Kragujevac Journal of Mathematics
Volume 36 Number 2 (2012), Pages 189–198.

A PAIR OF NON-SELF MAPPINGS IN CONE METRIC SPACES

STOJAN RADENOVIĆ

Abstract. In this paper we extend from metric to cone metric spaces a fixed point theorem for a pair of non-self maps proved by M. S. Khan et al. [M. S. Khan, K. Pathak and M. D. Khan, Some fixed point theorems in metrically convex spaces, Gregorian Mathematical Journal, Volume **7** (3) (2000), 523-530].

1. Introduction and preliminaries

Cone metric spaces were introduced in [3], where the authors described convergence in cone metric spaces, introduced completeness and proved some fixed point theorems of contractive mappings on these spaces. Recently, in [1], [4] and [13], some common fixed point theorems are proved for maps on cone metric spaces. The following definitions and results will be needed in the rest of this paper.

Let E be a real Banach space. A subset P of E is called a *cone* if and only if:

(i) P is closed, nonempty and $P \neq \{\theta\}$;
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;
(iii) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, a partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in intP$ (interior of P). A cone $P \subset E$ is *normal* if there is a number $k > 0$ such that for all $x, y \in P : \theta \preceq x \preceq y$ implies $\|x\| \leq k \|y\|$. The least positive number satisfying the previous condition is called the *normal constant* of P.

Definition 1.1. [3] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

Key words and phrases. Cone metric spaces; normal and non-normal cone; metrically convex; non-self maps; fixed point.
2010 *Mathematics Subject Classification.* Primary: 54H25, Secondary: 47H10.
Received: January 20, 2012.
Revised: October 25, 2012.

(d_1) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;
(d_2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(d_3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a *cone metric on* X and (X, d) is called a *cone metric space*.

A concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space, where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Let $\{x_n\}$ be a sequence in X, and $x \in X$. If, for every c in E with $\theta \ll c$, there exists an $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, $d(x_n, x) \ll c$, then it is said that x_n *converges to* x, denoted by $\lim_{n\to\infty} x_n = x$, or $x_n \to x, n \to \infty$. If, for every c in E, such that $\theta \ll c$, there is an $n_0 \in \mathbb{N}$ such that for all n, $m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is a *Cauchy sequence* in X. If every Cauchy sequence is convergent in X, then X is a *complete cone metric space*.

Let (X, d) be a cone metric space. Then the following properties are often useful, particularly when dealing with cone metric spaces in which the cone need not to be normal (for details see [7]):

(p_1) If $u \preceq v$ and $v \ll w$, then $u \ll w$.
(p_2) If $\theta \preceq u \ll c$, for each $c \in intP$, then $u = \theta$.
(p_3) If $a \preceq b + c$, for each $c \in intP$, then $a \preceq b$.
(p_4) If E is a real Banach space with a cone P, and if $a \preceq \lambda a$, where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.
(p_5) If $c \in intP$, $a_n \in E$, and $a_n \to \theta$, then there exists an n_0 such that, for all $n > n_0$, we have $a_n \ll c$. For details on cone metric spaces see three recent very important papers: [8], [9], [12].

In the following we suppose only that E is a Banach space, that P is a cone in E with $intP \neq \emptyset$, and that "$\preceq$" is a partial ordering with respect to P.

There exist a lot of fixed-point theorems for self-mappings defined on closed subsets on Banach spaces. However, for applications (numerical analysis, optimization, etc.) it is important to consider functions that are not self-mappings, and it is natural to search for sufficient conditions which would guarantee the existence of fixed points for such mappings–for details see: [5], [7] and [11]. In this paper we continue to study the non-self mappings in the frame of the cone metric spaces, started in [7] and [11].

Generalizing a theorem of [2], Khan et al., [10] proved the following result, valid in a complete metrically convex space.

