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HESSIAN DETERMINANTS OF COMPOSITE FUNCTIONS WITH APPLICATIONS FOR PRODUCTION FUNCTIONS IN ECONOMICS

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Abstract. B.-Y. Chen [7] derived an explicit formula for the Hessian determinants of composite functions defined by $f = F(h_1(x_1) + \cdots + h_n(x_n))$. In this paper, we introduce a new formula for the Hessian determinants of composite functions of the form

$$f = F(h_1(x_1) \times \cdots \times h_n(x_n)).$$

Several applications of the new formula to the well-known Cobb-Douglas production functions in economics are also given.

1. Introduction

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $f = f(x_1, \ldots, x_n)$, be a twice differentiable function. Then the *Hessian matrix* $\mathcal{H}(f)$ is the square matrix $\left(f_{x_i x_j}\right)$ of second-order partial derivatives of the function f. If the second-order partial derivatives of f are all continuous in a neighborhood D, then the Hessian of f is a symmetric matrix throughout D (cf. [7]).

For applications of Hessian matrices to production models in economics, we refer the reader to B.-Y. Chen's papers [7, 8]. In addition, the Hessian matrices have an important geometric interpretation as following.

Let $f = f(x_1, \ldots, x_n)$ be a twice differentiable real valued function. Then the Hessian matrix $\mathcal{H}(f)$ of f is singular if and only if the graph of f in $\mathbb{R}^{n+1}$ has null Gauss-Kronecker curvature [7].

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On the other hand, the *bordered Hessian matrix* of the function f is given by

$$\mathcal{H}^B(f) = \begin{pmatrix} 0 & f_{x_1} & \cdots & f_{x_n} \\ f_{x_1} & f_{x_1x_1} & \cdots & f_{x_1x_n} \\ \vdots & \vdots & \cdots & \vdots \\ f_{x_n} & f_{x_nx_1} & \cdots & f_{x_nx_n} \end{pmatrix},$$

where $f_{x_i} = \frac{\partial f}{\partial x_i}$, $f_{x_ix_j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ for all $i, j \in \{1, \ldots, n\}$.

The bordered Hessian matrices of functions have important applications in many areas of mathematics. For instance, the bordered Hessian matrices are used to analyze quasi-convexity and quasi-concavity of the functions. If the signs of the bordered principal diagonal determinants of the bordered Hessian matrix of a function are alternate (resp. negative), then the function is quasi-concave (resp. quasi-convex). For more detailed properties see [4, 12, 13, 14].

Another example is the application of the bordered Hessian matrices to elasticity of substitutions of production functions in economics. Explicitly, let $f = f(x_1, \ldots, x_n)$ be a production function. Then the *Allen's elasticity of substitution* of the $i-$th production variable with respect to the $j-$th production variable is defined by

$$A_{ij}(\mathbf{x}) = -\frac{(x_1 f_{x_1} + x_2 f_{x_2} + \cdots + x_n f_{x_n})}{x_i x_j} \frac{\mathcal{H}^B(f)_{ij}}{\det \mathcal{H}^B(f)}$$

for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$, $i, j \in \{1, \ldots, n\}$, $i \neq j$, where $\mathcal{H}^B(f)_{ij}$ is the co-factor of the element $f_{x_ix_j}$ in the determinant of $\mathcal{H}^B(f)$ [15, 17]. The authors [2, 3] called the bordered Hessian matrix $\mathcal{H}^B(f)$ by *Allen's matrix* and $\det \mathcal{H}^B(f)$ by *Allen determinant.*

Let f be a composite function of the form

$$f(\mathbf{x}) = F(h_1(x_1) \times \cdots \times h_n(x_n)). \tag{1.1}$$

In [3] for the composite functions of the form (1.1), an Allen Determinant Formula was obtained as follows

$$\det\left(\mathcal{H}^B(f)\right) = -u^{n+1}\left(\dot{F}\right)^{n+1} \sum_{j=1}^{n} \left(\frac{h_1'}{h_1}\right)' \cdots \left(\frac{h_{j-1}'}{h_{j-1}}\right)' \left(\frac{h_j'}{h_j}\right)^2 \left(\frac{h_{j+1}'}{h_{j+1}}\right)' \cdots \left(\frac{h_n'}{h_n}\right)',$$

where $h_j' = \frac{dh_j}{dx_j}$ and $\dot{F} = \dot{F}(u)$ for $u = h_1(x_1) \times \cdots \times h_n(x_n)$.

