A PAIR OF NON-SELF MAPPINGS IN CONE METRIC SPACES

STOJAN RADENOVIĆ

Abstract. In this paper we extend from metric to cone metric spaces a fixed point theorem for a pair of non-self maps proved by M. S. Khan et al. [M. S. Khan, K. Pathak and M. D. Khan, Some fixed point theorems in metrically convex spaces, Gregorian Mathematical Journal, Volume 7 (3) (2000), 523-530].

1. Introduction and preliminaries

Cone metric spaces were introduced in [3], where the authors described convergence in cone metric spaces, introduced completeness and proved some fixed point theorems of contractive mappings on these spaces. Recently, in [1], [4] and [13], some common fixed point theorems are proved for maps on cone metric spaces. The following definitions and results will be needed in the rest of this paper.

Let $E$ be a real Banach space. A subset $P$ of $E$ is called a cone if and only if:

(i) $P$ is closed, nonempty and $P \neq \{\theta\}$;
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;
(iii) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, a partial ordering $\preceq$ with respect to $P$ is defined by $x \preceq y$ if $y - x \in P$. We write $x < y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \text{int}P$ (interior of $P$). A cone $P \subset E$ is normal if there is a number $k > 0$ such that for all $x, y \in P : \theta \preceq x \preceq y$ implies $\|x\| \leq k \|y\|$. The least positive number satisfying the previous condition is called the normal constant of $P$.

Definition 1.1. [3] Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

Key words and phrases. Cone metric spaces; normal and non-normal cone; metrically convex; non-self maps; fixed point.

2010 Mathematics Subject Classification. Primary: 54H25, Secondary: 47H10.

Received: January 20, 2012.

Revised: October 25, 2012.
Let \(\{x_n\}\) be a sequence in \(X\), and \(x \in X\). If, for every \(c\) in \(E\) with \(\theta \ll c\), there exists an \(n_0 \in \mathbb{N}\) such that, for all \(n > n_0\), \(d(x_n, x) \ll c\), then it is said that \(x_n\) converges to \(x\), denoted by \(\lim_{n \to \infty} x_n = x\), or \(x_n \to x\), \(n \to \infty\). If, for every \(c\) in \(E\), such that \(\theta \ll c\), there is an \(n_0 \in \mathbb{N}\) such that for all \(m > n_0\), \(d(x_n, x_m) \ll c\), then \(\{x_n\}\) is a Cauchy sequence in \(X\). If every Cauchy sequence is convergent in \(X\), then \(X\) is a complete cone metric space.

Let \((X, d)\) be a cone metric space. Then the following properties are often useful, particularly when dealing with cone metric spaces in which the cone need not to be normal (for details see \([7]\)):

\[(p_1)\] If \(u \preceq v\) and \(v \ll w\), then \(u \ll w\).

\[(p_2)\] If \(\theta \preceq u \ll c\), for each \(c \in \text{int}P\), then \(u = \theta\).

\[(p_3)\] If \(a \preceq b + c\), for each \(c \in \text{int}P\), then \(a \preceq b\).

\[(p_4)\] If \(E\) is a real Banach space with a cone \(P\), and if \(a \preceq \lambda a\), where \(a \in P\) and \(0 \leq \lambda < 1\), then \(a = \theta\).

\[(p_5)\] If \(c \in \text{int}P\), \(a_n \in E\), and \(a_n \to \theta\), then there exists an \(n_0\) such that, for all \(n > n_0\), we have \(a_n \ll c\). For details on cone metric spaces see three recent very important papers: \([8]\), \([9]\), \([12]\).

In the following we suppose only that \(E\) is a Banach space, that \(P\) is a cone in \(E\) with \(\text{int}P \neq \emptyset\), and that “\(\preceq\)” is a partial ordering with respect to \(P\).

There exist a lot of fixed-point theorems for self-mappings defined on closed subsets on Banach spaces. However, for applications (numerical analysis, optimization, etc.) it is important to consider functions that are not self-mappings, and it is natural to search for sufficient conditions which would guarantee the existence of fixed points for such mappings—for details see: \([5]\), \([7]\) and \([11]\). In this paper we continue to study the non-self mappings in the frame of the cone metric spaces, started in \([7]\) and \([11]\).