Theorem 1.1. *Let X be a complete metrically convex space, and K a closed nonempty subset of X. Let $T : K \to X$ be the mapping satisfying the inequality*

$$d(Tx, Ty) \leq C \cdot \max\{d(x, Tx), d(y, Ty)\} + C' \cdot (d(x, Ty) + d(y, Tx)), \tag{1.1}$$

for all x, y in K, where C and C' are nonnegative reals such that:

$$\max\left\{\frac{C+C'}{1-C'}, \frac{C'}{1-C-C'}\right\} = h > 0, \quad \max\left\{\frac{1+C+C'}{1-C'}h, \frac{1+C'}{1-C-C'}h\right\} = h'$$
$$\text{and} \qquad \max\{h, h'\} = h'' < 1.$$

Further, $Tx \in K$, for every $x \in \partial K$. Then T has a unique fixed point in K.

2. Main result

The purpose of this paper is to prove the analogue of Theorem 1.1, for a pair of non-self maps in the frame of cone metric spaces, using the concept of metric convexity. We use only the definition of convergence in terms of the relation "$\ll$". The only assumption is that the interior of the cone P is nonempty - hence we use neither continuity of the vector metric d, nor Sandwich Theorem. We begin with the following definition.

Definition 2.1. Let (X, d) be a cone metric space, let K be a nonempty closed subset of X, and let $f, g : K \to X$. If f and g satisfy the condition

$$d(fx, fy) \preceq C \cdot u(x, y) + C' \cdot (d(fx, gy) + d(fy, gx)), \tag{2.1}$$

where $u(x, y) \in \{d(fx, gx), d(fy, gy)\}$, for all x, y in K, and where C, C' are nonnegative reals as in Theorem 1.1, then f is a *generalized g-contractive mapping* of K into X.

A pair of nonself-mappings (f, g), defined on a nonempty subset K of a cone metric space (X, d), is said to be *coincidentally commuting* if, for $fx, gx \in K$, $fx = gx$ implies that $fgx = gfx$. Note that, for $K = X$, this notion is reduced to the corresponding notion of Jungck and Rhoades for self-mappings.

We state and prove our main result as follows.

Theorem 2.1. *Let (X, d) be a complete cone metric space, let K be a nonempty closed subset of X such that, for each $x \in K$ and $y \notin K$, there exists a point $z \in \partial K$ such that*

$$d(x, z) + d(z, y) = d(x, y). \tag{2.2}$$

Suppose that $f, g : K \to X$ are such that f is a generalized g-contractive mapping of K into X, and

(i) *$\partial K \subseteq gK$, $fK \cap K \subset gK$;*
(ii) *$gx \in \partial K \Rightarrow fx \in K$;*
(iii) *gK is closed in X.*

Then there exists a point of coincidence p in K. Moreover, if (f, g) are coincidentally commuting, then p is the unique common fixed point of f and g.

Proof. First, we construct two sequences: $\{x_n\}$ in K and the sequence $\{y_n\}$ in $fK \subset X$ in the following way.

Let $x \in \partial K$ be arbitrary. There exists a point $x_0 \in K$ such that $x = gx_0$ as $\partial K \subset gK$. Since $gx_0 \in \partial K \Rightarrow fx_0 \in K$, we conclude that $fx_0 \in K \cap fK \subset gK$. Let $x_1 \in K$ be such that $y_1 = gx_1 = fx_0 \in K$. Let $y_2 = fx_1$. Suppose $y_2 \in K$. Then $y_2 \in K \cap fK \subset gK$, which implies that there exists a point $x_2 \in K$ such that $y_2 = gx_2$. Suppose $y_2 \notin K$. Then there exists a point $p \in \partial K$ such that $d(gx_1, p) + d(p, y_2) = d(gx_1, y_2)$. Since $p \in \partial K \subset gK$, there exists a point $x_2 \in K$ such that $p = gx_2$, so that the equation above takes the form $d(gx_1, gx_2) + d(gx_2, y_2) = d(gx_1, y_2)$. Put $y_3 = fx_2$. In this way, repeating the foregoing arguments, one obtains two sequences: $\{x_n\} \subset K$ and $\{y_n\} \subset fK \subset X$ such that:

(a) $y_{n+1} = fx_n$, for $n = 0, 1, 2, \ldots$;
(b) if $y_n \in K$, then $y_n = gx_n = fx_{n-1}$;
(c) if $y_n \notin K$, then $gx_n \in \partial K$ and $d(gx_{n-1}, gx_n) + d(gx_n, y_n) = d(gx_{n-1}, y_n)$.

