf{x}) = F(r(x_1, \ldots, x_n))$. Hence we write

$$\det(M(f)) = (F')^{n+1} \det(M(r))$$

for the Allen's matrices $M(f)$ and $M(r)$. Under the hypothesis of theorem we have that $0 = \det(M(r))$. Thus, putting the derivatives $r_i = \partial r / \partial x_i = p c_i^p x_i^{p-1}$, $r_{ii} = \partial^2 r / \partial x_i^2 = p(p-1) c_i^p x_i^{p-2}$ into the determinant of the Allen's matrix $M(r)$, we get

$$0 = \det(M(r)) = \begin{vmatrix} 0 & p c_1^p x_1^{p-1} & \cdots & p c_n^p x_n^{p-1} \\ p c_1^p x_1^{p-1} & p(p-1) c_1^p x_1^{p-2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ p c_n^p x_n^{p-1} & 0 & \cdots & p(p-1) c_n^p x_n^{p-2} \end{vmatrix}. \tag{3.1}$$

By calculating the determinant from the formula (3.1), we find

$$0 = \det(M(r)) = -p^{n+1} (p-1)^{n-1} \prod_{i=1}^{n} c_i^p x_i^{p-2} r(\mathbf{x}), \tag{3.2}$$

and therefore we deduce from (3.2) that r is linearly homogeneous.

The converse can be verified directly. □

In 1961, K. J. Arrow, H. B. Chenery, B. S. Minhas and R. M. Solow [2] introduced a two-factor production function given by

$$Q = F \cdot (aK^r + (1-a) L^r)^{\frac{1}{r}},$$

where Q is the output, F the factor productivity, a the share parameter, K and L the primary production factors, $r = (s-1)/s$, and $s = 1/(1-r)$ is the elasticity of substitution.

The *generalized ACMS production function of* n variables is given by

$$f(x_1, \ldots, x_n) = \gamma \left(a_1^p x_1^p + \cdots + a_n^p x_n^p\right)^{\frac{d}{p}},$$

where $p \neq 0$, $p < 1$, $d, \gamma > 0$ and $a_i > 0$ for all $i = 1, \ldots, n$.

Corollary 3.1. *The Allen's matrix of generalized ACMS production function are always regular.*

Proof. Let us choose that twice differentiable function $F(u)$ with $F'(u) \neq 0$, $F(u) = \gamma u^{\frac{d}{p}}$, for $\gamma, d > 0$ and $p \neq 0$, $p < 1$, and also twice differentiable function r as $r(x_1, \ldots, x_n) = c_1^p x_1^p + \cdots + c_n^p x_n^p$, where $c_1, \ldots, c_n > 0$ and $(x_1, \ldots, x_n) \in \mathbb{R}_+^n$. Then, we obtain the composite function

$$f = F \circ r = \gamma \left(c_1^p x_1^p + \cdots + c_n^p x_n^p\right)^{\frac{d}{p}},$$

which is a generalized ACMS production function. By Theorem 3.1, if the Allen's matrix of the generalized ACMS production function $f = F \circ r$ is singular, then p is equal to one which is not possible. This completes the proof. □

Theorem 3.2. *Let $F : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a twice differentiable function with $F'(u) \neq 0$ and q twice differentiable function given by $q : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$, $q(\mathbf{x}) = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Then the Allen matrix of the composite function $f = F \circ q$ is invertible unless one of the following occurs*

(i) $\sum_{i=1}^n \alpha_i = 0$ *or*
(ii) *at least one of the* $\alpha_1, \ldots, \alpha_n$ *vanishes.*

Proof. Let f be a twice differentiable composite function given by

$$f = F\left(x_1^{\alpha_1} \ldots x_n^{\alpha_n}\right) \tag{3.3}$$

for $F'(u) \neq 0$. We can apply the Lemma 2.1 for the composite function given by (3.3). Thus, we have a relation between Allen determinants of f and q as follows

$$\det(M(f)) = (F')^{n+1} \det(M(q)). \tag{3.4}$$

If the Allen's matrix $M(f)$ of f is singular, then it follows from (3.4) that

$$0 = \det(M(q)) = \begin{vmatrix} 0 & \frac{a_1}{x_1} q & \cdots & \frac{a_n}{x_n} q \\ \frac{a_1}{x_1} q & \frac{a_1(a_1-1)}{(x_1)^2} q & \cdots & \frac{a_1 a_n}{x_1 x_n} q \\ \vdots & \vdots & \cdots & \vdots \\ \frac{a_n}{x_n} q & \frac{a_1 a_n}{x_1 x_n} q & \cdots & \frac{a_n(a_n-1)}{(x_n)^2} q \end{vmatrix}. \tag{3.5}$$

By using Gauss elimination method for the determinant from the formula (3.5), we derive

$$0 = (-1)^n \, q(x_1, \ldots, x_n)^{n+1} \prod_{i=1}^{n} \frac{a_i}{(x_i)^2} \sum_{i=1}^{n} \alpha_i,$$

which gives cases (i) and (ii).