In this paper, we obtain a new formula for Hessian determinants $\mathcal{H}(f)$ of composite functions of the form (1.1). Several applications of the new formula to production functions in economics are also given.

2. Production models in economics

In economics, a *production function* is a mathematical expression which denotes the physical relations between the output generated of a firm, an industry or an economy and inputs that have been used. Explicitly, a production function is a map which has non-vanishing first derivatives defined by

$$f:\mathbb{R}_+^n \longrightarrow \mathbb{R}_+, \quad f=f\left(x_1,\ldots,x_n\right),$$

where f is the quantity of output, n are the number of inputs and $x_1,\ldots,x_n$ are the inputs.

A production function $f\left(x_1,\ldots,x_n\right)$ is said to be *homogeneous of degree* p or $p-$*homogenous* if

$$f\left(tx_1,\ldots,tx_n\right)=t^p f\left(x_1,\ldots,x_n\right) \tag{2.1}$$

holds for each $t\in\mathbb{R}_+$ for which (2.1) is defined. A homogeneous function of degree one is called *linearly homogeneous.* If $p>1$, the function exhibits increasing return to scale, and it exhibits decreasing return to scale if $p<1$. If it is homogeneous of degree 1, it exhibits constant return to scale [5].

Many important properties of homogeneous production functions in economics were interpreted in terms of the geometry of their graphs by [5, 9, 10, 18, 19].

In 1928, C. W. Cobb and P. H. Douglas introduced [11] a famous two-factor production function

$$Y=bL^kC^{1-k},$$

where b presents the total factor productivity, Y the total production, L the labor input and C the capital input. This function is nowadays called *Cobb-Douglas production function.*

The Cobb–Douglas production function with $n-$factor, also called *generalized Cobb–Douglas production function*, is given by

$$f\left(\mathbf{x}\right)=\gamma x_1^{\alpha_1}\cdots x_n^{\alpha_n},$$

where γ is a positive constant and $\alpha_1,\ldots,\alpha_n$ are nonzero constants [6].

3. Hessian determinant formula

Let us denote the first derivative of $h_i\left(x_i\right)$ with respect to x_i by a prime $(')$ and that of $F\left(u\right)$ with respect to u by a dot $\left(\dot{}\right)$.

Throughout this article, we assume that $h_1,\ldots,h_n:\mathbb{R}\longrightarrow\mathbb{R}$ are thrice differentiable functions with $h_i'\left(x_i\right)\neq 0$ and $F:I\subset\mathbb{R}\longrightarrow\mathbb{R}$ a twice differentiable function with $\dot{F}\left(u\right)\neq 0$ such that $I\subset\mathbb{R}$ is an interval of positive length.

The following provides an explicit formula for the Hessian determinant of the composite function given by (1.1).

Theorem 3.1. *The determinant of the Hessian matrix* $\mathcal{H}(f)$ *of the composite function* $f = F(h_1(x_1) \times \cdots \times h_n(x_n))$ *is given by*

$$\begin{aligned}\det(\mathcal{H}(f)) = \left(u\dot{F}\right)^n \Bigg\{ & \left(\frac{h_1'}{h_1}\right)' \cdots \left(\frac{h_n'}{h_n}\right)' \\ & + \left(1 + u\frac{\ddot{F}}{\dot{F}}\right) \sum_{j=1}^{n} \left(\frac{h_1'}{h_1}\right)' \cdots \left(\frac{h_{j-1}'}{h_{j-1}}\right)' \left(\frac{h_j'}{h_j}\right)^2 \left(\frac{h_{j+1}'}{h_{j+1}}\right)' \cdots \left(\frac{h_n'}{h_n}\right)' \Bigg\},\end{aligned} \tag{3.1}$$

where $h_j' = \frac{dh_j}{dx_j}$, $h_j'' = \frac{d^2h_j}{dx_j^2}$, $\dot{F} = \frac{dF}{du}$ *and* $\ddot{F} = \frac{d^2F}{du^2}$ *for* $u = h_1(x_1) \times \cdots \times h_n(x_n)$.

