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A NOTE ON GENERALIZED QUASI-BAER RINGS

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Abstract. A ring with identity is generalized quasi-Baer if for any ideal I of R, the right annihilator of I^n is generated by an idempotent for some positive integer n, depending on I. We study the generalized quasi-Baerness of $R[x;\sigma;\delta]$ over a generalized quasi-Baer ring R where σ is an automorphism of R.

1. Introduction

Throughout this paper R denotes an associative ring with identity. Recall that R is (quasi-) Baer if the right annihilator of every (right ideal) nonempty subset of R is generated as a right ideal by an idempotent. It is easy to see that the Baer and quasi-Baer properties are left-right symmetric for any ring. In [8] Kaplansky introduced Baer rings to abstract various properties of AW^*-algebras and von Neumann algebras. Clark defined quasi-Baer rings in [4] and used them to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra.

As a generalization of Baer rings, R is called a right (left) p.p.- ring if every principal right (left) ideal is projective. Equivalently, if the right (left) annihilator of any element of R is generated by an idempotent of R. R is called a p.p.-ring (also called a Ricart ring [10, p. 18]), if it is both right and left p.p.-ring. In [6] Huh et al. defined a ring R to be called generalized right p.p.-ring if for any $x \in R$ the right annihilator of x^n is generated by an idempotent for some positive integer n. von Neumann regular rings are p.p.-rings by Goodearl [5, Theorem 1.1], and π-regular rings are generalized p.p.-rings.

In [2] Birkenmeier et al. introduced the concept of principally quasi-Baer rings. A ring R is right (left) principally quasi-Baer if the right (left) annihilator of a principal right (left) ideal of R is generated by an idempotent. In [9] Moussavi et al. initiated

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the study of generalized right (principally) quasi-Baer rings. A ring R is generalized right (principally) quasi-Baer if for any (principal) right ideal I of R, the right annihilator of I^n is generated by an idempotent for some positive integer n, depending on I.

In [3, Theorem 1.2] Birkenmeier et al. showed that if a ring R is quasi-Baer then $R[x;\sigma]$ is a quasi-Baer ring when σ is an automorphism of R. To study quasi-Baerness (right principally quasi-Baerness) of Ore extensions of a quasi-Baer (right principally quasi-Baer) ring, in [7] Hong et al. proved that $R[x;\sigma;\delta]$ over a quasi-Baer ring R is a quasi-Baer ring when σ is an automorphism of R. In this paper we prove that $R[x;\sigma;\delta]$ over a generalized quasi-Baer ring is a generalized quasi-Baer ring.

2. Main Results

Given a ring R, right annihilator of I in R is $r_R(I) = \{a \in R \mid Ia = 0\}$. Left annihilator is defined analogously. The Ore extension $R[x;\sigma;\delta]$ is the polynomial ring over R, subject to $xr = \sigma(r)x + \delta(r)$ for any $r \in R$. Note that $R[x;\sigma;\delta]$ is written by $R[x;\sigma]$ and $R[x;\delta]$ when $\delta = 0$ and σ is the identity map, respectively. Following [7], if R is a semiprime ring with $\sigma(I) \subseteq I$ for any ideal I of R then R is a quasi-Baer ring if and only if $R[x;\sigma;\delta]$ is a quasi-Baer ring. By same method we prove following theorems.

Theorem 2.1. *If R is a generalized quasi-Baer ring, then $R[x;\sigma;\delta]$ is a generalized quasi-Baer ring.*

Proof. Let I be an ideal of $R[x;\sigma;\delta]$ and $J = \{a \in R \mid ax^n +$ terms of lower degree $\in I$ for some $0 \leq n \in \mathbb{Z}\}$. Then J is an ideal of R. Note that $\sigma(J) \subseteq J$, and so $J \subseteq \sigma^{-1}(J)$. Let $\acute{J} = \sum_{k\geq 1}\sigma^{-k}(J)$. Then $\acute{J}$ is also an ideal of R and $J \subseteq \acute{J}$. Moreover, $\sigma(\acute{J}) = \acute{J} = \sigma^{-1}(\acute{J})$. If $d \in \acute{J}$, $d = \sigma^{-k_1}(d_1)+\cdots+\sigma^{-k_i}(d_i)$, where $d_k \in J$ for $k \in \{1,\ldots,i\}$. Since $\sigma(J) \subseteq J$, $\sigma(d) = \sigma(\sigma^{-k_1}(d_1))+\cdots+\sigma(\sigma^{-k_i}(d_i)) = \sigma^{-k_1}(\sigma(d_1))+ \cdots + \sigma^{-k_i}(\sigma(d_i)) \in \acute{J}$. Thus $\sigma(\acute{J}) \subseteq \acute{J}$ and $\acute{J} \subseteq \sigma^{-1}(\acute{J})$, obtaning from the difinition of $\acute{J}$, $\sigma^{-1}(\acute{J}) \subseteq \acute{J}$. Hence, $\sigma^{-1}(\acute{J}) = \acute{J}$ and so $\sigma(\acute{J}) = \acute{J}$, obtaning $\sigma^u(\acute{J}) = \acute{J}$ for any integer u. By assumption, $r_R(\acute{J}^n) = eR$ for some $e^2 = e \in R$ and so $\acute{J}^n e = 0$. We will show that $r_{R[x;\sigma;\delta]}(I^n) = eR[x;\sigma;\delta]$. Let $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x;\sigma;\delta]$ and $f(x)g(x) = 0$ for any $f(x) = a_0 + a_1x + \cdots + a_mx^m \in I^n$. Note that for any $r \in R$,

