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NORMAL FAMILIES OF MEROMORPHIC FUNCTIONS SHARING A HOLOMORPHIC FUNCTION

CHENG-XIONG SUN

Abstract. Let k be a positive integer, and m be an even number. Suppose that $a(z)(\not\equiv 0)$ is a holomorphic function with zeros of multiplicity m in a domain D. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that each $f \in \mathcal{F}$ have zeros of multiplicity at least $k+1+m$ and poles of multiplicity at least $m+1$. It is mainly proved that for each pair $(f, g) \in \mathcal{F}$, if $ff^{(k)}$ and $gg^{(k)}$ share $a(z)$ IM, then $\mathcal{F}$ is normal in D. This result improves Hu and Meng's results published in Journal of Mathematical Analysis and Applications (2009, 2011), and also Jiang and Gao's result in Acta Matematica Scientia (2012).

1. Introduction and main results

Let D be a domain in $\mathbb{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined in the domain D. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for every sequence $\{f_n\} \subset \mathcal{F}$ there exists a subsequence $\{f_{n_j}\}$ converges spherically locally uniformly to a meromorphic function or ∞.

Let f and g be two meromorphic functions in D, and let $\phi(z)$ be a function. If the two equations $f(z) = \phi(z)$ and $g(z) = \phi(z)$ have the same solutions in D(ignoring multiplicity), then we say that f and g share a function $\phi(z)$ IM.

Now, we introduce a normality criterion related to a Hayman normal conjecture.

Theorem 1.1. [1] *Let $\mathcal{F}$ be a family of meromorphic function in D, and $a(\neq 0) \in \mathbb{C}$. If $f^n f' \neq a$, for each function $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D, where n is a positive integer.*

The result is due to L. Yang and G. Zhang [2] (for $n \geq 5$), Y. X. Gu [3] (for $n = 4, 3$), X. C. Pang [4] (for $n \geq 2$) and Chen and Fang [5] (for $n = 1$).

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In 2009, Q. Lu and Y. X. Gu [6] considered the general order derivative in Theorem 1.1 for $n = 1$ and proved the following result.

Theorem 1.2. [6] *Let $\mathcal{F}$ be a family of meromorphic function in D, and $a(\neq 0) \in \mathbb{C}$. If $ff^{(k)} \neq a$, for each function $f \in \mathcal{F}$, the zeros of f have multiplicities at least $k+2$, then $\mathcal{F}$ is normal in D, where n is a positive integer.*

In 2010, J. Xu and W. Cao [7] improved Theorem 1.2 by including meromorphic functions having zeros with multiplicities at least $k + 1$.

In 2011 D. W. Meng and P. Ch. Hu proved the following normality criteria.

Theorem 1.3. [8] *Take $a \in \mathbb{C} - \{0\}$ and take a positive integer k. Let $\mathcal{F}$ be a family of mreomorphic functions in the plane domain D such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least $k + 1$. For each pair $(f, g) \in \mathcal{F}$, if $ff^{(k)}$ and $gg^{(k)}$ share a IM, then $\mathcal{F}$ is normal in D.*

In 2009 D. W. Meng and P. Ch. Hu proved the following normality criteria.

Theorem 1.4. [9] *Take $a \in \mathbb{C} - \{0\}$ and take positive integers n and k with $n, k \geq 2$. Let $\mathcal{F}$ be a family of mreomorphic functions in the plane domain D such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least k. For each pair $(f, g) \in \mathcal{F}$, if $f(f^{(k)})^n$ and $g(g^{(k)})^n$ share a IM, then $\mathcal{F}$ is normal in D.*

Recently, Jiang and Gao improved Theorem 1.4 in the following manner.

Theorem 1.5. [10] *Let $n, k \geq 2, m \geq 0$ be three positive integers, and m be divisible by $n + 1$. Suppose that $a(z)(\not\equiv 0)$ is a holomorphic function with zeros of multiplicity m in a domain D. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that each $f \in \mathcal{F}$ have zeros of multiplicity at least $max\{k+m, 2m+2\}$. For each pair $(f, g) \in \mathcal{F}$, if $f(f^{(k)})^n$ and $g(g^{(k)})^n$ share $a(z)$ IM, then $\mathcal{F}$ is normal in D.*

Here, we want to generalize Theorem 1.3 by replacing the constant a by a function. In this direction we prove the following result.