Generalizing a theorem of \([2]\), Khan et al., \([10]\) proved the following result, valid in a complete metrically convex space.

**Theorem 1.1.** Let \(X\) be a complete metrically convex space, and \(K\) a closed nonempty subset of \(X\). Let \(T : K \to X\) be the mapping satisfying the inequality

\[(1.1)\quad d(Tx, Ty) \leq C \cdot \max\{d(x, Tx), d(y, Ty)\} + C' \cdot (d(x, Ty) + d(y, Tx)),\]
for all $x, y$ in $K$, where $C$ and $C'$ are nonnegative reals such that:

$$\max\left\{\frac{C+1}{1-C}, \frac{C'}{1-C-C'}\right\} = h > 0,$$

and

$$\max\left\{\frac{1+1}{1-C-C'}h, \frac{1+1}{1-C-C'}h\right\} = h'$$

Further, $Tx \in K$, for every $x \in \partial K$. Then $T$ has a unique fixed point in $K$.

2. Main result

The purpose of this paper is to prove the analogue of Theorem 1.1, for a pair of non-self maps in the frame of cone metric spaces, using the concept of metric convexity. We use only the definition of convergence in terms of the relation “$\ll$”. The only assumption is that the interior of the cone $P$ is nonempty - hence we use neither continuity of the vector metric $d$, nor Sandwich Theorem. We begin with the following definition.

**Definition 2.1.** Let $(X, d)$ be a cone metric space, let $K$ be a nonempty closed subset of $X$, and let $f, g : K \to X$. If $f$ and $g$ satisfy the condition

(2.1) \[ d(fx, fy) \leq C \cdot u(x, y) + C' \cdot (d(fx, gy) + d(fy, gx)), \]

where $u(x, y) \in \{d(fx, gx), d(fy, gy)\}$, for all $x, y$ in $K$, and where $C, C'$ are nonnegative reals as in Theorem 1.1, then $f$ is a generalized $g$-contractive mapping of $K$ into $X$.

A pair of nonself-mappings $(f, g)$, defined on a nonempty subset $K$ of a cone metric space $(X, d)$, is said to be coincidentally commuting if, for $fx, gx \in K$, $fx = gx$ implies that $fgx = gfx$. Note that, for $K = X$, this notion is reduced to the corresponding notion of Jungck and Rhoades for self-mappings.

We state and prove our main result as follows.

**Theorem 2.1.** Let $(X, d)$ be a complete cone metric space, let $K$ be a nonempty closed subset of $X$ such that, for each $x \in K$ and $y \notin K$, there exists a point $z \in \partial K$ such that

(2.2) \[ d(x, z) + d(z, y) = d(x, y). \]

Suppose that $f, g : K \to X$ are such that $f$ is a generalized $g$-contractive mapping of $K$ into $X$, and

(i) $\partial K \subseteq gK$, $fK \cap K \subseteq gK$;
(ii) $gx \in \partial K \Rightarrow fx \in K$;
(iii) $gK$ is closed in $X$.

Then there exists a point of coincidence $p$ in $K$. Moreover, if $(f, g)$ are coincidentally commuting, then $p$ is the unique common fixed point of $f$ and $g$.

**Proof.** First, we construct two sequences: $\{x_n\}$ in $K$ and the sequence $\{y_n\}$ in $fK \subset X$ in the following way.
Let $x \in \partial K$ be arbitrary. There exists a point $x_0 \in K$ such that $x = gx_0$ as $\partial K \subset gK$. Since $gx_0 \in \partial K \Rightarrow fx_0 \in K$, we conclude that $fx_0 \in K \cap fK \subset gK$. Let $x_1 \in K$ be such that $y_1 = gx_1 = fx_0 \in K$. Let $y_2 = fx_1$. Suppose $y_2 \in K$. Then $y_2 \in K \cap fK \subset gK$, which implies that there exists a point $x_2 \in K$ such that $y_2 = gx_2$. Suppose $y_2 \notin K$. Then there exists a point $p \in \partial K$ such that $d(gx_1, p) + d(p, y_2) = d(gx_1, y_2)$. Since $p \in \partial K \subset gK$, there exists a point $x_2 \in K$ such that $p = gx_2$, so that the equation above takes the form $d(gx_1, gx_2) + d(gx_2, y_2) = d(gx_1, y_2)$. Put $y_3 = fx_2$. In this way, repeating the foregoing arguments, one obtains two sequences: 

\[
\{x_n\} \subset K \text{ and } \{y_n\} \subset fK \subset X \text{ such that:}
\]