Put

$$S = \{gx_i \in \{gx_n\} : gx_i = y_i\} \text{ and } Q = \{gx_i \in \{gx_n\} : gx_i \neq y_i\}.$$

Two consecutive terms cannot lie in Q. We have to estimate $d(gx_n, gx_{n+1})$. If $d(gx_n, gx_{n+1}) = \theta$ for some n, then it is easy to show that $d(gx_n, gx_{n+k}) = \theta$ for all $k \geq 1$.

Suppose that $d(gx_n, gx_{n+1}) \succ \theta$ for all n. From the presented construction we distinguish three cases:

Case 1. If $gx_n \in S$ and $gx_{n+1} \in S$, then according to (a), (b) and (2.1) we have:

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(y_n, y_{n+1}) = d(fx_{n-1}, fx_n) \\ &\preceq C \cdot u_1(x_{n-1}, x_n) + C' \cdot (d(gx_n, gx_n) + d(gx_{n+1}, gx_{n-1})) \\ &= C \cdot u_1(x_{n-1}, x_n) + C' \cdot (\theta + d(gx_{n+1}, gx_{n-1})) \\ &= C \cdot u_1(x_{n-1}, x_n) + C' \cdot d(gx_{n+1}, gx_{n-1}), \end{aligned}$$

where

$$u_1(x_{n-1}, x_n) \in \{d(fx_{n-1}, gx_{n-1}), d(fx_n, gx_n)\} = \{d(gx_n, gx_{n-1}), d(gx_{n+1}, gx_n)\}.$$

Clearly, there are infinitely many n such that at least one of the following two possibilities holds:

$$\mathbf{I}: d(gx_n, gx_{n+1}) \preceq C \cdot d(gx_n, gx_{n-1}) + C' \cdot d(gx_{n+1}, gx_n) + C' \cdot d(gx_n, gx_{n-1}),$$

that is,

$$d(gx_n, gx_{n+1}) \preceq \frac{C + C'}{1 - C'} \cdot d(gx_{n-1}, gx_n).$$

$$\mathbf{II}: d(gx_n, gx_{n+1}) \preceq C \cdot d(gx_{n+1}, gx_n) + C' \cdot d(gx_{n+1}, gx_n) + C' \cdot d(gx_n, gx_{n-1}),$$

that is,

$$d(gx_n, gx_{n+1}) \preceq \frac{C'}{1 - C - C'} \cdot d(gx_{n-1}, gx_n).$$

Case 2. Let $gx_n \in S$, $gx_{n+1} \in Q$. Then $y_n = gx_n$, $y_{n+1} \notin K$, $gx_{n+1} \in \partial K$ such that $d(y_{n+1}, gx_{n+1}) + d(gx_{n+1}, gx_n) = d(y_{n+1}, gx_n)$. We have

$$\begin{aligned} d(gx_n, gx_{n+1}) &\preceq d(gx_n, gx_{n+1}) + d(gx_{n+1}, y_{n+1}) \\ &= d(gx_n, y_{n+1}) = d(fx_n, gx_n) = d(fx_{n-1}, fx_n) \\ &\preceq C \cdot u_2(x_{n-1}, x_n) + C' \cdot (d(gx_n, gx_n) + d(fx_n, gx_{n-1})) \\ &= C \cdot u_2(x_{n-1}, x_n) + C' \cdot (\theta + d(fx_n, gx_{n-1})) \\ &= C \cdot u_2(x_{n-1}, x_n) + C' \cdot d(fx_n, gx_{n-1}), \end{aligned} \tag{2.3}$$

where

$$u_2(x_{n-1}, x_n) \in \{d(fx_{n-1}, gx_{n-1}), d(fx_n, gx_n)\} = \{d(gx_n, gx_{n-1}), d(fx_n, gx_n)\}.$$