Proof. Let f be a twice differentiable composite function given by

$$f(\mathbf{x}) = F(h_1(x_1) \times \cdots \times h_n(x_n)) \tag{3.2}$$

for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. It follows from (3.2) that

$$f_{x_i} = \frac{\partial f}{\partial x_i} = \frac{h_i'}{h_i} u\dot{F}, \; f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{h_i' h_j'}{h_i h_j} u \left[\dot{F} + u\ddot{F}\right], \; 1 \leq i \neq j \leq n, \tag{3.3}$$

and

$$f_{x_i x_i} = \frac{\partial^2 f}{\partial x_i^2} = \frac{h_i''}{h_i} u\dot{F} + \left(\frac{h_i'}{h_i}\right)^2 u^2 \ddot{F}. \tag{3.4}$$

By using (3.3) and (3.4) the determinant of Hessian matrix $\mathcal{H}(f)$ of the composite function given by (3.2) is

$$\det(\mathcal{H}(f)) =$$
$$\begin{vmatrix} \frac{h_1''}{h_1}u\dot{F} + \left(\frac{h_1'}{h_1}\right)^2 u^2\ddot{F} & \frac{h_1'h_2'}{h_1h_2}u\left[\dot{F} + u\ddot{F}\right] & \frac{h_1'h_3'}{h_1h_3}u\left[\dot{F} + u\ddot{F}\right] & \cdots & \frac{h_1'h_n'}{h_1h_n}u\left[\dot{F} + u\ddot{F}\right] \\ \frac{h_1'h_2'}{h_1h_2}u\left[\dot{F} + u\ddot{F}\right] & \frac{h_2''}{h_2}u\dot{F} + \left(\frac{h_2'}{h_2}\right)^2 u^2\ddot{F} & \frac{h_2'h_3'}{h_2h_3}u\left[\dot{F} + u\ddot{F}\right] & \cdots & \frac{h_2'h_n'}{h_2h_n}u\left[\dot{F} + u\ddot{F}\right] \\ \frac{h_1'h_3'}{h_1h_3}u\left[\dot{F} + u\ddot{F}\right] & \frac{h_2'h_3'}{h_2h_3}u\left[\dot{F} + u\ddot{F}\right] & \frac{h_3''}{h_3}u\dot{F} + \left(\frac{h_3'}{h_3}\right)^2 u^2\ddot{F} & \cdots & \frac{h_3'h_n'}{h_3h_n}u\left[\dot{F} + u\ddot{F}\right] \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{h_1'h_n'}{h_1h_n}u\left[\dot{F} + u\ddot{F}\right] & \frac{h_2'h_n'}{h_2h_n}u\left[\dot{F} + u\ddot{F}\right] & \frac{h_3'h_n'}{h_3h_n}u\left[\dot{F} + u\ddot{F}\right] & \cdots & \frac{h_n''}{h_n}u\dot{F} + \left(\frac{h_n'}{h_n}\right)^2 u^2\ddot{F} \end{vmatrix}$$

Now we apply Gauss elimination method for the determinant from the last equality. We replace the second column by second column minus $\frac{h_1h_2'}{h_1'h_2}$ times the first column;