Theorem 1.6. *Let k be a positive integer, and m be an even number. Suppose that $a(z)(\not\equiv 0)$ is a holomorphic function with zeros of multiplicity m in a domain D. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that each $f \in \mathcal{F}$ have zeros of multiplicity at least $k + 1 + m$ and poles of multiplicity at least $m + 1$. For each pair $(f, g) \in \mathcal{F}$, if $ff^{(k)}$ and $gg^{(k)}$ share $a(z)$ IM, then $\mathcal{F}$ is normal in D.*

Example 1.1. Let $D = \{z : |z| < 1\}$. Let $\mathcal{F} = \{f_n(z)\}$ where $f_n(z) = e^{nz}$, $z \in D$, $n = 1, 2, \ldots$. Then for distinct positive integers n, l, $f_n f_n^{(k)}$ and $f_l f_l^{(k)}$ share 0 IM, but $\mathcal{F}$ fails to be normal at $z = 0$.

Example 1.2. Let $D = \{z : |z| < 1\}$. Let $\mathcal{F} = \{f_n(z)\}$ where $f_n(z) = \frac{z^4}{4(z^2 - \frac{1}{n})^2}$, $z \in D$, $n = 1, 2, \ldots$. Then $f_n f_n'(z) = -\frac{z^7}{4n(z^2 - \frac{1}{n})^5}$. Thus for distinct positive integers n, l, $f_n f_n'$ and $f_l f_l'$ share 0 IM, but $\mathcal{F}$ fails to be normal at $z = 0$, since $f_n(0) = 0$ and $f_n(\frac{1}{\sqrt{n}}) = \infty$.

Remark 1.1. Examples 1.1 and 1.2 show the condition that $a(z)(\not\equiv 0)$ is necessary and Example 1.2 shows that the even number m is sharp in Theorem 1.6.

Remark 1.2. It seems reasonable to conjecture that the conclusion of Theorem 1.6 still holds when m is odd number.

Let us set some notations. we use $\longrightarrow$ to stand for convergence, $\overset{\chi}{\Rightarrow}$ to stand for spherical local uniform convergence in $D \subset \mathbb{C}$.

2. Some lemmas

To prove Theorem 1.6, we require the following lemmas.

Lemma 2.1. [11] *Let $\mathcal{F}$ be a family of functions meromorphic in the unit disc Δ , all of whose zeros have multiplicity at least k; Suppose that there exists $A \geqslant 1$ such that $\left|f^{(k)}(z)\right| \leqslant A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal at $z_0 \in D$, there exist, for each $0 \leqslant \alpha \leqslant k$,*

(i) *points $z_n,\ z_n \to z_0, z_0 \in \Delta$,*
(ii) *functions $f_n \in \mathcal{F}$, and*
(iii) *positive numbers $\rho_n \to 0^+$*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\xi) = g_n(\xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where $g(\xi)$ is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros of $g(\xi)$ are of multiplicity at least k, such that $g^{\#}(\xi) \le g^{\#}(0) = kA + 1$.Moreover, g has order at most 2.

Lemma 2.2. *Let function $f(z)$ be meromorphic and transcendental in the plane, all of whose zeros have multiplicity $k+1$ at least, and $a(z)(\not\equiv 0)$ be a polynomial. Then the differential monomial $f(z)f^{(k)}(z) - a(z)$ has infinitely zeros, where k is an integer number.*

Proof. Set

$$F := F(z) = f(z)f^{(k)}(z) - a(z) \tag{2.1}$$

and

$$F_1 := \frac{F}{a} = \frac{f(z)f^{(k)}(z)}{a(z)} - 1. \tag{2.2}$$

By differentiating the equation (2.2), we get $f\beta = \frac{F_1'}{F_1}$ where

$$\beta = \frac{F_1'}{F_1}\frac{f^{(k)}}{a} - (\frac{1}{a})' f^{(k)} - \frac{1}{a}\frac{f'}{f} f^{(k)} - \frac{1}{a} f^{(k+1)}. \tag{2.3}$$

