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ON THE DOMINATION AND TOTAL DOMINATION NUMBERS OF CAYLEY SUM GRAPHS OVER $\mathbb{Z}_n$

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Abstract. Let G be a finite Abelian group and S be a subset of G. The Cayley sum graph $\mathrm{Cay}^+(G,S)$ of G with respect to S is a graph whose vertex set is G and two vertices g and h are joined by an edge if and only if $g+h \in S$. In this paper, we prove some basic facts on the domination and total domination numbers of Cayley sum graphs. Then, we find the sharp bounds for domination number of $\mathrm{Cay}^+(\mathbb{Z}_n, S)$, where $S = \{1, 2, \dots, k\}$ and n, k are positive integers with $1 \leq k \leq (n-1)/2$.

1. Introduction

Let G be a finite Abelian group and S be a subset of G. The Cayley sum graph $\mathrm{Cay}^+(G,S)$ is the graph having the vertex set G and the edge set $\{\{g,h\} \mid g,h \in G, g+h \in S\}$. If S is a multiset, then $\mathrm{Cay}^+(G,S)$ contains multiple edges, and if there exists $g \in G$ with $2g \in S$, then the edge $\{g,g\}$ is a semiedge, i.e. an edge with one endpoint. Unlike a loop, a semiedge contributes just one to both the valency of its endpoint and the corresponding diagonal entry of the adjacency matrix. With this convention, $\mathrm{Cay}^+(G,S)$ is a regular graph with valency $|S|$.

Cayley sum graphs also are known under names addition Cayley graphs [5, 6, 7], addition graphs [2] and sum graphs [3]. The study of Cayley sum graphs has been the object of some papers, for example, independence number [1], hamiltonicity [2, 6], expander properties [3], clique number [4] and connectivity [5].

Recall that a set $D \subseteq V$ of vertices in a graph $\mathcal{G} = (V,E)$ is called a dominating set if every vertex $v \in V$ is either an element of D or is adjacent to an element of D. A dominating set D is a minimal dominating set if no proper subset is a dominating set. The domination number $\gamma(\mathcal{G})$ of a graph $\mathcal{G}$ is the minimum cardinality of a dominating set in $\mathcal{G}$ and the corresponding dominating set is called a γ-set.

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A set $T \subseteq V(\mathcal{G})$ is a total dominating set if every vertex in $V(\mathcal{G})$ is adjacent to a vertex in T. The total domination number of a graph $\mathcal{G}$ denoted by $\gamma_t(\mathcal{G})$ is the minimum cardinality of all total dominating sets. Clearly, $\gamma(\mathcal{G}) \leq \gamma_t(\mathcal{G})$.

In this paper, we give some results on the (total) domination number of Cayley sum graphs. Then, we find some (total) dominating sets and (total) domination numbers in graph $\mathrm{Cay}^+(\mathbb{Z}_n, \{1, 2, \ldots, k\})$, where n and k are positive integers with $1 \leq k \leq (n-1)/2$.

2. Main Results

Hereafter, n is a positive integer, $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ and $+$ stands for modulo n addition in $\mathbb{Z}_n$. If g and h are adjacent in $\mathrm{Cay}^+(G, S)$, i.e. $g + h \in S$, we write $g \sim h$. Also, $N(g) = \{h \in G \mid g + h \in S\}$ and $N[g] = N(g) \cup \{g\}$ denote the set of open neighbours and closed neighbours of g, respectively.

In what follows, we prove some basic facts on the (total) domination number of Cayley sum graphs.

Theorem 2.1. *Let G be a finite Abelian group of order n and S be a subset of G. Then,*

$$\lceil \frac{n}{|S| + 1} \rceil \leq \gamma(\mathrm{Cay}^+(G, S)) \leq n - |S| + 1.$$

Proof. First, we consider the lower bound. If $g \in G$ is a vertex such that $2g \in S$, then g can dominate $|S|$ vertices. If $g \in G$ is a vertex such that $2g \notin S$, then g can dominate itself and $|S|$ other vertices. So, $\gamma(\mathrm{Cay}^+(G, S)) \geq \lceil \frac{n}{|S| + 1} \rceil$.