then we derive

$$\det(\mathcal{H}(f)) =$$
$$\begin{vmatrix} \frac{h_1''}{h_1}u\dot{F} + \left(\frac{h_1'}{h_1}\right)^2 u^2\ddot{F} & -\frac{h_1h_2'}{h_1'h_2}\left(\frac{h_1'}{h_1}\right)' u\dot{F} & \frac{h_1'h_3'}{h_1h_3}u\left[\dot{F}+u\ddot{F}\right] & \cdots & \frac{h_1'h_n'}{h_1h_n}u\left[\dot{F}+u\ddot{F}\right] \\ \frac{h_1'h_2'}{h_1h_2}u\left[\dot{F}+u\ddot{F}\right] & \left(\frac{h_2'}{h_2}\right)' u\dot{F} & \frac{h_2'h_3'}{h_2h_3}u\left[\dot{F}+u\ddot{F}\right] & \cdots & \frac{h_2'h_n'}{h_2h_n}u\left[\dot{F}+u\ddot{F}\right] \\ \frac{h_1'h_3'}{h_1h_3}u\left[\dot{F}+u\ddot{F}\right] & 0 & \frac{h_3''}{h_3}u\dot{F} + \left(\frac{h_3'}{h_3}\right)^2 u^2\ddot{F} & \cdots & \frac{h_3'h_n'}{h_3h_n}u\left[\dot{F}+u\ddot{F}\right] \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{h_1'h_n'}{h_1h_n}u\left[\dot{F}+u\ddot{F}\right] & 0 & \frac{h_3'h_n'}{h_3h_n}u\left[\dot{F}+u\ddot{F}\right] & \cdots & \frac{h_n''}{h_n}u\dot{F} + \left(\frac{h_n'}{h_n}\right)^2 u^2\ddot{F} \end{vmatrix}$$

By similar elementary transformations, we get

$$\det(\mathcal{H}(f)) =$$
$$\begin{vmatrix} \frac{h_1''}{h_1}u\dot{F} + \left(\frac{h_1'}{h_1}\right)^2 u^2\ddot{F} & -\frac{h_1h_2'}{h_1'h_2}\left(\frac{h_1'}{h_1}\right)' u\dot{F} & -\frac{h_1h_3'}{h_1'h_3}\left(\frac{h_1'}{h_1}\right)' u\dot{F} & \cdots & -\frac{h_1h_n'}{h_1'h_n}\left(\frac{h_1'}{h_1}\right)' u\dot{F} \\ \frac{h_1'h_2'}{h_1h_2}u\left[\dot{F}+u\ddot{F}\right] & \left(\frac{h_2'}{h_2}\right)' u\dot{F} & 0 & \cdots & 0 \\ \frac{h_1'h_3'}{h_1h_3}u\left[\dot{F}+u\ddot{F}\right] & 0 & \left(\frac{h_3'}{h_3}\right)' u\dot{F} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{h_1'h_n'}{h_1h_n}u\left[\dot{F}+u\ddot{F}\right] & 0 & 0 & \cdots & \left(\frac{h_n'}{h_n}\right)' u\dot{F} \end{vmatrix}.$$

After calculating the determinant in the previous formula, we obtain

$$\begin{aligned} \det(\mathcal{H}(f)) = &\left(u\dot{F}\right)^n \frac{h_1''}{h_1}\left(\frac{h_2'}{h_2}\right)'\left(\frac{h_3'}{h_3}\right)'\cdots\left(\frac{h_n'}{h_n}\right)' + \left(u\dot{F}\right)^n \times \\ &\times \left\{ \left(\frac{h_1'}{h_1}\right)'\left(\frac{h_2'}{h_2}\right)^2\left(\frac{h_3'}{h_3}\right)'\cdots\left(\frac{h_n'}{h_n}\right)' + \left(\frac{h_1'}{h_1}\right)'\left(\frac{h_2'}{h_2}\right)'\left(\frac{h_3'}{h_3}\right)^2\cdots\left(\frac{h_n'}{h_n}\right)' \right. \\ &\qquad \left. + \cdots + \left(\frac{h_1'}{h_1}\right)'\left(\frac{h_2'}{h_2}\right)'\left(\frac{h_3'}{h_3}\right)'\cdots\left(\frac{h_n'}{h_n}\right)^2 \right\} \\ &+ (u)^{n+1}\left(\dot{F}\right)^{n-1}\ddot{F}\left\{ \left(\frac{h_1'}{h_1}\right)^2\left(\frac{h_2'}{h_2}\right)'\left(\frac{h_3'}{h_3}\right)'\cdots\left(\frac{h_n'}{h_n}\right)' \right. \\ &+ \left(\frac{h_1'}{h_1}\right)'\left(\frac{h_2'}{h_2}\right)^2\left(\frac{h_3'}{h_3}\right)'\cdots\left(\frac{h_n'}{h_n}\right)' + \left(\frac{h_1'}{h_1}\right)'\left(\frac{h_2'}{h_2}\right)'\left(\frac{h_3'}{h_3}\right)^2\cdots\left(\frac{h_n'}{h_n}\right)' \\ &\left. + \cdots + \left(\frac{h_1'}{h_1}\right)'\left(\frac{h_2'}{h_2}\right)'\left(\frac{h_3'}{h_3}\right)'\cdots\left(\frac{h_n'}{h_n}\right)^2 \right\}. \end{aligned}$$