Noting $a(z)(\not\equiv 0)$ is a polynomial and the zeros of f are of multiplicity at least $k+1$, then $N(r, \frac{1}{a}) = O(logr)$ and $N_{k)}(r, \frac{1}{f}) = S(r, f)$. As the preceding of proof of Lemma 2.2 from [12], we can get the conclusion of Lemma 2.2. □

Lemma 2.3. [13] *Let* $R = \frac{A}{B}$ *be a rational function and* B *be non-constant. Let* $(R)_\infty = \deg(A) - \deg(B)$*, and* k *be a positive integer. Then* $(R^{(k)})_\infty \le (R)_\infty - k$*.*

Lemma 2.4. *Let* $a(z)$ *be a non-zero polynomial of degree* m*, and* k *be a positive integer. Let* f *be a non-constant rational function, all of whose zeros and poles(if exists)have multiplicity at least* $k+m+1$ *and* $m+1$*, then the function* $ff^{(k)} - a(z)$ *has at least one zero.*

Proof. We consider the following cases.

Case 1. f is a non-constant polynomial. Since f is a non-constant polynomial with zeros of multiplicity at least $k+m+1$, then $\deg(ff^{(k)}) \ge k+2m+2$, thus $\deg(ff^{(k)}) > \deg(a(z))$, so the function $ff^{(k)} - a(z)$ has at least one zero.

Case 2. f is a non-polynomial rational function. Write

$$f(z) = A\frac{(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}}, \tag{2.4}$$

where $A(\ne 0)$ is a constant and $m_i \ge k+m+1$; $n_j \ge m+1$ $(i = 1,2,\ldots,s; j = 1,2,\ldots,t)$ are positive integers.

We put

$$M = m_1 + m_2 + \cdots + m_s \ge (k+m+1)s, \tag{2.5}$$

$$N = n_1 + n_2 + \cdots + n_t \ge (m+1)t. \tag{2.6}$$

From (2.4) we get

$$f^{(k)}(z) = \frac{(z-\alpha_1)^{m_1-k}(z-\alpha_2)^{m_2-k}\cdots(z-\alpha_s)^{m_s-k}g(z)}{(z-\beta_1)^{n_1+k}(z-\beta_2)^{n_2+k}\cdots(z-\beta_t)^{n_t+k}}, \tag{2.7}$$

where g is a polynomial. From (2.4) and (2.7) we get

$$ff^{(k)}(z) = \frac{(z-\alpha_1)^{2m_1-k}(z-\alpha_2)^{2m_2-k}\cdots(z-\alpha_s)^{2m_s-k}g(z)}{(z-\beta_1)^{2n_1+k}(z-\beta_2)^{2n_2+k}\cdots(z-\beta_t)^{2n_t+k}}. \tag{2.8}$$

Suppose the function $ff^{(k)} - a(z)$ has no zero, this is $ff^{(k)} - a(z) \ne 0$. Then we get from (2.8)

$$ff^{(k)}(z) = a(z) + \frac{B}{(z-\beta_1)^{2n_1+k}(z-\beta_2)^{2n_2+k}\cdots(z-\beta_t)^{2n_t+k}}, \tag{2.9}$$

where $B(\ne 0)$ is a constant. From (2.8) and (2.9) we obtain respectively

$$[ff^{(k)}(z)]^{(m+1)} = \frac{(z-\alpha_1)^{2m_1-k-m-1}(z-\alpha_2)^{2m_2-k-m-1}\cdots(z-\alpha_s)^{2m_s-k-m-1}g_1(z)}{(z-\beta_1)^{2n_1+k+m+1}(z-\beta_2)^{2n_2+k+m+1}\cdots(z-\beta_t)^{2n_t+k+m+1}} \tag{2.10}$$

and

$$[ff^{(k)}(z)]^{(m+1)} = \frac{g_2(z)}{(z-\beta_1)^{2n_1+k+m+1}(z-\beta_2)^{2n_2+k+m+1}\cdots(z-\beta_t)^{2n_t+k+m+1}}, \tag{2.11}$$

where g_1, g_2 are polynomials. From (2.4) and (2.7) we get $(f)_\infty = M - N$ and $(f^{(k)})_\infty = M - N - k(s+t) + \deg(g)$. Since by Lemma 2.3 $(f^{(k)})_\infty \le (f)_\infty - k$, then $\deg(g) \le k(s+t-1)$. From (2.8) and(2.10) we obtain $(ff^{(k)})_\infty = 2(M-N) - k(s+t) + \deg(g)$ and

$$(ff^{(k)})_\infty^{(m+1)} = 2(M-N) - k(m+1)(s+t) + \deg(g_1).$$

By Lemma 2.3, we get $\deg(g_1) \le (k+m+1)(s+t-1)$.