(a) $y_{n+1} = fx_n$, for $n = 0, 1, 2, \ldots$;

(b) if $y_n \in K$, then $y_n = gx_n = fx_{n-1}$;

(c) if $y_n \notin K$, then $gx_n \in \partial K$ and $d(gx_{n-1}, gx_n) + d(gx_n, y_n) = d(gx_{n-1}, y_n)$.

Put

\[
S = \{gx_i \in \{gx_n\} : gx_i = y_i\} \text{ and } Q = \{gx_i \in \{gx_n\} : gx_i \neq y_i\}.
\]

Two consecutive terms cannot lie in $Q$. We have to estimate $d(gx_n, gx_{n+1})$. If $d(gx_n, gx_{n+1}) = \theta$ for some $n$, then it is easy to show that $d(gx_n, gx_{n+k}) = \theta$ for all $k \geq 1$.

Suppose that $d(gx_n, gx_{n+1}) \succ \theta$ for all $n$. From the presented construction we distinguish three cases:

**Case 1.** If $gx_n \in S$ and $gx_{n+1} \in S$, then according to (a), (b) and (2.1) we have:

\[
d(gx_n, gx_{n+1}) = d(y_n, y_{n+1}) = d(fx_{n-1}, fx_n) \\
\leq C \cdot u_1(x_{n-1}, x_n) + C' \cdot (d(gx_n, gx_n) + d(gx_{n+1}, gx_{n-1})) \\
= C \cdot u_1(x_{n-1}, x_n) + C' \cdot (d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n-1})) \\
= C \cdot u_1(x_{n-1}, x_n) + C' \cdot d(gx_n, gx_{n+1}) + C' \cdot d(gx_n, gx_{n-1}),
\]

where

\[
u_1(x_{n-1}, x_n) \in \{d(fx_{n-1}, gx_n), d(fx_n, gx_n)\} = \{d(gx_n, gx_{n-1}), d(gx_{n+1}, gx_n)\}.
\]

Clearly, there are infinitely many $n$ such that at least one of the following two possibilities holds:

**I:** $d(gx_n, gx_{n+1}) \leq C \cdot d(gx_n, gx_{n-1}) + C' \cdot d(gx_{n+1}, gx_n) + C' \cdot d(gx_n, gx_{n-1})$, that is,

\[
d(gx_n, gx_{n+1}) \leq \frac{C + C'}{1 - C'} \cdot d(gx_{n-1}, gx_n).
\]

**II:** $d(gx_n, gx_{n+1}) \leq C \cdot d(gx_{n+1}, gx_n) + C' \cdot d(gx_{n+1}, gx_n) + C' \cdot d(gx_n, gx_{n-1})$, that is,

\[
d(gx_n, gx_{n+1}) \leq \frac{C'}{1 - C - C'} \cdot d(gx_{n-1}, gx_n).
\]
Case 2. Let $gx_n \in S$, $gx_{n+1} \in Q$. Then $y_n = gx_n$, $y_{n+1} \notin K$, $gx_{n+1} \in \partial K$ such that $d(y_{n+1}, gx_{n+1}) + d(gx_{n+1}, gx_n) = d(y_{n+1}, gx_n)$. We have

$$d(gx_n, gx_{n+1}) \leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, y_{n+1})$$

$$= d(gx_n, y_{n+1}) = d(fx_n, gx_n) = d(fx_{n-1}, fx_n)$$

$$\leq C \cdot u_2(x_{n-1}, x_n) + C' \cdot (d(gx_n, gx_{n+1}) + d(fx_n, gx_{n-1}))$$

$$= C \cdot u_2(x_{n-1}, x_n) + C' \cdot (\theta + d(fx_n, gx_{n-1}))$$

$$\leq C \cdot u_2(x_{n-1}, x_n) + C' \cdot d(fx_n, gx_{n-1}), \quad (2.3)$$

where

$$u_2(x_{n-1}, x_n) \in \{d(fx_{n-1}, gx_{n-1}), d(fx_n, gx_n)\} = \{d(gx_n, gx_{n-1}), d(fx_n, gx_n)\}.$$