After adding and substracting $\left(u\dot{F}\right)^n\left(\frac{h_1'}{h_1}\right)^2\left(\frac{h_2'}{h_2}\right)'\left(\frac{h_3'}{h_3}\right)'\cdots\left(\frac{h_n'}{h_n}\right)'$ we deduce

$$\det\left(\mathcal{H}\left(f\right)\right)=\left(u\dot{F}\right)^n\left\{\prod_{j=1}^{n}\left(\frac{h_j'}{h_j}\right)'+\right.$$
$$\left.+\left(1+u\frac{\ddot{F}}{\dot{F}}\right)\sum_{j=1}^{n}\left(\frac{h_1'}{h_1}\right)'\cdots\left(\frac{h_{j-1}'}{h_{j-1}}\right)'\left(\frac{h_j'}{h_j}\right)^2\left(\frac{h_{j+1}'}{h_{j+1}}\right)'\cdots\left(\frac{h_n'}{h_n}\right)'\right\}.$$

This completes the proof of the formula (3.1). □

4. Characterizations of CD production functions

Next, we provide the following characterization of the generalized Cobb-Douglas production function with constant return to scale via the Theorem 3.1.

Theorem 4.1. *Let $F\left(u\right)$ be a twice differentiable function with $\dot{F}\left(u\right)\neq 0$ and let f be a composite function given by*

$$f=F\left(\left(x_1+\zeta_1\right)^{\alpha_1}\times\cdots\times\left(x_n+\zeta_n\right)^{\alpha_n}\right) \tag{4.1}$$

for some constants α_i,ζ_i.The Hessian matrix $\mathcal{H}\left(f\right)$ of f is singular if and only if either

- (i) *at least one of the $\alpha_1,\ldots,\alpha_n$ vanishes, or*
- (ii) *up to suitable translations of $x_1,\ldots,x_n$, f is a generalized Cobb-Douglas production function with constant return to scale.*

Proof. Let us assume that the Hessian matrix of f is singular. By the hypothesis of the theorem, we have $h_j\left(x_j\right)=\left(x_j+\zeta_j\right)^{\alpha_j}$. Thus we get

$$h_j'\left(x_i\right)=\alpha_j\left(x_j+\zeta_j\right)^{\alpha_j-1},\quad h_j''\left(x_j\right)=\alpha_j\left(\alpha_j-1\right)\left(x_j+\zeta_j\right)^{\alpha_j-2}$$

for all $j\in\{1,\ldots,n\}$. After applying the formula (3.1), we write

$$0=\left(-1\right)^{n-1}\left(u\dot{F}\right)^n\prod_{j=1}^{n}\frac{\alpha_j}{\left(x_j+\zeta_j\right)^2}\left\{\left(-1+\sum_{j=1}^{n}\alpha_j\right)+u\frac{\ddot{F}}{\dot{F}}\left(\sum_{j=1}^{n}\alpha_j\right)\right\}, \tag{4.2}$$

where $u=\left(x_1+\zeta_1\right)^{\alpha_1}\times\cdots\times\left(x_n+\zeta_n\right)^{\alpha_n}$. Since $u\neq 0$ and $\dot{F}\neq 0$, the equation (4.2) reduces to

$$0=\prod_{j=1}^{n}\frac{\alpha_j}{\left(x_j+\zeta_j\right)^2}\left(-1+\sum_{j=1}^{n}\alpha_j+u\frac{\ddot{F}}{\dot{F}}\sum_{j=1}^{n}\alpha_j\right). \tag{4.3}$$