From (2.9) and (2.11) we get $(ff^{(k)})_\infty = -2N - kt$ and $(ff^{(k)})_\infty^{(m+1)} = \deg(g_1) - 2N - (k+m+1)t$. By Lemma 2.3, we obtain $\deg(g_2) \le (m+1)(t-1)$.

From (2.1) and (2.11) we see that

$$(z-\alpha_1)^{2m_1-k-m-1}(z-\alpha_2)^{2m_2-k-m-1}\cdots(z-\alpha_s)^{2m_s-k-m-1}$$

is a factor of g_2. Then $2M - (k+m+1)s \le \deg(g_2) \le (m+1)(t-1)$, which implies $2M \le (k+m+1)s + (m+1)(t-1)$. From (2.5) and (2.6) we obtain $2M \le M + N - (m+1)$. This implies

$$M < N. \tag{2.12}$$

From (2.8) and (2.9) we can get

$$2N + kt + m = 2M - ks + \deg(g) \le 2M - ks + k(s+t-1),$$

which implies $2N \le 2M - k - m$, this is $N < M$, which contradicts (2.12). This proves Lemma 2.4. □

Lemma 2.5. *Let $a(z)$ be a non-zero polynomial of degree m, and k be a positive integer. Let f be a non-constant rational function, all of whose zeros and poles(if exists)have multiplicity at least $k+m+1$ and $m+1$, then the function $ff^{(k)} - a(z)$ has at least two zeros.*

Proof. By lemma 2.4, we deduce that the function $ff^{(k)} - a(z)$ has at least one zero. Suppose, to the contrary, the function $ff^{(k)} - a(z)$ has exactly one root.

First we suppose that f is a non-constant polynomial. We set $ff^{(k)} - a(z) = C(z-z_0)^n$, where $C(\neq 0)$ is a constant and n is a positive integer satisfying $n \ge k + 2 + 2m \ge 2m + 3$. Then

$$[ff^{(k)} - a(z)]^{(m+1)} = [ff^{(k)}]^{(m+1)} = Cn(n-1)\cdots(n-m)(z-z_0)^{n-m-1}.$$

So $[ff^{(k)}]^{(m+1)}$ has exactly one zero at z_0. Since f is a non-constant polynomial with zeros of multiplicity at least $k+m+1$, then z_0 is a zero of f, it follows that $[ff^{(k)}]^{(m)}(z_0) = 0$. Noting that

$$[ff^{(k)} - a(z)]^{(m)} = [ff^{(k)}]^{(m)} - [a(z)]^{(m)} = Cn(n-1)\cdots(n-m+1)(z-z_0)^{n-m}.$$

Then $[a(z_0)]^{(m)} = 0$, which is a contradiction, since $a(z)$ is a non-zero polynomial with $\deg(a(z)) = m$. Therefore f is a non-polynomial rational function, We can express f by (2.4) again. Since the function $ff^{(k)} - a(z)$ has exactly one zero, we get from (2.8)

$$ff^{(k)}(z) = a(z) + \frac{D(z-z_0)^l}{(z-\beta_1)^{2n_1+k}(z-\beta_2)^{2n_2+k}\cdots(z-\beta_t)^{2n_t+k}}. \tag{2.13}$$

where $D(\neq 0)$ is a constant and l is a positive integer.

We consider the following two cases.

Case 1. $m \geq l$. From (2.8) and (2.13) we can get

$$2N + kt + m = 2M - ks + \deg(g) \leq 2M - ks + k(s + t - 1).$$

which implies $2N \leq 2M - k - m$, that is $N < M$, From (2.13) we obtain

$$[ff^{(k)}(z)]^{(m+1)} = \frac{g_3(z)}{(z-\beta_1)^{2n_1+k+m+1}(z-\beta_2)^{2n_2+k+m+1}\cdots(z-\beta_t)^{2n_t+k+m+1}}. \tag{2.14}$$

where g_3 are polynomials with $\deg(g_3) \leq (m+1)t - (m-l+1)$.