First, we have the following two cases:

**I:**

$$d(fx_n, gx_n) \leq C \cdot d(gx_n, gx_{n-1}) + C' \cdot d(fx_n, gx_{n-1})$$

$$\leq C \cdot d(gx_n, gx_{n-1}) + C' \cdot d(fx_n, gx_n) + C' \cdot d(gx_n, gx_{n-1}).$$

It follows that

$$d(fx_n, gx_n) \leq \frac{C + C'}{1 - C'} \cdot d(gx_n, gx_{n-1})$$

i.e.,

$$d(gx_n, gx_{n+1}) \leq \frac{C + C'}{1 - C'} \cdot d(gx_n, gx_{n-1}). \quad (2.4)$$

**II:** In the similar manner we obtain

$$d(fx_n, gx_n) \leq C \cdot d(fx_n, gx_n) + C' \cdot d(fx_n, gx_n) + C' \cdot d(gx_n, gx_{n-1}),$$

that is,

$$d(fx_n, gx_n) \leq \frac{C'}{1 - C - C'} \cdot d(gx_n, gx_{n-1}). \quad (2.5)$$

From (2.4) and (2.5) we get

$$d(fx_n, gx_n) \leq h \cdot d(gx_n, gx_{n-1}), \quad (2.6)$$

where $h = \max \left\{ \frac{C + C'}{1 - C'}, \frac{C'}{1 - C - C'} \right\}$. Now, according to (2.3) we obtain

$$d(gx_n, gx_{n+1}) \leq h \cdot d(gx_n, gx_{n-1}). \quad (2.7)$$

Case 3. Let $gx_n \in Q$, $gx_{n+1} \in S$. Then $y_{n+1} = fx_n$, $y_n \notin K$ and $gx_n \in \partial K$, such that $d(y_n, gx_n) + d(gx_n, y_{n+1}) = d(y_n, y_{n+1})$, where $y_{n+1} = fx_{n-1} =$
Now, in view of Case 2, we have
\[
d(g_{n+1}, g_{n+2}) \leq d(g_{n+1}, y_n) + d(y_n, g_{n+1}) \\
\leq d(g_{n+1}, g) + d(g, y_n) + d(y_n, g_{n+1}) \\
= d(g_{n+1}, y_n) + d(y_n, g_{n+1})
\]
(2.8)
\[
\leq h'd(g_{n-2}, g_{n-1}) + d(f_{n-1}, f_n) \\
\leq h'd(g_{n-2}, g_{n-1}) + Cu_3(x_{n-1}, x_n) + C' (d(f_{n-1}, g_{n-1}) + d(f_n, g_{n-1})),
\]
where \(u_3(x_{n-1}, x_n) \in \{d(f_{n-1}, g_{n-1}), d(f_n, g_{n-1}) = d(g_n, g_{n+1})\}\). Further, it follows that
\[
d(f_{n-1}, g_{n+1}) + d(f_n, g_{n-1}) = d(f_{n-1}, g_{n+1}) + d(g_{n+1}, g_{n-1}) \\
\leq d(g_{n+1}, g) + d(g, g_{n-1}) + d(f_{n-1}, g_{n-1}) \\
= d(g_{n+1}, g) + d(g_{n-1}, f_{n-1}) \\
\leq d(g_n, g_{n+1}) + h \cdot d(g_{n-2}, g_{n-1}).
\]

Now, from (2.9) we obtain:

\begin{align*}
\text{I} : & \quad d(g_n, g_{n+1}) \leq h \cdot d(g_{n-2}, g_{n-1}) + C \cdot h'd(g_{n-2}, g_{n-1}) \\
& \quad + C' \cdot (d(g_n, g_{n+1}) + h \cdot d(g_{n-2}, g_{n-1})),
\end{align*}

that is,
\[
d(g_n, g_{n+1}) \leq \frac{1 + C + C'}{1 - C'} \cdot h \cdot d(g_{n-2}, g_{n-1}).
\]

