SEIDEL ENERGY OF ITERATED LINE GRAPHS OF REGULAR GRAPHS

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ABSTRACT. The Seidel matrix $S(G)$ of a graph G is the square matrix whose (i, j)-entry is equal to -1 or 1 if the i-th and j-th vertices of G are adjacent or non-adjacent, respectively, and is zero if $i = j$. The Seidel energy of G is the sum of the absolute values of the eigenvalues of $S(G)$. We show that if G is regular of order n and of degree $r \geq 3$, then for each $k \geq 2$, the Seidel energy of the k-th iterated line graph of G depends solely on n and r. This result enables the construction of pairs of non-cospectral, Seidel equienergetic graphs of the same order.

1. INTRODUCTION

Let G be a simple, undirected graph with n vertices and m edges. The vertices of G are labeled as $v_1, v_2, \ldots, v_n$. The *adjacency matrix* of G is the square matrix $A(G) = (a_{ij})$, in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$ otherwise. The eigenvalues of $A(G)$ will be referred to as the *adjacency eigenvalues* of G, and labeled as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. These form the *adjacency spectrum* of G [4, 5].

The *Seidel matrix* of the graph G is the $n \times n$ real symmetric matrix $S(G) = (s_{ij})$, where $s_{ij} = -1$ if the vertices v_i and v_j are adjacent, $s_{ij} = 1$ if the vertices v_i and v_j are not adjacent, and $s_{ij} = 0$ if $i = j$. The eigenvalues of the Seidel matrix, labeled as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$, are said to be the *Seidel eigenvalues* of G and their collection is the *Seidel spectrum* of G [1]. Two non-isomorphic graphs are said to be *Seidel cospectral* if their Seidel spectra coincide.

The *Seidel energy* of the graph G is defined as [8]

$$(1.1) \qquad SE = SE(G) = \sum_{i=1}^{n} |\sigma_i| \ .$$

The Seidel energy is conceived in full analogy with the ordinary graph energy $E(G)$, which is defined as [7]

$$(1.2) \qquad E(G) = \sum_{i=1}^{n} |\lambda_i|$$

and whose theory is nowadays extensively elaborated [10].

The study of the Seidel energy started only quite recently [8], and only some of its basic properties have been established so far [6, 8, 11].

Two graphs G_1 and G_2 are said to be *equienergetic* if $E(G_1) = E(G_2)$. In a trivial manner, cospectral graphs are equienergetic. Much work has been done on the study of non-cospectral equienergetic graphs; for details see [10].

Two graphs G_1 and G_2 will be said to be *Seidel equienergetic* if $SE(G_1) = SE(G_2)$. Of course, Seidel cospectral graphs are Seidel equienergetic. We are interested in Seidel non-cospectral, Seidel equienergetic graphs, having same number of vertices.

Denote by $\overline{G}$ the complement of the graph G. Then, evidently, $S(G) = A(\overline{G}) - A(G)$ implying $S(\overline{G}) = -S(G)$. Consequently, if the Seidel eigenvalues of a graph G are σ_i, $i = 1, 2, \ldots, n$, then the Seidel eigenvalues of $\overline{G}$ are $-\sigma_i$, $i = 1, 2, \ldots, n$. Thus, by Eq. (1.1), $SE(\overline{G}) = SE(G)$, that is G and $\overline{G}$ are Seidel equienergetic.

In what follows we search for less trivial pairs of Seidel equienergetic graphs.

In this paper, based on a result that relates the Seidel eigenvalues of regular graphs with its adjacency eigenvalues, we show that SE of second and higher line graphs of regular graphs is fully determined by the order n and regularity r. By this, infinitely many pairs of Seidel equienergetic graphs can be constructed.

2. On spectra of regular graphs and their line graphs

The line graph [9] of the graph G will be denoted by $L(G)$. For $k = 1, 2, \ldots$, the k-th iterated line graph of G, denoted by $L^k(G)$, is defined recursively as $L^k(G) = L(L^{k-1}(G))$, with $L^0(G) = G$ and $L^1(G) = L(G)$.