From (2.10) and (2.14) we see that

$$(z-\alpha_1)^{2m_1-k-m-1}(z-\alpha_2)^{2m_2-k-m-1}\cdots(z-\alpha_s)^{2m_s-k-m-1}$$

is a factor of g_3. Then $(m+1)t - (m-l+1) \geq 2M - (k+m+1)s$. From (2.5) and (2.6) we get $M \leq N - (m-l+1)s$. Then $M \leq N - (m-l+1)s < M - (m-l+1)$. This implies $m < l - 1$, a contradiction.

Case 2. $m < l$. From (2.13) we get

$$[ff^{(k)}(z)]^{(m+1)} = \frac{D(z-z_0)^{l-m-1}g_4(z)}{(z-\beta_1)^{2n_1+k+m+1}(z-\beta_2)^{2n_2+k+m+1}\cdots(z-\beta_t)^{2n_t+k+m+1}}. \tag{2.15}$$

where g_4 are polynomials with $\deg(g_4) \leq (m+1)t$.

Since $\alpha_i \neq z_0$ for $i = 1, 2, \ldots, s$, from (2.10) and (2.15) we see that

$$(z-\alpha_1)^{2m_1-k-m-1}(z-\alpha_2)^{2m_2-k-m-1}\cdots(z-\alpha_s)^{2m_s-k-m-1}$$

is a factor of g_4. Therefore $2M - ks - (m+1)s \leq \deg(g_4) \leq (m+1)t$, then from (2.5) and (2.6) we can deduce $M \leq N$.

Now we consider the following subcases.

Subcase 2.1. Let $l \neq 2N+kt+m$. From (2.8) and (2.13) we obtain $2N+kt+m \leq 2M - ks + \deg(g) \leq 2M + k(t-1)$, then $2N \leq 2M - k - m$, which implies $N < M$, a contradiction.

Subcase 2.2. Let $l = 2N+kt+m$. If $N < M$, then proceeding as case 1 we arrive at a contradiction. So $M \leq N$. Since $\alpha_i \neq z_0$ for $i = 1, 2, \ldots, s$, from (2.10) and (2.15) we see that $(z-z_0)^{l-m-1}$ is a factor of g_1. Thus $l-m-1 \leq \deg(g_1) \leq (k+m+1)(s+t-1)$, then $2N + kt + m = l \leq (k+m+1)(s+t-1) + m + 1$. From (2.5) and (2.6) we can deduce $2N \leq M + N - k \leq 2N - k$, which implies $-k \geq 0$, a contradiction. This proves Lemma 2.5. □

3. Proof of Theorem 1.6.

Proof. For any point $z_0 \in D$, either $a(z_0) = 0$ or $a(z_0) \neq 0$. We consider two cases.

Case 1. $a(z_0) \neq 0$. Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. Let $\alpha = \frac{k}{2}$. Then by Lemma 2.1, there exists a sequence of complex numbers $z_n \longrightarrow z_0$, a sequence of functions $f_n \in \mathcal{F}$ and a sequence of positive numbers $\rho_n \longrightarrow 0^+$ such that

$$h_n(\xi) = \rho_n^{-\frac{k}{2}} f_n(z_n + \rho_n\xi) \stackrel{\chi}{\Rightarrow} h(\xi),$$

where $h(\xi)$ is a non-constant meromorphic functions in $\mathbb{C}$. Also the order of $h(\xi)$ does not exceed 2 and by Hurwitz's theorem $h(\xi)$ has no zero of mulitiplicity less than $k+m+1$.