First, we have the following two cases:

I:

$$\begin{aligned} d(fx_n, gx_n) &\preceq C \cdot d(gx_n, gx_{n-1}) + C' \cdot d(fx_n, gx_{n-1}) \\ &\preceq C \cdot d(gx_{n-1}, gx_n) + C' \cdot d(fx_n, gx_n) + C' \cdot d(gx_n, gx_{n-1}). \end{aligned}$$

It follows that

$$d(fx_n, gx_n) \preceq \frac{C + C'}{1 - C'} \cdot d(gx_{n-1}, gx_n),$$

i.e.,

$$d(gx_n, gx_{n+1}) \preceq \frac{C + C'}{1 - C'} \cdot d(gx_{n-1}, gx_n). \tag{2.4}$$

II: In the similar manner we obtain

$$d(fx_n, gx_n) \preceq C \cdot d(fx_n, gx_n) + C' \cdot d(fx_n, gx_n) + C' \cdot d(gx_n, gx_{n-1}),$$

that is,

$$d(fx_n, gx_n) \preceq \frac{C'}{1 - C - C'} \cdot d(gx_{n-1}, gx_n). \tag{2.5}$$

From (2.4) and (2.5) we get

$$d(fx_n, gx_n) \preceq h \cdot d(gx_{n-1}, gx_n), \tag{2.6}$$

where $h = \max\left\{\frac{C+C'}{1-C'}, \frac{C'}{1-C-C'}\right\}$. Now, according to (2.3) we obtain

$$d(gx_n, gx_{n+1}) \preceq h \cdot d(gx_{n-1}, gx_n). \tag{2.7}$$

Case 3. Let $gx_n \in Q$, $gx_{n+1} \in S$. Then $y_{n+1} = fx_n = gx_{n+1} \in K$, $y_n \notin K$ and $gx_n \in \partial K$, such that $d(y_n, gx_n) + d(gx_n, y_{n-1}) = d(y_n, y_{n-1})$, where $y_{n-1} = fx_{n-2} =$

$gx_{n-1} \in K$. Now, in view of Case 2, we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &\preceq d(gx_n, y_n) + d(y_n, gx_{n+1}) \\ &\preceq d(gx_{n-1}, gx_n) + d(gx_n, y_n) + d(y_n, gx_{n+1}) \\ &= d(gx_{n-1}, y_n) + d(y_n, gx_{n+1}) \\ &\preceq hd(gx_{n-2}, gx_{n-1}) + d(fx_{n-1}, fx_n) \quad (2.8) \\ &\preceq hd(gx_{n-2}, gx_{n-1}) + Cu_3(x_{n-1}, x_n) \\ &\quad + C' \cdot (d(fx_{n-1}, gx_n) + d(fx_n, gx_{n-1})), \quad (2.9) \end{aligned}$$

where $u_3(x_{n-1}, x_n) \in \{d(fx_{n-1}, gx_{n-1}), d(fx_n, gx_n) = d(gx_n, gx_{n+1})\}$. Further, it follows that

$$\begin{aligned} d(fx_{n-1}, gx_n) + d(fx_n, gx_{n-1}) &= d(fx_{n-1}, gx_n) + d(gx_{n+1}, gx_{n-1}) \\ &\preceq d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1}) + d(fx_{n-1}, gx_n) \\ &= d(gx_{n+1}, gx_n) + d(gx_{n-1}, fx_{n-1}) \\ &\preceq d(gx_n, gx_{n+1}) + h \cdot d(gx_{n-2}, gx_{n-1}). \end{aligned}$$