The line graph of a regular graph is a regular graph. In particular, the line graph of a regular graph G of order n_0 and of degree r_0 is a regular graph of order $n_1 = (r_0 n_0)/2$ and of degree $r_1 = 2 r_0 - 2$. Consequently, the order and degree of $L^k(G)$ are [2, 3]:

$$n_k = \frac{1}{2} r_{k-1} n_{k-1} \qquad \text{and} \qquad r_k = 2 r_{k-1} - 2$$

where n_i and r_i stand for the order and degree of $L^i(G)$, $i = 1, 2, \ldots$. Therefore,

$$(2.1) \qquad r_k = 2^k r_0 - 2^{k+1} + 2$$

and

$$(2.2) \qquad n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = n_0 \prod_{i=0}^{k-1} (2^{i-1} r_0 - 2^i + 1).$$

The basic spectral properties of regular graphs are well known. Here we state some results needed for the present considerations.

Theorem 2.1. [4, 5] *If G is an r-regular graph, then its maximum adjacency eigenvalue is equal to r.*

Theorem 2.2. [4, 5] *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r, then the adjacency eigenvalues of $L(G)$ are*

$$\begin{cases} \lambda_i + r - 2, & i = 1, 2, \ldots, n, \\ -2, & \dfrac{n(r-2)}{2} \ times. \end{cases}$$

Theorem 2.3. [12, 13] *If G is a regular graph of order n and of degree $r \geq 3$, then*

$$E(L^2(G)) = 2nr(r-2).$$

Theorem 2.4. [1] *Let G be an r-regular graph on n vertices. If $r, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of G, then its Seidel eigenvalues are $n - 1 - 2r$ and $-1 - 2\lambda_i$, $i = 2, 3, \ldots, n$.*

3. Seidel energy

We first state a result for Seidel energy that is formally similar to Theorem 2.3.

Theorem 3.1. *If G is a regular graph of order n and of degree $r \geq 3$, then*

$$SE(L^2(G)) = \begin{cases} 4nr(r-2) + nr - 16r + 22, & if \ \ n \geq \dfrac{16r - 22}{r(r-1)}, \\ \\ 3nr(r-2), & if \ \ n < \dfrac{16r - 22}{r(r-1)}. \end{cases}$$

Proof. Let $r, \lambda_2, \ldots, \lambda_n$ be the adjacency eigenvalues of the regular graph G. Then from Theorem 2.2, the adjacency eigenvalues of $L(G)$ are

(3.1)
$$\begin{cases} 2r - 2, & \\ \lambda_i + r - 2, & i = 2, 3, \ldots, n, \\ -2, & \dfrac{n(r-2)}{2} \ times. \end{cases}$$

In view of the fact that $L(G)$ is also a regular graph on $nr/2$ vertices and of degree $2r - 2$, applying Theorem 2.2 to Eq. (3.1), the adjacency eigenvalues of $L^2(G)$ are found to be

(3.2)
$$\begin{cases} 4r - 6, & \\ \lambda_i + 3r - 6, & i = 2, 3, \ldots, n, \\ 2r - 6, & \dfrac{n(r-2)}{2} \ times, \\ -2, & \dfrac{nr(r-2)}{2} \ times. \end{cases}$$

Because $L^2(G)$ is a regular graph on $nr(r-1)/2$ vertices and of degree $4r-6$, applying Theorem 2.4 to Eq. (3.2), the Seidel eigenvalues of $L^2(G)$ are calculated as

$$(3.3) \qquad \begin{cases} \frac{1}{2}nr(r-1)+11-8r, & \\ -2\lambda_i-6r+11, & i=2,3,\dots,n, \\ -4r+11, & \dfrac{n(r-2)}{2} \text{ times,} \\ 3, & \dfrac{nr(r-2)}{2} \text{ times.} \end{cases}$$