On every compact subset of $\mathbb{C}$ which contains no poles of h, we have

$$h_n(\xi)h_n^{(k)}(\xi)-a(\xi)=f_n(z_n+\rho_n\xi)f_n^{(k)}(z_n+\rho_n\xi)-a(z_n+\rho_n\xi)\overset{\chi}{\Rightarrow}h(\xi)h^{(k)}(\xi)-a(z_0).$$

If $hh^{(k)}\equiv a(z_0)$, then h has no poles and zeros, and thus h is entire function. Noting that $\frac{1}{h^2(\xi)}\equiv\frac{1}{a(z_0)}\frac{h^{(k)}(\xi)}{h(\xi)}$, thus

$$2T(r,h)=2m(r,h)\leq log^+\frac{1}{|a(z_0)|}+m(r,\frac{h^{(k)}}{h})=o\{T(r,h)\}(r\to\infty),$$

which is impossible. Hence $hh^{(k)}\not\equiv a(z_0)$.

By lemmas 2.2 and 2.5, the function $hh^{(k)}-a(z_0)$ has at least two distinct zeros ξ_0, ξ_0^*, say. We choose a small $\delta>0$ such that $D_1\cap D_2=\emptyset$ and $hh^{(k)}-a(z_0)$ has no other zeros in $D_1\cup D_2$ except for ξ_0 and ξ_0^*, where $D_1=\{\xi:|\xi-\xi_0|<\delta\}$ and $D_2=\{\xi:|\xi-\xi_0^*|<\delta\}$. By Hurwitz's theorem, there exit points $\xi_n\in D_1$ and $\xi_n^*\in D_2$ such that

$$f_n(z_n+\rho_n\xi_n)f_n^{(k)}(z_n+\rho_n\xi_n)-a(z_n+\rho_n\xi_n)=0$$

and

$$f_n(z_n+\rho_n\xi_n^*)f_n^{(k)}(z_n+\rho_n\xi_n^*)-a(z_n+\rho_n\xi_n^*)=0$$

for sufficiently large n.

By the assumption of Theorem 1.6, we see that for any integer l and for all n we get

$$f_l(z_n+\rho_n\xi_n)f_l^{(k)}(z_n+\rho_n\xi_n)-a(z_n+\rho_n\xi_n)=0$$

and

$$f_l(z_n+\rho_n\xi_n^*)f_l^{(k)}(z_n+\rho_n\xi_n^*)-a(z_n+\rho_n\xi_n^*)=0.$$

Fix l and take $n\to\infty$, and note $z_n+\rho_n\xi_n\to z_0$, $z_n+\rho_n\xi_n^*\to z_0$, then

$$f_l(z_0)f_l^{(k)}(z_0)-a(z_0)=0.$$

Since the zeros of $ff^{(k)}-a(\xi)$ has no accumulation point, so for sufficiently large n we get $z_n+\rho_n\xi_n=z_0$, $z_n+\rho_n\xi_n^*=z_0$. Hence, $\xi_n=\xi_n^*=\frac{z_0-z_n}{\rho_n}$.

This contradicts with $\xi_n\in D_1$ and $\xi_n^*\in D_2$ and $D_1\cap D_2=\emptyset$. Thus $\mathcal{F}$ is normal at z_0.

Case 2. $a(z_0)=0$. Let $z_0=0$, $D=\Delta=\{z:|z|<1\}$ and $a(z)=z^m+a_{m+1}z^{m+1}+\cdots=z^m\phi(z)$, $\phi(0)=1,\phi(z)\neq 1,z\in\{z:0<|z|<1\}$.

Since m is an even number, then we obtain a new family as follows

$$\mathcal{F}_1=\{F:=F(z)=\frac{f(z)}{z^{\frac{m}{2}}},f\in\mathcal{F}\}.$$

Suppose that $\mathcal{F}_1$ is not normal at $z_0 = 0$. Then by Lemma 2.1, there exists a sequence of complex numbers $z_n \longrightarrow 0$, a sequence of functions $F_n \in \mathcal{F}_1$ and a sequence of positive numbers $\rho_n \longrightarrow 0^+$ such that

$$h_n(\xi) = \rho_n^{-\frac{k}{2}} F_n(z_n + \rho_n\xi) \overset{\chi}{\Rightarrow} h(\xi),$$

where $h(\xi)$ is a non-constant meromorphic functions in $\mathbb{C}$. Also the order of $h(\xi)$ does not exceed 2 and by Hurwitz's theorem $h(\xi)$ has no zero of mulitiplicity less than $k + m + 1$.

Now we distinguish the following subcases.