(2.10)

\begin{align*}
\text{II} : & \quad d(g_n, g_{n+1}) \leq h \cdot d(g_{n-2}, g_{n-1}) + C \cdot d(g_n, g_{n+1}) \\
& \quad + C' \cdot (d(g_n, g_{n+1}) + h \cdot d(g_{n-2}, g_{n-1})),
\end{align*}

that is
\[
d(g_n, g_{n+1}) \leq \frac{1 + C'}{1 - C} \cdot h \cdot d(g_{n-2}, g_{n-1}).
\]

(2.11)

Taking \(h' = \max \{\frac{1+C+C'}{1-C'}, \frac{1+C'}{1-C} \cdot h\}\), we get from (2.10) and (2.11)
\[
d(g_n, g_{n+1}) \leq h' \cdot d(g_{n-2}, g_{n-1}).
\]

(2.12)

Now, from (2.7) and (2.12) we have in all Cases 1–3
\[
d(g_n, g_{n+1}) \leq h'' \cdot w_n,
\]
where \(h'' = \max \{h, h'\}\) and \(w_n \in \{d(g_{n-2}, g_{n-1}), d(g_{n-1}, g_n)\}\).

From (2.13), following the procedure of Assad and Kirk [2] (see also [5], [7], [11]), we shall show, by induction, that for all \(n > 1\),
\[
d(g_n, g_{n+1}) \leq h''^{n-1} \cdot w_2,
\]
where \(w_2 \in \{d(g_0, g_1), d(g_1, g_2)\}\).
For \( n = 2 \), according to (2.13) we have, \( d(gx_1, gx_3) \leq h'' \cdot w_2 \), where
\[
 w_2 \in \{ d(gx_2, gx_2-1), d(gx_2-1, gx_2) \} = \{ d(gx_0, gx_1), d(gx_1, gx_2) \},
\]
that is \( d(gx_2, gx_3) \leq h''^{\frac{n-1}{2}} \cdot w_2 \), since \( h'' < h''^{\frac{n}{2}} \). Hence (2.14) holds.

Similarly, again according to (2.13), for \( n = 3 \) we have \( d(gx_3, gx_4) \leq h'' \cdot w_3 \), where
\[
 w_3 \in \{ d(gx_3, gx_3-1), d(gx_3-1, gx_3) \} = \{ d(gx_1, gx_2), d(gx_2, gx_3) \}.
\]
If \( w_3 = d(gx_1, gx_2) \), it follows that
\[
d(gx_3, gx_4) \leq h''d(gx_1, gx_2) = h''^{\frac{n-1}{2}}w_2,
\]
so (2.14) holds. If \( w_3 = d(gx_2, gx_3) \) we have
\[
d(gx_3, gx_4) \leq h'' \cdot d(gx_2, gx_3) \leq h''^2w_2 < h''^{\frac{n-1}{2}}w_2
\]
(because \( h'' \in (0, 1) \)), and (2.14) also holds.

Therefore, (2.14) holds for \( n = 2 \) and \( n = 3 \). Suppose now that (2.14) holds for some \( n \) and \( n + 1 \). Then again from (2.13) it follows that \( d(gx_{n+2}, gx_{n+3}) \leq h'' \cdot w_{n+2} \), where \( w_{n+2} \in \{ d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}) \} \), that is, we get the following two cases:
\[
d(gx_{n+2}, gx_{n+3}) \leq h'' \cdot d(gx_n, gx_{n+1}) \quad \text{and} \quad d(gx_{n+2}, gx_{n+3}) \leq h'' \cdot d(gx_{n+1}, gx_{n+2}).
\]
Since (2.14) holds for \( n \) and \( n + 1 \), we obtain in the first case
\[
d(gx_{n+2}, gx_{n+3}) \leq h'' \cdot d(gx_n, gx_{n+1}) \leq h'' \cdot h''^{\frac{n-1}{2}} \cdot w_2 = h''^{\frac{n+1}{2}}w_2,
\]
and in the second case
\[
d(gx_{n+2}, gx_{n+3}) \leq h'' \cdot d(gx_{n+1}, gx_{n+2}) \leq h'' \cdot h''^{\frac{n}{2}} \cdot w_2 \leq h''^{\frac{n+1}{2}}w_2,
\]
where \( w_2 \in \{ d(gx_0, gx_1), d(gx_1, gx_2) \} \).