Now, from (2.9) we obtain:

$$\begin{aligned} \mathbf{I}\ :\ & d(gx_n, gx_{n+1}) \preceq h \cdot d(gx_{n-2}, gx_{n-1}) + C \cdot hd(gx_{n-2}, gx_{n-1}) \\ & + C' \cdot (d(gx_n, gx_{n+1}) + h \cdot d(gx_{n-2}, gx_{n-1})), \end{aligned}$$

that is,

$$d(gx_n, gx_{n+1}) \preceq \frac{1 + C + C'}{1 - C'} \cdot h \cdot d(gx_{n-2}, gx_{n-1}). \tag{2.10}$$

$$\begin{aligned} \mathbf{II}\ :\ & d(gx_n, gx_{n+1}) \preceq h \cdot d(gx_{n-2}, gx_{n-1}) + C \cdot d(gx_n, gx_{n+1}) \\ & + C' \cdot (d(gx_n, gx_{n+1}) + h \cdot d(gx_{n-2}, gx_{n-1})), \end{aligned}$$

that is

$$d(gx_n, gx_{n+1}) \preceq \frac{1 + C'}{1 - C - C'} \cdot h \cdot d(gx_{n-2}, gx_{n-1}). \tag{2.11}$$

Taking $h' = \max\left\{\frac{1+C+C'}{1-C'} \cdot h, \frac{1+C'}{1-C-C'} \cdot h\right\}$, we get from (2.10) and (2.11)

$$d(gx_n, gx_{n+1}) \preceq h' \cdot d(gx_{n-2}, gx_{n-1}). \tag{2.12}$$

Now, from (2.7) and (2.12) we have in all Cases 1–3

$$d(gx_n, gx_{n+1}) \preceq h'' \cdot w_n, \tag{2.13}$$

where $h'' = \max\{h, h'\}$ and $w_n \in \{d(gx_{n-2}, gx_{n-1}), d(gx_{n-1}, gx_n)\}$.

From (2.13), following the procedure of Assad and Kirk [2] (see also [5], [7], [11]), we shall show, by induction, that for all $n > 1$,

$$d(gx_n, gx_{n+1}) \preceq h''^{\frac{n-1}{2}} \cdot w_2, \tag{2.14}$$

where $w_2 \in \{d(gx_0, gx_1), d(gx_1, gx_2)\}$.

For $n = 2$, according to (2.13) we have, $d(gx_2, gx_3) \preceq h'' \cdot w_2$, where

$$w_2 \in \{d(gx_{2-2}, gx_{2-1}), d(gx_{2-1}, gx_2)\} = \{d(gx_0, gx_1), d(gx_1, gx_2)\},$$

that is $d(gx_2, gx_3) \preceq h''^{\frac{2-1}{2}} \cdot w_2$, since $h'' < h''^{\frac{1}{2}}$. Hence (2.14) holds.

Similarly, again according to (2.13), for $n = 3$ we have $d(gx_3, gx_4) \preceq h'' \cdot w_3$, where

$$w_3 \in \{d(gx_{3-2}, gx_{3-1}), d(gx_{3-1}, gx_3)\} = \{d(gx_1, gx_2), d(gx_2, gx_3)\}.$$

If $w_3 = d(gx_1, gx_2)$, it follows that

$$d(gx_3, gx_4) \preceq h'' d(gx_1, gx_2) = h''^{\frac{3-1}{2}} w_2,$$

so (2.14) holds. If $w_3 = d(gx_2, gx_3)$ we have

$$\begin{aligned} d(gx_3, gx_4) &\preceq h'' \cdot d(gx_2, gx_3) \\ &\preceq h''^2 w_2 < h''^{\frac{3-1}{2}} w_2 \end{aligned}$$

(because $h'' \in (0, 1)$), and (2.14) also holds.