Subcase 2.1. $\frac{z_n}{\rho_n} \to \infty$. By simple calculation, we have

$$\begin{aligned} f_n^{(k)}(z) &= z^{\frac{m}{2}} F_n^{(k)}(z) + \sum_{l=1}^{k} C_k^l (z^{\frac{m}{2}})^{(l)} F_n^{(k-l)}(z) \\ &= z^{\frac{m}{2}} F_n^{(k)}(z) + \sum_{l=1}^{k} C_l z^{\frac{m}{2}-l} F_n^{(k-l)}(z), \end{aligned}$$

where

$$C_l = \begin{cases} \frac{m}{2}(\frac{m}{2}-1)\cdots(\frac{m}{2}-l+1), l \le \frac{m}{2} \\ 0 \qquad\qquad\qquad\qquad\quad , l > \frac{m}{2}. \end{cases}$$

Since $f_n(z) = z^{\frac{m}{2}} F_n(z)$, then we have

$$\begin{aligned} f_n(z) f_n^{(k)}(z) &= f_n(z) z^{\frac{m}{2}} F_n^{(k)}(z) + f_n(z) \sum_{l=1}^{k} C_k^l (z^{\frac{m}{2}})^{(l)} F_n^{(k-l)}(z) \\ &= z^m F_n(z) F_n^{(k)}(z) + z^{\frac{m}{2}} F_n(z) \sum_{l=1}^{k} C_l z^{\frac{m}{2}-l} F_n^{(k-l)}(z), \end{aligned}$$

and

$$\begin{aligned} \frac{f_n(z) f_n^{(k)}(z)}{z^m} &= F_n(z) F_n^{(k)}(z) + \sum_{l=1}^{k} C_l \frac{F_n(z) F_n^{(k-l)}(z)}{z^l}, \\ \frac{f_n(z) f_n^{(k)}(z)}{a(z)} &= \left[F_n(z) F_n^{(k)}(z) + \sum_{l=1}^{k} C_l \frac{F_n(z) F_n^{(k-l)}(z)}{z^l} \right] \frac{1}{\phi(z)}. \end{aligned}$$

Noting that $h_n^{(k-l)}(\xi) = \rho_n^{\frac{k}{2}} F_n^{(k-l)}(z_n + \rho_n\xi)$, $l = 0, 1, \ldots, k$, then

$$\begin{aligned} &\frac{f_n(z_n + \rho_n\xi) f_n^{(k)}(z_n + \rho_n\xi)}{a(z_n + \rho_n\xi)} = \\ &\left[h_n(\xi) h_n^{(k)}(\xi) + \sum_{l=1}^{k} C_l \frac{h_n(\xi) h_n^{(k-l)}(\xi)}{(\frac{z_n}{\rho_n} + \xi)^l} \right] \frac{1}{\phi(z_n + \rho_n\xi)}. \end{aligned} \tag{3.1}$$

On the other hand, we have

$$\lim_{n\to\infty}\frac{C_l}{(\frac{z_n}{\rho_n}+\xi)^l}=0,\ \lim_{n\to\infty}\frac{1}{\phi(z_n+\rho_n\xi)}=1. \tag{3.2}$$

From (3.1) and (3.2) we get

$$\frac{f_n(z_n+\rho_n\xi)f_n^{(k)}(z_n+\rho_n\xi)}{a(z_n+\rho_n\xi)}\overset{\chi}{\Rightarrow}h(\xi)h^{(k)}(\xi),$$

on a every compact subset of $\mathbb{C}$ which contains no poles of $h(\xi)$. By lemmas 2.2 and 2.5. With similar discussion to the proof of Case 1, we can get a contradiction.