For the upper bound, if there exists $g \in G$ such that $2g \notin S$, then g dominates $N[g]$ vertices and the vertices in $G \backslash N[g]$ dominate themselves. Hence, $G \backslash N(g)$ is a dominating set of cardinality $n - |S|$. If $2G \subseteq S$, then g dominates $N[g] = N(g)$ vertices, for every $g \in G$, and the vertices in $G \backslash N(g)$ dominate themselves. Thus, $G \backslash N(g) \cup \{g\}$ is a dominating set of cardinality $n - |S| + 1$. Therefore, $\gamma(\mathrm{Cay}^+(G, S)) \leq n - |S| + 1$. □

The bounds for $\gamma(\mathrm{Cay}^+(G, S))$ obtained in Theorem 2.1, can not be reduced in general. More specifically, the lower bound is attained for the graph $\mathcal{G}_1 = \mathrm{Cay}^+(\mathbb{Z}_4, \{1, 3\})$ and the upper bound is attained for the graph $\mathcal{G}_2 = \mathrm{Cay}^+(\mathbb{Z}_5, \{0, 1, 2, 4\})$.

Theorem 2.2. *Let G be a finite Abelian group of order n and S be a subset of G, such that* $\mathrm{Cay}^+(G, S)$ *has no component of size* 1. *Then,*

$$\gamma_t(\mathrm{Cay}^+(G, S)) \leq n - |S| + 1.$$

Proof. Let $g \in G$ and put $X = G \backslash N[g]$. If $X = \emptyset$, then $|S| = n - 1$ or $|S| = n$. Thus, $\gamma_t(\mathrm{Cay}^+(G, S)) = 2$ or $\gamma_t(\mathrm{Cay}^+(G, S)) = 1$. If $X \neq \emptyset$, then by considering I as the set of all components of size 1 of $\mathrm{Cay}^+(G, S)[X]$, the subgraph of $\mathrm{Cay}^+(G, S)$ induced by X, we have two cases;

Case 1: $I = \emptyset$. If $2g \in S$, then $X \cup \{g\}$ is a total dominating set and so, $\gamma_t(\mathrm{Cay}^+(G,S)) \leq |X| + 1 = n - |S| + 1$. If $2g \notin S$, then $X \cup \{g\} \cup \{h\}$ is a total dominating set, where $h \in N(g)$, and thus $\gamma_t(\mathrm{Cay}^+(G,S)) \leq |X|+2 = n-|S|+1$.

Case 2: $I \neq \emptyset$. We denote by $C(I)$ a subset of $N(g)$ with the smallest cardinality such that each $i \in I$ is adjacent to an element of $C(I)$. Clearly, $|C(I)| \leq |I|$. Then, $(X\backslash I) \cup C(I) \cup \{g\}$ is a total dominating set and so, $\gamma_t(\mathrm{Cay}^+(G,S)) \leq |X\backslash I| + |C(I)| + 1 \leq |X| + 1 \leq n - |S| + 1$. □

In [6], Lev proved that if S is a subset of a finite Abelian group G, then $\mathrm{Cay}^+(G,S)$ is connected if and only if S is not contained in a coset of a proper subgroup of G, except, perhaps, for the non-zero coset of a subgroup of index 2. In the following theorem, we find a bound for the total domination number of a connected Cayley sum graph.

Theorem 2.3. *Let G be a finite Abelian group of order n and S be a subset of G of size less than $n-1$ such that $\mathrm{Cay}^+(G,S)$ is connected. If $S \cap 2G = \emptyset$, then $\gamma_t(\mathrm{Cay}^+(G,S)) \leq n - |S|$ and if $S \cap 2G \neq \emptyset$, then $\gamma_t(\mathrm{Cay}^+(G,S)) \leq n - |S| + 1$.*