Conversely, it is straightforward to verify that each one of cases (i) and (ii) implies that f has a singular Allen's matrix. □

In 1928, C. W. Cobb and P. H. Douglas introduced in [13] a famous two-factor production function $Y = bL^k C^{1-k}$, where b presents the total factor productivity, Y the total production, L the labor input and C the capital input. This function is nowadays called *Cobb-Douglas production function.* In its generalized form the Cobb-Douglas production function may be expressed as $f(x_1, \ldots, x_n) = bx_1^{\alpha_1} \ldots x_n^{\alpha_n}$, where $b > 0$ and $\alpha_i > 0$ for all $i = 1, \ldots, n$.

Corollary 3.2. *The Allen's matrix of generalized Cobb-Douglas production function are always regular.*

Proof. Let q be a twice differentiable function given by $q(x_1, \ldots, x_n) = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, for $\alpha_1, \ldots, \alpha_n > 0$, and $F(u) = bu$, $b > 0$. Thus we have generalized Cobb-Douglas production function as follows $f = F \circ q = bx_1^{\alpha_1} \ldots x_n^{\alpha_n}$. By Theorem 3.2, if the Allen's matrix of the generalized Cobb-Douglas production function is singular, then we get either $\alpha_1 + \cdots + \alpha_n = 0$ or at least one of the $\alpha_1, \ldots, \alpha_n$ vanishes. However, both cases are not possible for a generalized Cobb-Douglas production function. Therefore the proof is completed. □

4. Geometric interpretations of allen determinants

Let M^n be a hypersurface of a Euclidean space $\mathbb{E}^{n+1}$. For general references on the geometry of hypersurfaces see [3, 4].

The *Gauss map* $\nu : M^n \longrightarrow S^{n+1}$ maps M^n to the unit hypersphere S^n of $\mathbb{E}^{n+1}$. The differential $d\nu$ of the Gauss map ν is known as *shape operator* or Weingarten map. Denote by T_pM^n the tangent space of M^n at the point $p \in M^n$. Then, for $v, w \in T_pM^n$, the shape operator A_p at the point $p \in M^n$ is defined by $g(A_p(v), w) = g(d\nu(v), w)$, where g is the induced metric tensor on M^n from the Euclidean metric on $\mathbb{E}^{n+1}$.

The determinant of the shape operator A_p is called the *Gauss-Kronocker* curvature. When $n = 2$, the Gauss-Kronocker curvature is simply called *Gauss curvature.*

Let (N, g) be a Riemannian manifold. For more detailed properties of geometric structures on Riemannian manifolds, see [14]. A *Riemannian connection*, also called Levi-Civita connection, on the Riemannian manifold (N, g) is an affine connection which is compatible with metric, i. e. $\nabla g = 0$ and symmetric, i.e, $\nabla_X Y - \nabla_Y X = [X, Y]$ for any vector fields X and Y on N, where $[,]$ is the Lie bracket.

The *Riemannian curvature tensor* R is given in terms of ∇ by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

A Riemannian manifold is called a *flat space* if its Riemann curvature tensor vanishes identically.

Let σ be a two dimensional subspace of the tangent space T_pN and let $u, v \in \sigma$ be two linearly independent vectors such that $\sigma = Sp(u, v)$. Then the *sectional curvature* of σ at the point $p \in N$ is a real number defined by

$$K(u, v) = K(\sigma) = \frac{g(R(u, v)u, v)}{g(u, u)g(v, v) - g(u, v)^2}.$$

The *Ricci tensor* of a Riemannian manifold N at a point $p \in N$ is defined to be the trace of the linear map $T_pN \longrightarrow T_pN$ given by $w \longmapsto R(w, u)v$. A Riemannian manifold is called *Ricci-flat* if its Ricci tensor vanishes identically. Ricci-flat 3-manifolds are always flat.

The following result is well-known from [4, 7].