Thus by induction we conclude that (2.14) holds for all \( n > 1 \).

From (2.14) and by the triangle inequality, for \( n > m \) we have:
\[
d(gx_n, gx_m) \leq d(gx_n, gx_{n-1}) + d(gx_{n-1}, gx_{n-2}) + \cdots + d(gx_{m+1}, gx_m) \leq \left( h''^\frac{n-2}{2} + h''^\frac{n-3}{2} + \cdots + h''^\frac{n-1}{2} \right) \cdot w_2 \leq \frac{\sqrt{h''^{n-1}}}{1 - \sqrt{h''}} : w_2 \to \theta, \text{ as } m \to \infty.
\]

According to (p5) and (p1) it follows that, for \( \theta < c \) and large \( m \), \( d(gx_n, gx_m) < c \), i.e., \( \{gx_n\} \) is a Cauchy sequence. Since \( gx_n \in K \cap gK \) and \( K \cap gK \) is complete, there exists a point \( p \in K \cap gK \) such that \( gx_n \to p \). Let \( q \) in \( K \) be such that \( gq = p \). By the construction of \( \{gx_n\} \), there exists a subsequence \( \{g_{n(k)}\} \) such that
$g x_{n(k)} = y_{n(k)} = f x_{n(k)-1}$ and hence $f x_{n(k)-1} \to p$. We now prove that $f g = p$. We have
\[ d(f g, p) \leq d(f, f x_{n(k)-1}) + d(f x_{n(k)-1}, p) \]
\[ \leq C \cdot u_{n(k)} + C' (d(f g x_{n(k)-1}), d(f x_{n(k)-1}, g g) + d(f x_{n(k)-1}, p), \]
where $u_{n(k)} \in \{d(f g, g g), d(f x_{n(k)-1}, g x_{n(k)-1})\}$. Since $y_{n(k)} = f x_{n(k)-1} \to p$, as $k \to \infty$, we obtain the following two cases:

1) $u_{n(k)} = d(f g, g g)$. Then
\[ d(f g, p) \leq C \cdot d(f q, g g) + C' d(f g x_{n(k)-1}) + C' d(f x_{n(k)-1}, g g) + d(f x_{n(k)-1}, p) \]
\[ \leq C \cdot d(f q, p) + C \cdot d(p, g g) + C' d(f q, p) + C' d(p, g x_{n(k)-1}) \]
\[ + C' d(f x_{n(k)-1}, p) + d(f x_{n(k)-1}, p), \]
that is,
\[ d(f q, p) \leq \frac{C' d(p, g x_{n(k)-1})}{1 - C - C'} + \frac{(1 + C') d(f x_{n(k)-1}, p)}{1 - C - C'}. \]

Put $a = \frac{C'}{1 - C - C'}$ and $b = \frac{1}{1 - C - C'}$. Let $\theta \ll c$ be given. Since $g x_{n(k)-1} \to p$ and $f x_{n(k)-1} \to p$, we can choose a positive integer $k_0$ such that, for all $k \geq k_0$, we have
\[ d(f x_{n(k)-1}, p) \leq \frac{c}{2 a} \quad \text{and} \quad d(p, g x_{n(k)-1}) \leq \frac{c}{2 (a + b)}. \]
Thus, we get
\[ d(f q, p) \leq a \frac{c}{2 a} + (a + b) \frac{c}{2 (a + b)} = c, \]
in the case 1).