Proof. Let g, X and I be as in the proof of Theorem 2.2. Since $|S| \leq n-2$, $X \neq \emptyset$. We have two cases. If $I = \emptyset$, by connectivity of $\mathrm{Cay}^+(G,S)$, some $x \in X$ is adjacent to some $h \in N(g)$. Let V and Δ be the vertex set and maximum degree of the component of $\mathrm{Cay}^+(G,S)[X]$ which contains x, respectively. By Theorem 2.2, $\mathrm{Cay}^+(G,S)[V]$ has a total dominating set T of cardinality at most $|V|-\Delta+1$. If $\Delta = 1$ and $2x \in S$, then $|V| = 1$ and so $\{g,h\} \cup (X\backslash V)$ is a total dominating set. Thus, $\gamma_t(\mathrm{Cay}^+(G,S)) \leq 2+|X|-|V|$. Now, if $2g \in S$, then $|X| = n-|S|$ and so $\gamma_t(\mathrm{Cay}^+(G,S)) \leq n-|S|+1$, else $|X| = n - |S| - 1$ and so $\gamma_t(\mathrm{Cay}^+(G,S)) \leq n - |S|$. If $\Delta = 1$ and $2x \notin S$, then $|V| = 2$ and so $\{g,h,x\} \cup (X\backslash V)$ is a total dominating set. Thus, similar to above discussion, $\gamma_t(\mathrm{Cay}^+(G,S)) \leq n - |S| + 1$ or $\gamma_t(\mathrm{Cay}^+(G,S)) \leq n - |S|$. If $\Delta > 1$, then $\{g,h\} \cup T \cup (X\backslash V)$ is a total dominating set and thus $\gamma_t(\mathrm{Cay}^+(G,S)) \leq 2 + |T| + |X| - |V| \leq |X| - \Delta + 3 = |X| + 1 + (2-\Delta) \leq |X| + 1$. Again, by similar discussion mentioned above, $\gamma_t(\mathrm{Cay}^+(G,S)) \leq n-|S|+1$ or $\gamma_t(\mathrm{Cay}^+(G,S)) \leq n-|S|$. The proof of the remaining case ($I \neq \emptyset$) is similar to that of Theorem 2.2. □

Now, we find some results on $\gamma(\mathrm{Cay}^+(\mathbb{Z}_n,S))$ and $\gamma_t(\mathrm{Cay}^+(\mathbb{Z}_n,S))$, where $S = \{1,2,\ldots,k\}$ and $1 \leq k \leq (n-1)/2$.

Theorem 2.4. *Let $\mathcal{G} = \mathrm{Cay}^+(\mathbb{Z}_n, \{1,2,\ldots,k\})$, where n,k are positive integers with $1 \leq k \leq (n-1)/2$. Then, $\lceil \frac{n}{k+1} \rceil \leq \gamma(\mathcal{G}) \leq \lceil \frac{n}{k} \rceil$.*

Proof. Consider the set $D = \{0, k, 2k, \ldots, (l-1)k\}$, where $l = \lceil n/k \rceil$. We claim that D is a dominating set of $\mathcal{G}$. For an arbitrary vertex $c \in \mathbb{Z}_n$, if $c \in S$, then $c \sim 0$. If $c \in \mathbb{Z}_n \setminus S$, then by division algorithm, we have $n - c = ik + r$ such that $0 \leq r \leq k-1$ and $0 \leq i \leq l-2$. So, $c + (i+1)k = c + ik + k = n - r + k \equiv k - r < k \pmod n$ and thus $c \sim (i+1)k \in D$ which implies that $c \in N((i+1)k)$ and D is a dominating set. Therefore, by Theorem 2.1, $\lceil \frac{n}{k+1} \rceil \leq \gamma(\mathcal{G}) \leq \lceil \frac{n}{k} \rceil$. □

The bounds are given in Theorem 2.4 are sharp. Consider two Cayley sum graphs $\mathcal{G}_1 = \mathrm{Cay}^+(\mathbb{Z}_{15}, \{1,2,3,4\})$ and $\mathcal{G}_2 = \mathrm{Cay}^+(\mathbb{Z}_8, \{1,2,3\})$. The lower and upper bound is attained for $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

Let $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ be a graph isomorphism and D be a γ-set of $\mathcal{G}$. One can easily show that $\varphi(D)$ is a dominating set of $\mathcal{H}$. So, $\gamma(\mathcal{H}) \le |\varphi(D)| = \gamma(\mathcal{G})$. Similarly, by considering $\varphi^{-1} : \mathcal{H} \longrightarrow \mathcal{G}$, we have $\gamma(\mathcal{G}) \le \gamma(\mathcal{H})$. Therefore we have the following lemma.