Therefore, (2.14) holds for $n = 2$ and $n = 3$. Suppose now that (2.14) holds for some n and $n+1$. Then again from (2.13) it follows that $d(gx_{n+2}, gx_{n+3}) \preceq h'' \cdot w_{n+2}$, where $w_{n+2} \in \{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})\}$, that is, we get the following two cases:

$$\begin{aligned} d(gx_{n+2}, gx_{n+3}) &\preceq h'' \cdot d(gx_n, gx_{n+1}) \text{ and} \\ d(gx_{n+2}, gx_{n+3}) &\preceq h'' \cdot d(gx_{n+1}, gx_{n+2}). \end{aligned}$$

Since (2.14) holds for n and $n + 1$, we obtain in the first case

$$d(gx_{n+2}, gx_{n+3}) \preceq h'' \cdot d(gx_n, gx_{n+1}) \preceq h'' \cdot h''^{\frac{n-1}{2}} \cdot w_2 = h''^{\frac{n+1}{2}} w_2,$$

and in the second case

$$d(gx_{n+2}, gx_{n+3}) \preceq h'' \cdot d(gx_{n+1}, gx_{n+2}) \preceq h'' \cdot h''^{\frac{n}{2}} \cdot w_2 \preceq h''^{\frac{n+1}{2}} w_2,$$

where $w_2 \in \{d(gx_0, gx_1), d(gx_1, gx_2)\}$.

Thus by induction we conclude that (2.14) holds for all $n > 1$.

From (2.14) and by the triangle inequality, for $n > m$ we have:

$$\begin{aligned} d(gx_n, gx_m) &\preceq d(gx_n, gx_{n-1}) + d(gx_{n-1}, gx_{n-2}) + \cdots + d(gx_{m+1}, gx_m) \\ &\preceq \left(h''^{\frac{n-2}{2}} + h''^{\frac{n-3}{2}} + \cdots + h''^{\frac{m-1}{2}}\right) \cdot w_2 \\ &\preceq \frac{\sqrt{h''}^{m-1}}{1 - \sqrt{h''}} \cdot w_2 \to \theta, \text{ as } m \to \infty. \end{aligned}$$

According to (p_5) and (p_1) it follows that, for $\theta \ll c$ and large m, $d(gx_n, gx_m) \ll c$, i.e., $\{gx_n\}$ is a Cauchy sequence. Since $gx_n \in K \cap gK$ and $K \cap gK$ is complete, there exists a point $p \in K \cap gK$ such that $gx_n \to p$. Let q in K be such that $gq = p$. By the construction of $\{gx_n\}$, there exists a subsequence $\left\{g_{n(k)}\right\}$ such that

$gx_{n(k)} = y_{n(k)} = fx_{n(k)-1}$ and hence $fx_{n(k)-1} \to p$. We now prove that $fq = p$. We have

$$d(fq,p) \preceq d\left(fq, fx_{n(k)-1}\right) + d\left(fx_{n(k)-1}, p\right)$$
$$\preceq C \cdot u_{n(k)} + C'\left(d\left(fq, gx_{n(k)-1}\right) + d\left(fx_{n(k)-1}, gq\right)\right) + d\left(fx_{n(k)-1}, p\right),$$

where $u_{n(k)} \in \left\{d(fq,gq), d\left(fx_{n(k)-1}, gx_{n(k)-1}\right)\right\}$. Since $y_{n(k)} = fx_{n(k)-1} \to p$, as $k \to \infty$, we obtain the following two cases:

1) $u_{n(k)} = d(fq,gq)$. Then

$$\begin{aligned} d(fq,p) &\preceq C \cdot d(fq,gq) + C'd\left(fq, gx_{n(k)-1}\right) + C'd\left(fx_{n(k)-1}, gq\right) + d\left(fx_{n(k)-1}, p\right) \\ &\preceq C \cdot d(fq,p) + C \cdot d(p,gq) + C'd(fq,p) + C'd\left(p, gx_{n(k)-1}\right) \\ &\quad + C'd\left(fx_{n(k)-1}, p\right) + d\left(fx_{n(k)-1}, p\right), \end{aligned}$$

that is,

$$d(fq,p) \preceq \frac{C'd\left(p, gx_{n(k)-1}\right)}{1-C-C'} + \frac{(1+C')d\left(fx_{n(k)-1}, p\right)}{1-C-C'}.$$