Proposition 4.1. *For the production hypersurface of* $\mathbb{E}^{n+1}$ *defined by*

$$L(\mathbf{x}) = (x_1, \ldots, x_n, f(x_1, \ldots, x_n)),$$

we have the following three statements.

(i) *The Gauss-Kronecker curvature* G *is* $G = \frac{\det(f_{ij})}{w^{n+2}}$, *with* $w = \sqrt{1 + \sum_{i=1}^{n} f_i^2}$.

(ii) *The sectional curvature* K_{ij} *of the plane section spanned by* $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}$ *is given by* $K_{ij} = \frac{f_{ii}f_{jj} - f_{ij}^2}{w^2\left(1+f_i^2+f_j^2\right)}$.

(iii) *The Riemannian curvature tensor* R *and the metric tensor* g *satisfy*

$$g\left(R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right) = \frac{f_{il}f_{jk} - f_{ik}f_{jl}}{w^4}.$$

One of the B.-Y. Chen's geometric interpretions regarding homothetic production functions from [12] is as follows.

Corollary 4.1. *Let* $f(x, y) = F(h(x, y))$ *be a homothetic production function. Then the graph of* f *is a flat surface if and only if either*

(i) $f(x, y)$ *is linearly homogenous, or*

(ii) $F(u)$ *is a strictly increasing function and* $h(x, y)$ *is a perfect substitute.*

Corollary 4.2. *Let* $f(u, v) = F(h(u, v))$ *be a homothetic production function of* 2 *variables with* $F'' \neq 0$. *Then the graph of* f *is a flat surface then the Allen's matrix of* f *is singular.*

Proof. Let $f = F \circ h$ be a homothetic production function of 2 variables with $F'' \neq 0$. Let us assume that the graph of f is a flat surface. Then, by Corollary 4.1, F is a strictly increasing function and $h(u, v)$ is a perfect substitute, i. e. h is of the form $c_1u + c_2v$, for some nonzero constants c_1, c_2. It means from Theorem 2.1 that the Allen's matrix of f is singular. $\square$

Remark 4.1. The converse of the Corollary 4.2 only holds in case the homogeneity degree of $h(u, v)$ is one.

Theorem 4.1. *Let $f = F\left(h\left(x_1, x_2, x_3\right)\right)$ be twice differentiable homothetic production function of 3 variables. If the graph of h in $\mathbb{E}^4$ is a flat space, then the Allen's matrix of $f = F \circ h$ is singular.*

Proof. Let $f = F \circ h$ be twice differentiable homothetic production function of 3 variables. By Lemma 2.1, we write

$$\det\left(M\left(f\right)\right) = \left(F'\right)^4 \det\left(M\left(h\right)\right) \tag{4.1}$$

for the Allen's matrices $M\left(f\right)$ and $M\left(h\right)$. Then the Allen's matrix of the function h is

$$M\left(h\right) = \begin{pmatrix} 0 & h_1 & h_2 & h_3 \\ h_1 & h_{11} & h_{12} & h_{13} \\ h_2 & h_{21} & h_{22} & h_{23} \\ h_3 & h_{31} & h_{32} & h_{33} \end{pmatrix}.$$

Calculating the determinant for the matrix $M\left(h\right)$, we derive

$$\begin{aligned} \det\left(M\left(h\right)\right) = & -h_1^2\left(h_{22}h_{33} - h_{23}^2\right) - h_2^2\left(h_{11}h_{33} - h_{13}^2\right) \\ & - h_3^2\left(h_{11}h_{22} - h_{12}^2\right) + 2h_1h_2\left(h_{12}h_{33} - h_{13}h_{23}\right) \\ & - 2h_1h_3\left(h_{12}h_{23} - h_{13}h_{22}\right) + 2h_2h_3\left(h_{11}h_{23} - h_{13}h_{12}\right). \end{aligned} \tag{4.2}$$

On the other hand, if the production hypersurface of h is a flat space, then, by the statement (iii) of Proposition 4.1, we have

$$h_{kn}h_{ml} = h_{kl}h_{mn}, \tag{4.3}$$

i. e. all brackets in the formula (4.2) are zero. From (4.2) and (4.3) we obtain $\det\left(M\left(h\right)\right) = 0$, and by (4.1), it completes the proof. □

Remark 4.2. Theorem 4.1 also holds in case the graph of h in $\mathbb{E}^4$ is Ricci-flat or has vanishing sectional curvature function.

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