From the equation (4.3), it is easily seen that either at least one of the $\alpha_1,\ldots,\alpha_n$ vanishes or

$$1-\sum_{j=1}^{n}\alpha_j=u\frac{\ddot{F}}{\dot{F}}\sum_{j=1}^{n}\alpha_j. \tag{4.4}$$

For (4.4), if F is a linear function, then $\sum_{j=1}^{n}\alpha_j = 1$, which implies that, up to suitable translations of $x_1, \dots, x_n$, f is a generalized Cobb-Douglas production function with constant return to scale. If F is a non-linear function, then by (4.4) we derive

$$\frac{\ddot{F}}{\dot{F}} - \frac{1-\sum_{j=1}^{n}\alpha_j}{\sum_{j=1}^{n}\alpha_j}\frac{1}{u} = 0,$$

which implies that

$$F = \frac{\delta}{\gamma+1}(u)^{\gamma+1} + \varepsilon, \tag{4.5}$$

where γ, δ are nonzero constants and ε some constant such that

$$\gamma = \frac{1-\sum_{j=1}^{n}\alpha_j}{\sum_{j=1}^{n}\alpha_j}. \tag{4.6}$$

Combining (4.1), (4.5) and (4.6) gives that, up to suitable translations of $x_1, \dots, x_n$, f is a generalized Cobb-Douglas production function with constant return to scale.

Conversely, it is straightforward to verify that cases (i) and (ii) imply that f has vanishing Hessian determinant. □

Theorem 4.2. *Let $F = u^r$ be a power function such that $r \neq 0, 1$ and let f be a composite function given by*

$$f = F(h_1(x_1) \times \cdots \times h_n(x_n)). \tag{4.7}$$

The Hessian matrix $\mathcal{H}(f)$ of f is singular if and only if either

(i) $f = F(\gamma e^{\alpha_1 x_1 + \alpha_2 x_2} \times h_3(x_3) \times \cdots \times h_n(x_n))$ *for nonzero constants $\gamma, \alpha_1, \alpha_2$, or*

(ii) *up to suitable translations of $x_1, \dots, x_n$, f is a generalized Cobb-Douglas production function with constant return to scale.*

Proof. Let us assume that the Hessian matrix of f is singular. Then we have $\det(\mathcal{H}(f)) = 0$. From the hypothesis of theorem, we get

$$\dot{F} = ru^{r-1} \text{ and } \ddot{F} = r(r-1)u^{r-2}. \tag{4.8}$$

After substituting (4.8) into the formula (3.1), we derive

$$0 = \prod_{j=1}^{n}\left(\frac{h_j'}{h_j}\right)' + r\sum_{j=1}^{n}\left(\frac{h_1'}{h_1}\right)' \cdots \left(\frac{h_{j-1}'}{h_{j-1}}\right)'\left(\frac{h_j'}{h_j}\right)^2\left(\frac{h_{j+1}'}{h_{j+1}}\right)' \cdots \left(\frac{h_n'}{h_n}\right)'. \tag{4.9}$$

For (4.9) we have two cases:

Case (a): At least one of $\left(\frac{h_1'}{h_1}\right)',\ldots,\left(\frac{h_n'}{h_n}\right)'$ vanishes. Without loss of generality, we may assume that

$$\left(\frac{h_1'}{h_1}\right)'=0. \tag{4.10}$$

Then from (4.9), we find

$$0=\left(\frac{h_1'}{h_1}\right)^2\left(\frac{h_2'}{h_2}\right)'\left(\frac{h_3'}{h_3}\right)'\cdots\left(\frac{h_n'}{h_n}\right)'. \tag{4.11}$$

Without loss of generality, we may assume from (4.11) that

$$\left(\frac{h_2'}{h_2}\right)'=0. \tag{4.12}$$

After solving (4.10) and (4.12), we obtain $h_j(x_j)=\gamma_j e^{\alpha_j x_j}$, $(j=1,2)$ for nonzero constants γ_j, α_j. This gives the statement (i).