Put $a = \frac{C'}{1-C-C'}$ and $b = \frac{1}{1-C-C'}$. Let $\theta \ll c$ be given. Since $gx_{n(k)-1} \to p$ and $fx_{n(k)-1} \to p$, we can choose a positive integer k_0 such that, for all $k \geq k_0$, we have

$$d\left(fx_{n(k)-1}, p\right) \ll \frac{c}{2a} \quad \text{and} \quad d\left(p, gx_{n(k)-1}\right) \ll \frac{c}{2(a+b)}.$$

Thus, we get

$$d(fq,p) \preceq a\frac{c}{2a} + (a+b)\frac{c}{2(a+b)} = c,$$

in the case **1**).

2) $u_{n(k)} = d\left(fx_{n(k)-1}, gx_{n(k)-1}\right)$. Then

$$\begin{aligned} d(fq,p) &\preceq C \cdot d\left(fx_{n(k)-1}, gx_{n(k)-1}\right) + C'd\left(fq, gx_{n(k)-1}\right) \\ &\quad + C'd\left(fx_{n(k)-1}, gq\right) + d\left(fx_{n(k)-1}, p\right) \\ &\preceq C \cdot d\left(fx_{n(k)-1}, gx_{n(k)-1}\right) + C'd(fq,p) + C'd\left(p, gx_{n(k)-1}\right) \\ &\quad + C'd\left(fx_{n(k)-1}, p\right) + d\left(fx_{n(k)-1}, p\right), \end{aligned}$$

i.e.,

$$\begin{aligned} d(fq,p) &\preceq \frac{Cd\left(fx_{n(k)-1}, gx_{n(k)-1}\right)}{1-C'} + \frac{C'd\left(p, gx_{n(k)-1}\right)}{1-C'} \\ &\quad + \frac{C'd\left(fx_{n(k)-1}, p\right)}{1-C'} + \frac{d\left(fx_{n(k)-1}, p\right)}{1-C'} \\ &\preceq \frac{1+C+C'}{1-C'} d\left(fx_{n(k)-1}, p\right) + \frac{C+C'}{1-C'} d\left(p, gx_{n(k)-1}\right). \end{aligned}$$

Take now $A = \frac{1}{1-C'}$ and $B = \frac{C+C'}{1-C'}$. Let $\theta \ll c$ be given. There exists a positive integer k_1 such that, for all $k \geq k_1$, we have

$$d\left(fx_{n(k)-1}, p\right) \ll \frac{c}{2(A+B)} \quad \text{and} \quad d\left(p, gx_{n(k)-1}\right) \ll \frac{c}{2B}.$$

Thus, we get

$$d(fq, p) \ll (A+B)\frac{c}{2(A+B)} + B\frac{c}{2B} = c,$$

in the case **2**).

In both cases we obtain $d(fq,p) \ll c$, for each $c \in intP$. Using (p_2) it follows that $d(fq,p) = \theta$, i.e. $fq = p$.

Suppose now that f and g are coincidentally commuting. Then $p = fq = gq \Rightarrow fp = fgq = gfq = gp$. Then, from (2.1),

$$d(fp,p) = d(fp,fq) \preceq C \cdot u(p,q) + C' \cdot (d(fp,gq) + d(fq,gp)),$$

where

$$u(p,q) \in \{d(fp,gp), d(fq,gq)\} = \{\theta,\theta\} = \{\theta\}.$$

Hence, we get the following:

$$\begin{aligned} d(fp,fq) &\preceq C \cdot \theta + C' \cdot (d(fp,gq) + d(fq,gp)) \\ &= 2C' \cdot d(fp,fq). \end{aligned}$$