Lemma 2.1. *If two graphs are isomorphic, then their domination numbers are equal.*

Theorem 2.5. *Let n and r be positive integers. Then,*

$$\gamma(\mathrm{Cay}^+(\mathbb{Z}_n, A_i)) = \gamma(\mathrm{Cay}^+(\mathbb{Z}_n, A_{i+2r})),$$

where $A_i = \{i, i+1, \ldots, i+k-1\}$, for every $i, k \in \mathbb{Z}_n$.

Proof. Define $\varphi : \mathrm{Cay}^+(\mathbb{Z}_n, A_i) \longrightarrow \mathrm{Cay}^+(\mathbb{Z}_n, A_{i+2r})$ given by $\varphi(m) = m + r$. Clearly, φ is a bijection. It remains to show that φ is adjacent preserving. In $\mathrm{Cay}^+(\mathbb{Z}_n, A_i)$, if $a \sim b$, then $\varphi(a) + \varphi(b) = a + b + 2r \in A_i + 2r = A_{i+2r}$ and so in $\mathrm{Cay}^+(\mathbb{Z}_n, A_{i+2r})$ we have $\varphi(a) \sim \varphi(b)$. Therefore, φ is a graph isomorphism. Now, Lemma 2.1 implies the result. □

The converse of above theorem does not necessary hold. For example, let $n = 6$, $A_1 = \{1,2,3\}$ and $A_2 = \{2,3,4\}$. We have $\gamma(\mathrm{Cay}^+(\mathbb{Z}_6, A_1)) = \gamma(\mathrm{Cay}^+(\mathbb{Z}_6, A_2))$, but there is no $r \in \mathbb{Z}_6$ such that $A_2 = A_{1+2r}$.

Theorem 2.6. *Let n and r be two positive integers such that $(r, n) = 1$. If A is a subset of $\mathbb{Z}_n$, then*

$$\gamma(\mathrm{Cay}^+(\mathbb{Z}_n, A)) = \gamma(\mathrm{Cay}^+(\mathbb{Z}_n, rA)).$$

Proof. Define $\varphi : \mathrm{Cay}^+(\mathbb{Z}_n, A) \longrightarrow \mathrm{Cay}^+(\mathbb{Z}_n, rA)$ given by $\varphi(m) = rm$. Clearly, φ is a bijection. It suffices to show that φ is adjacent preserving. In $\mathrm{Cay}^+(\mathbb{Z}_n, A)$, if $a \sim b$, then $\varphi(a) + \varphi(b) = ra + rb = r(a+b) \in rA$ and so in $\mathrm{Cay}^+(\mathbb{Z}_n, rA)$ we have $\varphi(a) \sim \varphi(b)$. Therefore, φ is a graph isomorphism and thus, by Lemma 2.1, the result holds. □

Theorem 2.7. *Let $S = \{1, 2, \ldots, k\}$ be a subset of $\mathbb{Z}_n$, where $n \in \mathbb{N}$ and $k \in \mathbb{Z}_n$. Then,*

$$\gamma_t(\mathrm{Cay}^+(\mathbb{Z}_n, S)) = \lceil \frac{n}{k} \rceil.$$

Proof. Let T_0 be a γ_t-set of $\mathrm{Cay}^+(\mathbb{Z}_n, S)$ such that $\gamma_t(\mathrm{Cay}^+(\mathbb{Z}_n, S)) = |T_0|$. Every element in T_0 is adjacent to at most $k-1$ vertices in $V(\mathcal{G}) \setminus T_0$. Also, by definition, every vertex in T_0 is adjacent to some vertices in T_0. Thus, $|T_0|(k-1) + |T_0| \ge n$ and so, $|T_0|k \ge n$ which implies that $|T_0| = \gamma_t(\mathcal{G}) \ge \lceil \frac{n}{k} \rceil$.