Subcase 2.2. $\frac{z_n}{\rho_n}\to\alpha,\alpha\in\mathbb{C}$. Then we have

$$\frac{F_n(\rho_n\xi)}{\rho_n^{\frac{k}{2}}}=\frac{F_n(z_n+\rho_n(\xi-\frac{z_n}{\rho_n}))}{\rho_n^{\frac{k}{2}}}\overset{\chi}{\Rightarrow}h(\xi-\alpha),$$

on every compact subset of $\mathbb{C}$ which contains no poles of $h(\xi-\alpha)$. Clearly, all zeros of $h(\xi-\alpha)$ have multiplicity at least $k+m+1$, and $\xi=0$ is a pole of $h(\xi-\alpha)$ with multiplicity at least $\frac{m}{2}$. Set

$$G_n(\xi):=\frac{f_n(\rho_n\xi)}{\rho_n^{\frac{k+m}{2}}}=\frac{F_n(\rho_n\xi)}{\rho_n^{\frac{k}{2}}}=\frac{F_n(\rho_n\xi)}{\rho_n^{\frac{k}{2}}}\frac{(\rho_n\xi)^{\frac{m}{2}}}{\rho_n^{\frac{m}{2}}}\overset{\chi}{\Rightarrow}\xi^{\frac{m}{2}}h(\xi-\alpha)=G(\xi),$$

on every compact subset of $\mathbb{C}$ which contains no poles of $G(\xi)$.

Clearly, $G(\xi)$ is a non-constant meromorphic function, which have multiple zeros at least $k+m+1$. Since $\xi=0$ is a pole of $h(\xi-\alpha)$ with multiplicity at least $\frac{m}{2}$, then $G(0)\neq 0$. Thus we have

$$G_n(\xi_n)G_n^{(k)}(\xi_n)-\frac{a(\rho_n\xi_n)}{\rho_n^m}\overset{\chi}{\Rightarrow}G(\xi)G^{(k)}(\xi)-\xi^m.$$

With similar discussion to the proof of Case 1, we can conclude $G(\xi)G^{(k)}(\xi)-\xi^m\not\equiv 0$. It follows from lemmas 2.2 and 2.5 that $G(\xi)G^{(k)}(\xi)-\xi^m$ has two distinct zeros at least. By the similar arguments in Case 1, we obtain a contradiction.

Hence $\mathcal{F}_1$ is normal at $z_0=0$.

It remains to show that $\mathcal{F}$ is normal at $z_0=0$. For $f_n(z)\in\mathcal{F}$, let $F_n(z)=\frac{f_n(z)}{z^{\frac{m}{2}}}$, then $\{F_n(z)\}\subset\mathcal{F}_1$. Since $\mathcal{F}_1$ is normal at $z_0=0$, then there exists $\Delta_\delta=\{z:|z|<\delta\}$ and a subsequence of $\{F_n(z)\}$(still express it as $\{F_n(z)\}$) such that $\{F_n(z)\}$ converges spherically locally uniformly to a meromorphic function $F(z)$ or ∞.

Here, we discuss the following two cases.

Case A. When n is large enough, $f_n(z)\neq 0$. Then $F(0)=\infty$, then we have $\delta_1>0$ such that $F_n(z)\geq 1$ for each $z\in\Delta(0,\delta_1)$. So $\frac{1}{f_n}$ is a holomorphic function in $\Delta(0,\delta_1)$. Thus when $z=\frac{\delta_1}{2}$, we get

$$|\frac{1}{f_n(z)}|=|\frac{1}{F_n(z)}\frac{1}{z^{\frac{m}{2}}}|\leq(\frac{2}{\delta_1})^{\frac{m}{2}}.$$

By the maximum principle and Montel's theorem, there exists subsequence of $\{f_n(z)\}$ (still express it as $\{f_n(z)\}$) converges spherically locally uniformly.

Therefore $\mathcal{F}$ is normal at $z_0 = 0$.

Case B. $f_n(z) = 0$. Then we get $F(0) = 0$, since $F_n(z) = \frac{f_n(z)}{z^{\frac{m}{2}}} \overset{\chi}{\Rightarrow} F(z)$, and hence there exists a positive number r with $0 < r < \delta$ such that $F(z)$ is holomorphic in Δ_r and has a unique zero $z = 0$ in Δ_r. Therefore, we have $f_n(z) \Rightarrow z^{\frac{m}{2}} F(z)$ in Δr since $F_n(z)$ converges spherically locally uniformly to a holomorphic function $F(z)$ in Δ_r. Thus $\mathcal{F}$ is normal at $z_0 = 0$.

These shows that $\mathcal{F}$ is normal in D. □

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Department of Mathematics,
Xuanwei Senior School,
Yunnan Xuanwei 655400,PR China
E-mail address: ynxwjz@hotmail.com