By Theorem 2.1, all adjacency eigenvalues of a regular graph of degree r satisfy the condition $-r \le \lambda_i \le r$. If $r \ge 3$, then $4r-11 \ge 0$, and because of $\lambda_i \ge -r$, we have $2\lambda_i+6r-11 \ge 0$. Bearing this in mind, the Seidel energy of $L^2(G)$ is computed from Eq. (3.3) as:

$$\begin{aligned} SE(L^2(G)) &= \left| \frac{nr(r-1)}{2}+11-8r \right| + \sum_{i=2}^{n}(2\lambda_i+6r-11) \\ &\quad + \frac{n(r-2)}{2}(4r-11) + \frac{nr(r-2)}{2}(3) \\ &= \left| \frac{nr(r-1)}{2}+11-8r \right| - 2r + (n-1)(6r-11) \\ &\quad + \frac{n(r-2)}{2}(4r-11) + \frac{3nr(r-2)}{2} \end{aligned}$$

since $\sum_{i=2}^{n} \lambda_i = -r$.

If $\frac{nr(r-1)}{2}+11-8r \ge 0$ i.e., $n \ge \frac{16r-22}{r(r-1)}$, then the above equality yields

$$SE(L^2(G)) = 4nr(r-2) + nr - 16r + 22.$$

If $\frac{nr(r-1)}{2}+11-8r < 0$ i.e., $n < \frac{16r-22}{r(r-1)}$, then we get $SE(L^2(G)) = 3nr(r-2)$. $\square$

Corollary 3.1. *Let G be a regular graph of order n_0 and of degree $r_0 \ge 3$. Let n_k and r_k be the order and degree, respectively, of the k-th iterated line graph $L^k(G)$ of G, $k \ge 1$. Then*

$$SE(L^{k+1}(G)) = \begin{cases} 4n_k(r_k-2)+2n_k-8(r_k+2)+22, & \text{if } n_k \ge \dfrac{8r_k-6}{r_k}, \\ \\ 3n_k(r_k-2), & \text{if } n_k < \dfrac{8r_k-6}{r_k}. \end{cases}$$

From Corollary 3.1 we see that the Seidel energy of the k-th iterated line graph of a regular graph G is fully determined by the order n_0 and degree r_0 of G.

4. SEIDEL EQUIENERGETIC GRAPHS

Theorem 4.1. *Let G_1 and G_2 be two regular graphs of the same order and of the same degree. Then for any $k \geq 1$ the following holds:*

(i) *$L^k(G_1)$ and $L^k(G_2)$ are of the same order, same degree and have the same number of edges.*

(ii) *$L^k(G_1)$ and $L^k(G_2)$ are Seidel cospectral if and only if G_1 and G_2 are adjacency cospectral.*

Proof. Statement (i) follows from Eqs. (2.1) and (2.2), and the fact that the number of edges of $L^k(G)$ is equal to the number of vertices of $L^{k+1}(G)$. It was earlier stated in [12, 13]. Statement (ii) follows from Theorems 2.2 and 2.4. ☐

Theorem 4.1 and Corollary 3.1 infer the following.

Theorem 4.2. *Let G_1 and G_2 be two Seidel non-cospectral regular graphs of the same order and of the same degree $r \geq 3$. Then for any $k \geq 2$, the iterated line graphs $L^k(G_1)$ and $L^k(G_2)$ form a pair of Seidel non-cospectral, Seidel equienergetic graphs of equal order and with equal number of edges.*

On should recall that the following fully analogous result was earlier obtained for ordinary graph energy, based on adjacency eigenvalues, cf. Eq. (1.2).

Theorem 4.3. [13] *Let G_1 and G_2 be two non-cospectral regular graphs of the same order and of the same degree $r \geq 3$. Then for any $k \geq 2$, the iterated line graphs $L^k(G_1)$ and $L^k(G_2)$ form a pair of non-cospectral, equienergetic graphs of equal order and with equal number of edges.*

Comparing Theorems 4.2 and 4.3 a deep analogy between ordinary graph energy and Seidel energy can be envisaged.

From Theorem 4.2, it is now easy to generate large families of Seidel equienergetic graphs. For instance, there are 2, 5, 19, and 85 connected regular graphs of degree 3 and of order 6, 8, 10, and 12, respectively. No two of these are Seidel cospectral as they are adjacency non-cospectral (see [4], pp. 268–269). Their second and higher iterated line graphs form families consisting of size 2, 5, 19, 85, . . . , Seidel equienergetic graphs.

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