Since $2C' < 1$, it follows that $fp = fq$. Hence, p is a common fixed point of f and g. Uniqueness of the common fixed point follows easily from (2.1). □

Setting $g = I_X$, the identity mapping of X in Theorem 2.1, we obtain the following:

Theorem 2.2. *Let (X,d) be a complete cone metric space, and let K be a nonempty closed subset of X such that, for each $x \in K$ and $y \notin K$, there exists a point $z \in \partial K$ such that $d(x,z) + d(z,y) = d(x,y)$. Suppose that $f : K \to X$ satisfies the condition*

$$d(fx,fy) \preceq C \cdot u(x,y) + C'(d(fx,y) + d(x,fy)),$$

where $u(x,y) \in \{d(x,fx), d(y,fy)\}$, for all x,y in K. Here C, C' are nonnegative reals as in the Theorem 1.1, and f has the additional property that for each $x \in \partial K$, the boundary of $K, fx \in K$. Then f has a unique fixed point.

Remark 2.1. Setting $E = \mathbb{R}$, $P = [0,+\infty)$, $\|\cdot\| = |\cdot|$ in Theorem 2.2 we obtain the main result from [10], i.e., Theorem 1.1 above. This shows that Theorem 2.1 is more general, since the main theorem from [10] can be obtained as its special case. We believe that the results of our paper can be extended to obtain a common fixed point theorem for a family of non-self mappings in the frame of the cone metric spaces, analogous to [6] for metrically convex spaces.

Acknowledgement: Author is thankful to the Ministry of Education, Science and Technological Development of Serbia.

References

[1] M. Abbas, G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. **341** (2008), 416–420.

[2] N. A. Assad and W. A. Kirk, *Fixed point theorems for set-valued mappings of contractive type*, Pacific J. Math. **43** (3) (1972), 553–562.

[3] L.G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2) (2007), 1468–1476.

[4] D. Ilić, V. Rakočević, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl. **341** (2008), 876–882.

[5] M. Imdad and S. Kumar, *Rhoades-Type Fixed-Point Theorems for a Pair of Nonself Mappings,* Computers and Mathematics with Applications **46** (2003), 919–927.

[6] M. Imdad, Ladlay Khan, *Some common fixed point theorems for a family of mappings in metrically convex spaces*, Nonlinear Analysis **67** (2007), 2717–2726.

[7] S. Janković, Z. Kadelburg, S. Radenović and B. E. Rhoades, *Assad-Kirk-type Fixed-Point Theorems for a Pair of Nonself Mappings in Cone Metric Spaces*, Fixed Point Theory and Applications, Volume 2009, Article ID 761086, 16 pages, doi:10.1155/2009/761086.

[8] S. Janković, Z. Kadelburg, S. Radenović, *On cone metric spaces: A survey*, Nonlinear Analysis (2010), doi: 10.1016/j.na.2010.12.014

[9] Z. Kadelburg, S. Radenović, V. Rakočević, *A note on the equivalence of some metric and cone metric fixed point results*, Appl. Math. Lett. **24** (2011), 370–374.

[10] M. S. Khan, H. K. Pathak, and M. D. Khan, *Some fixed point theorems in metrically convex spaces*, Gregorian Mathematical Journal, Volume **7** (3) (2000), 523–530.

[11] S. Radenović, B. E. Rhoades, *A fixed point theorem for two non-self mappings in cone metric spaces*, Computers and Mathematics with Applications, **57** (2009), 1701–1707.

[12] S. Radenović, Z. Kadelburg, *Quasi-contractions on symmetric and cone symmetric spaces*, Banach J. Math. Anal. **5** (1) (2011), 38–50.

[13] P. Vetro, *Common fixed points in cone metric spaces*, Rend. Circ. Math. Palermo, **56** (2007), no 3, 464–468.

Faculty of Mechanical Engineering,
University of Belgrade,
Serbia
E-mail address: radens@beotel.rs