2) $u_{n(k)} = d(f x_{n(k)-1}, g x_{n(k)-1})$. Then
\[ d(f q, p) \leq C \cdot d(f x_{n(k)-1}, g x_{n(k)-1}) + C' d(f q, g x_{n(k)-1}) \]
\[ + C' d(f x_{n(k)-1}, g g) + d(f x_{n(k)-1}, p) \]
\[ \leq C \cdot d(f x_{n(k)-1}, g x_{n(k)-1}) + C' d(f q, p) + C' d(p, g x_{n(k)-1}) \]
\[ + C' d(f x_{n(k)-1}, p) + d(f x_{n(k)-1}, p), \]
i.e.,
\[ d(f q, p) \leq \frac{C d(f x_{n(k)-1}, g x_{n(k)-1})}{1 - C'} + \frac{C' d(p, g x_{n(k)-1})}{1 - C'} \]
\[ + \frac{C' d(f x_{n(k)-1}, p)}{1 - C'} + \frac{d(f x_{n(k)-1}, p)}{1 - C'} \]
\[ \leq \frac{1 + C + C'}{1 - C'} d(f x_{n(k)-1}, p) + \frac{C + C'}{1 - C'} d(p, g x_{n(k)-1}). \]
Take now $A = \frac{1}{1-C}$ and $B = \frac{C+C'}{1-C}$. Let $\theta \ll c$ be given. There exists a positive integer $k_1$ such that, for all $k \geq k_1$, we have
\[
d\left(fx_{n(k)-1}, p\right) \ll \frac{c}{2(A+B)} \quad \text{and} \quad d\left(p, gx_{n(k)-1}\right) \ll \frac{c}{2B}.
\]
Thus, we get
\[
d(fq, p) \ll (A+B) \frac{c}{2(A+B)} + B \frac{c}{2B} = c,
\]
in the case 2).

In both cases we obtain $d(fq, p) \ll c$, for each $c \in \text{int} P$. Using (p2) it follows that $d(fq, p) = \theta$, i.e. $fq = p$.

Suppose now that $f$ and $g$ are coincidentally commuting. Then $p = fq = gq \Rightarrow fp = fgq = gfq = gp$. Then, from (2.1),
\[
d(fp, p) = d(fp, fq) \leq C \cdot u(p, q) + C' \cdot (d(fp, gq) + d(fq, gp))
\]
where
\[
u(p, q) \in \{d(fp, gp), d(fq, gq)\} = \{
\theta, \theta \} = \{\theta\}.
\]
Hence, we get the following:
\[
d(fp, fq) \leq C \cdot \theta + C' \cdot (d(fp, gq) + d(fq, gp))
\]
\[
= 2C' \cdot d(fp, fq).
\]
Since $2C' < 1$, it follows that $fp = fq$. Hence, $p$ is a common fixed point of $f$ and $g$. Uniqueness of the common fixed point follows easily from (2.1).

Setting $g = I_X$, the identity mapping of $X$ in Theorem 2.1, we obtain the following:

**Theorem 2.2.** Let $(X, d)$ be a complete cone metric space, and let $K$ be a nonempty closed subset of $X$ such that, for each $x \in K$ and $y \notin K$, there exists a point $z \in \partial K$ such that $d(x, z) + d(z, y) = d(x, y)$. Suppose that $f : K \rightarrow X$ satisfies the condition
\[
d(fx, fy) \leq C \cdot u(x, y) + C' \cdot (d(fx, y) + d(x, fy)),
\]
where $u(x, y) \in \{d(x, fx), d(y, fy)\}$, for all $x, y \in K$. Here $C, C'$ are nonnegative reals as in the Theorem 1.1, and $f$ has the additional property that for each $x \in \partial K$, the boundary of $K$, $fx \in K$. Then $f$ has a unique fixed point.

**Remark 2.1.** Setting $E = \mathbb{R}$, $P = [0, +\infty)$, $\|\cdot\| = |\cdot|$ in Theorem 2.2 we obtain the main result from [10], i.e., Theorem 1.1 above. This shows that Theorem 2.1 is more general, since the main theorem from [10] can be obtained as its special case. We believe that the results of our paper can be extended to obtain a common fixed point theorem for a family of non-self mappings in the frame of the cone metric spaces, analogous to [6] for metrically convex spaces.

**Acknowledgement:** Author is thankful to the Ministry of Education, Science and Technological Development of Serbia.
References


Faculty of Mechanical Engineering,
University of Belgrade,
Serbia
E-mail address: radens@beotel.rs