$$rx^i = x^i\sigma^{-i}(r) - \left(\sum_{s+t=i-1}\sigma^s\delta\sigma^t(\sigma^{-i}(r))\right)x^{i-1}$$
$$-\cdots-\left(\sum_{s+t=i-1}\delta^s\sigma\delta^t(\sigma^{-i}(r))) - \delta^i(\sigma^{-i}(r)\right).$$

We can rewrite $f(x) = c_0 + xc_1 + \cdots + x^mc_m$, where $c_m = \prod_{i=1}^n \sigma^{\sum_{k=i}^n(-m_k)}(a_{m_i}) \in \acute{J}^n$.

Thus we have the following

$$(c_0 + xc_1 + \cdots + x^mc_m)(b_0 + b_1x + \cdots + b_nx^n) = 0. \tag{2.1}$$

Claim 2.1. $\acute{J}^n b_n = 0$ and $b_n = eb_n$.

From equation (2.1), we have $c_m b_n = 0$. Since

$$c_m = \prod_{i=1}^{n} \sigma^{\sum_{k=i}^{n}(-m_k)}(a_{m_i}) \in \acute{J}^n,$$

$$\prod_{i=1}^{n} \sigma^{\sum_{k=i}^{n}(-m_k)}(a_{m_i})(b_n) \in \acute{J}^n b_n,$$

we have $a_{m_1}\sigma^{m_1}(a_{m_2})\sigma^{m_1+m_2}\cdots(a_{m_{m-1}})\sigma^{m_1+m_2+\cdots+m_{m-1}}(a_{m_m})\sigma^{m_1+m_2+\cdots+m_m}(b_n) = 0$. Then for any integer $k \geq 0$, $\sigma^{-k}(a_{m_1})\sigma^{-k+m_1}(a_{m_2}) + \cdots + \sigma^{-k+m_1+\cdots+m_m}(b_n) = 0$. Thus $\acute{J}^n\sigma^{-k+m_1+c+m_m}(b_n) = 0$. Since $\sigma^{k-m_1-\cdots-m_m}(\acute{J}^n) = \acute{J}^n$ then $\acute{J}^n b_n = 0$. Hence $b_n \in r_R(\acute{J}^n) = eR$, and therefore $b_n = eb_n$.

Claim 2.2. $c_i e = 0$ for any $0 \leq i \leq m$.

Since $c_m \in \acute{J}^n$, $f(x)e = (c_0 + xc_1 + \cdots + x^m c_m)e = c_0 e + xc_1 e + \cdots + x^{m-1}c_{m-1}e \in I^n$. Then $\sigma^{m-1}(c_{m-1}e) \in J^n \subseteq \acute{J}^n$, and so $c_{m-1}e \in \sigma^{-(m-1)}(\acute{J}^n) \subseteq \acute{J}^n$. Hence, $c_{m-1}e = (c_{m-1}e)e \in \acute{J}^n e = 0$. Continuing this process, We have $c_i e = 0$ for any $0 \leq i \leq m$.

Claim 2.3. $g(x) \in eR[x;\sigma;\delta]$.

From Claims 2.1 and 2.2, we have

$$\begin{aligned} 0 &= f(x)g(x) \\ &= x^m c_m b_n x^n + (x^m c_m b_{n-1}x^{n-1} + x^{m-1}c_{m-1}b_n x^n) + \cdots + c_0 b_0 \\ &= (x^m c_m b_{n-1}x^{n-1} + x^{m-1}c_{m-1}eb_n x^n) + \cdots + c_0 b_0 \\ &= x^m c_m b_{n-1}x^{n-1} + \cdots + c_0 b_0. \end{aligned} \tag{2.2}$$