Now, we claim that $T = \{0, k, 2k, \ldots, (\lceil \frac{n}{k} \rceil - 1)k\}$ is a total dominating set. By Theorem 2.4, it is enough to show that every element of T is adjacent to some elements of T. Put $l := \lceil \frac{n}{k} \rceil$. So, $n = (l-1)k + r$, where $0 \le r < k$. For $i = 1, 2, \ldots, l-1$, let ik be an arbitrary element of T. Since $lk = n + k - r \equiv k - r \leq k \pmod{n}$ and $(ik) + ((l-i)k) = lk$, we conclude that $ik \sim (l-i)k$. Also, $0 \sim k$. So, T is a total dominating set. Thus $\gamma_t(\mathcal{G}) \leq |T| = \lceil \frac{n}{k} \rceil$, as required. □

Theorem 2.8. *Let $S = \{1, 2, \ldots, k\} \subseteq \mathbb{Z}_n$ and $T = \{0, k, 2k, \ldots, (\lceil \frac{n}{k} \rceil - 1)k\}$, where $k \in \mathbb{Z}_n$ and $n \in \mathbb{N}$. Then, $T_i = T + i$ is a total dominating set of $\mathrm{Cay}^+(\mathbb{Z}_n, S)$, for all $i \in \{0, 1, \ldots, k-1\}$.*

Proof. Let c be an arbitrary vertex in $\mathrm{Cay}^+(\mathbb{Z}_n, S)$. If $c \in \{1, 2, \ldots, n-i-1\}$, then $n - c - i = tk + r$, where $0 \leq r \leq k-1$ and so $c + (t+1)k + i = n + k - r \equiv k - r \in S \pmod{n}$ implies that c is adjacent to $(t+1)k + i \in T_i$. If $c = n - i$, then $c + i + k = n - i + i + k = n + k \equiv k \in S \pmod{n}$ implies that c is adjacent to $i + k \in T_i$. If $c \in \{n-i+1, \ldots, n\}$, then c is adjacent to $i \in T_i$, because $c + i \in S$. Therefore, T_i is a total dominating set, for every $i \in \{0, 1, \ldots, k-1\}$. □

Consider the notations used in Theorem 2.8. Let c be an arbitrary vertex in $\mathrm{Cay}^+(\mathbb{Z}_n, S)$. Then, there exist integers t and i such that $c = tk + i$ and $0 \leq i \leq k-1$. Clearly, $c \in T_i$, where $0 \leq i \leq k-1$. This shows that every vertex in $\mathrm{Cay}^+(\mathbb{Z}_n, S)$ is contained in some γ_t−sets.

Theorem 2.9. *Let $\mathrm{Cay}^+(\mathbb{Z}_n, S_1)$ and $\mathrm{Cay}^+(\mathbb{Z}_m, S_2)$ be two connected graphs, where $S_1 \subseteq \mathbb{Z}_n$ and $S_2 \subseteq \mathbb{Z}_m$. Then,*

$$\gamma_t(\mathrm{Cay}^+(\mathbb{Z}_n, S_1))\ \gamma_t(\mathrm{Cay}^+(\mathbb{Z}_m, S_2)) \geq \gamma_t(\mathrm{Cay}^+(\mathbb{Z}_n \times \mathbb{Z}_m, S_1 \times S_2)).$$

Proof. Suppose that T_1 and T_2 are γ_t-sets of $\mathrm{Cay}^+(\mathbb{Z}_n, S_1)$ and $\mathrm{Cay}^+(\mathbb{Z}_m, S_2)$, respectively. We claim that $T = T_1 \times T_2$ is a γ_t-set of $\mathrm{Cay}^+(\mathbb{Z}_n \times \mathbb{Z}_m, S_1 \times S_2)$. For an arbitrary element $(x, y) \in \mathbb{Z}_n \times \mathbb{Z}_m$, there exist $t_1 \in T_1$ and $t_2 \in T_2$ such that x is adjacent to t_1 and y is adjacent to t_2. Therefore, (x, y) is adjacent to (t_1, t_2) in T and thus, T is a total dominating set of $\mathrm{Cay}^+(\mathbb{Z}_n \times \mathbb{Z}_m, S_1 \times S_2)$. □

Note that, in Theorem 2.9, the equality can be hold. For example, consider $\mathrm{Cay}^+(\mathbb{Z}_2, \{1\})$ and $\mathrm{Cay}^+(\mathbb{Z}_3, \{1, 2\})$. By Theorem 2.7, we have $\gamma_t(\mathrm{Cay}^+(\mathbb{Z}_2, \{1\})) = \gamma_t(\mathrm{Cay}^+(\mathbb{Z}_3, \{1, 2\})) = 2$. On the other hand $\gamma_t(\mathrm{Cay}^+(\mathbb{Z}_2 \times \mathbb{Z}_3, \{(1,1), (1,2)\})) = 4$, as desired.

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