Case (b): $\left(\frac{h_1'}{h_1}\right)',\ldots,\left(\frac{h_n'}{h_n}\right)'$ are nonzero. Then from (4.9), by dividing with the product $\left(\frac{h_1'}{h_1}\right)'\cdots\left(\frac{h_n'}{h_n}\right)'$, we write

$$0=1+r\left(\frac{\left(\frac{h_1'}{h_1}\right)^2}{\left(\frac{h_1'}{h_1}\right)'}+\cdots+\frac{\left(\frac{h_n'}{h_n}\right)^2}{\left(\frac{h_n'}{h_n}\right)'}\right). \tag{4.13}$$

Taking partial derivative of (4.13) with respect to x_i, we find

$$2\left(\left(\frac{h_i'}{h_i}\right)'\right)^2=\left(\frac{h_i'}{h_i}\right)\left(\frac{h_i'}{h_i}\right)''. \tag{4.14}$$

By solving (4.14), we get

$$h_j(x_j)=\gamma_j(x_j+\zeta_j)^{\alpha_j}, \tag{4.15}$$

where γ_j,α_j are nonzero constants with $\sum_{j=1}^n\alpha_j=\frac{1}{r}$, and ζ_j some constants. Combining (4.7) and (4.15) gives the statement (ii).

Converse is straightforward to verify that cases (i) and (ii) imply that f has vanishing Hessian determinant. □

5. Further applications

We provide the following as further applications of Theorem 3.1.

Theorem 5.1. *Let f be a twice differentiable composite function given by*

$$f=\ln\left(h_1(x_1)\times\cdots\times h_n(x_n)\right).$$

The Hessian matrix $\mathcal{H}(f)$ of f is singular if and only if at least one of the $h_1(x_1),\ldots,h_n(x_n)$ is of the form $\gamma_j e^{\alpha_j x_j}$ for nonzero constants γ_j,α_j.

Proof. Let assume that the Hessian matrix $\mathcal{H}(f)$ of f is singular. Then under the hypothesis of the theorem, we get

$$F(u) = \ln u, \ \dot{F}(u) = \frac{1}{u}, \ \ddot{F} = -\frac{1}{u^2}.$$

After applying the formula (3.1), we derive $0 = \prod_{j=1}^{n} \left(\frac{h_j'}{h_j}\right)'$. Because of $h_j'(x_j) \neq 0$, at least one of $\left(\frac{h_j'}{h_j}\right)'$ vanishes which implies that at least one of h_j is of the form $\gamma_j e^{\alpha_j x_j}$ for nonzero constants γ_j, α_j.

Converse is easy to verify. □

Corollary 5.1. *Let $f = F(h_1(x_1) \times \cdots \times h_n(x_n))$ be a twice differentiable composite function. If at least two of $h_1(x_1), \ldots, h_n(x_n)$ is of the form $\gamma_j e^{\alpha_j x_j}$ for nonzero constants γ_j, α_j, then the Hessian matrix $\mathcal{H}(f)$ of f is singular.*

Proof. Let $f = F(h_1(x_1) \times \cdots \times h_n(x_n))$ be a twice differentiable composite function such that at least two of $h_1(x_1), \ldots, h_n(x_n)$ is of the form $\gamma_j e^{\alpha_j x_j}$ for nonzero constants γ_j, α_j. Without lose of generality, we may assume that

$$h_1 = \gamma_1 e^{\alpha_1 x_1} \text{ and } h_2 = \gamma_2 e^{\alpha_2 x_2}$$

Thus we get

$$\left(\frac{h_1'}{h_1}\right)' = 0 \text{ and } \left(\frac{h_2'}{h_2}\right)' = 0. \tag{5.1}$$

Substituting (5.1) into (3.1) gives the proof. □

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