So, $\sigma^m(c_m b_{m-1}) = 0$ and we have that $c_m b_{n-1} = 0$. Then by Claim 2.1, $\acute{J}^n b_{n-1} = 0$, and it follows that $b_{n-1} = eb_{n-1}$. Then equation (2.2) becomes

$$\begin{aligned} 0 &= (x^m c_m b_{n-2}x^{n-2} + x^{m-1}c_{m-1}b_{n-1}x^{n-1} + x^{m-2}c_{m-2}b_n x^n) + \cdots + c_0 b_0 \\ &= (x^m c_m b_{n-2}x^{n-2} + x^{m-1}c_{m-1}eb_{n-1}x^{n-1} + x^{m-2}c_{m-2}eb_n x^n) + \cdots + c_0 b_0 \\ &= x^m c_m b_{n-2}x^{n-2} + \cdots + c_0 b_0. \end{aligned}$$

Thus $c_m b_{n-2} = 0$, and so $\acute{J}^n b_{n-2} = 0$. Hence, we also have $b_{n-2} = eb_{n-2}$. Continuing this process, we have $b_j = eb_j$ for any $0 \leq j \leq n$. Consequently, we have $g(x) = b_0 + b_1 x + \cdots + b_n x^n = eb_0 + eb_1 x + \cdots + eb_n x^n \in eR[x;\sigma;\delta]$.
Conversely, let $f(x) = c_0 + xc_1 + \cdots + x^m c_m \in I^n$. Then by Claim 2.2, $f(x)e = 0$. Thus $I^n e = 0$, and so $eR[x;\sigma;\delta] \subseteq r_{R[x;\sigma;\delta]}(I^n)$. Consequently, $R[x;\sigma;\delta]$ is a generalized quasi-Baer ring. $\square$

In the following proposition we see that the converse of Theorem 2.1 is correct when R be a semiprime ring.

Proposition 2.1. *Let R be a semiprime ring with $\sigma(I) \subseteq I$ for any ideal I of R. If $R[x;\sigma;\delta]$ is a generalized quasi-Baer ring, then R is a generalized quasi-Baer ring.*

Proof. Let J be an ideal of R. Since $R[x;\sigma;\delta]$ is a generalized quasi-Baer ring, then $r_{R[x;\sigma;\delta]}((JR[x;\sigma;\delta])^n) = e(x)R[x;\sigma;\delta]$ for $e(x)^2 = e(x) = e_0 + e_1x + \cdots + e_tx^t \in R[x;\sigma;\delta]$ and a positive integer n. Since $e(x)R[x;\sigma;\delta]$ is an ideal, $(e(x) - 1)R[x;\sigma;\delta]e(x) = 0$. Hence we have $e_tR\sigma^t(e_t) = 0$. Since R is semiprime, $\sigma^t(e_t)Re_t = 0$. Then $\sigma^t(e_t)R\sigma^t(e_t) \subseteq \sigma^t(e_t)R\sigma^t(Re_tR) \subseteq \sigma^t(e_t)Re_tR = 0$ and so $\sigma^t(e_t)R\sigma^t(e_t) = 0$. Since σ is an automorphism of R and R is semiprime, $e_t = 0$. Continuing this process, we have $e(x) = e_0$, where $e_0^2 = e_0 \in R$. Thus $r_{R[x;\sigma;\delta]}((JR[x;\sigma;\delta])^n) = e_0R[x;\sigma;\delta]$. We claim that $r_R(J^n) = e_0R$. Obviously $e_0R \subseteq r_R(J^n)$. If $b \in r_R(J^n)$ then $J^nb = 0$. Since $\sigma^s(b) \in RbR$, $J^n\sigma^s(b) \subseteq J^nbR = 0$ and so $J^n\sigma^s(b) = 0$ for any integer $s \geq 0$. Then by [7, Lemma 9], we have $J^n\sigma^{m_1}\delta^{n_1}\cdots\sigma^{m_u}\delta^{n_u}(b) = 0$ for any integers $m_v, n_v \geq 0$. Thus $J^nR[x;\sigma;\delta]b = 0$ and so $b \in r_{R[x;\sigma;\delta]}((JR[x;\sigma;\delta])^n) = e_0R$. Hence, $r_R(J^n) \subseteq e_0R$. Consequently, R is a generalized quasi-Baer ring. □

Corollary 2.1. *Let R be a semiprime ring with $\sigma(I) \subseteq I$ for any ideal I of R. Then $R[x;\sigma;\delta]$ is a generalized quasi Baer ring if and only if $R[x;\sigma;\delta]$ is a quasi-Baer ring.*

Proof. If $R[x;\sigma;\delta]$ is a generalized quasi Baer ring then R is a generalized quasi-Baer ring. Since R is a semiprime ring by [9, Proposition 2.2], R is a quasi-Baer ring. Then by [7, theorem 1], $R[x;\sigma;\delta]$ is a quasi-Baer ring. The converse is clear. □

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