PULLBACK DIAGRAM OF HILBERT MODULES OVER $H^*$-ALGEBRAS

M. KHANEHGIR$^1$, M. AMYARI$^1$, AND M. MORADIAN KHIBARY$^1$

Abstract. In this paper, we generalize the construction of a pullback diagram in the framework of Hilbert modules over $H^*$-algebras. More precisely we prove that if a commutative diagram of Hilbert $H^*$-modules and morphisms

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\Phi_1} & Y_1 \\
\downarrow{\Psi_1} & & \downarrow{\Psi_2} \\
X_2 & \xrightarrow{\Phi_2} & Y_2
\end{array}
\]

is pullback and $\Psi_2$ is a surjection, then (i) $\Psi_1$ is a surjection and (ii) $\ker(\Phi_1) \cap \ker(\Psi_1) = \{0\}$. Conversely, if (i) and (ii) hold, $\psi_1(\tau(A_1))$ is $\tau_{A_2}$-closed and $\Psi_2$ is injective, then the above diagram is pullback.

1. Introduction and Preliminaries

Pedersen [9] studied pullback diagrams of $C^*$-algebras. He found conditions under which a commutative diagram of $C^*$-algebras and morphisms is pullback. Then Amyari and Chakoshi [2] studied it in the framework of Hilbert $C^*$-modules. In reference [8], we study pullback diagram of $H^*$-algebras and morphisms. We also find conditions for pullbackness such a commutative diagram and its underlying trace classes. In this paper, we generalized the notion of pullback diagram in the framework of Hilbert $H^*$-modules and describe some new relations between faithful Hilbert modules over commutative proper $H^*$-algebras and morphisms.

Some properties of pullback diagrams are stable under Hilbert modules over $H^*$-algebras. We use these properties to discover new ones for pullback diagram of Hilbert modules over $H^*$-algebras. An $H^*$-algebra, introduced by Ambrose [1] is a complex...
algebra $A$ with a conjugate-linear mapping $*: A \rightarrow A$ and an inner product $\langle \cdot , \cdot \rangle$ such that it is a Hilbert space and satisfies $a^{**} = a$, $(ab)^* = b^*a^*$, $\langle ab, c \rangle = \langle a, cb^* \rangle$ and $\langle ab, c \rangle = \langle b, a^*c \rangle$ for all $a, b, c \in A$. Recall that $A_0 = \{ a \in A : aA = \{0\} \} = \{ a \in A : Aa = \{0\} \}$ is called the annihilator ideal of $A$. A proper $H^*$-algebra is an $H^*$-algebra with zero annihilator ideal. The trace-class $\tau(A)$ of an $H^*$-algebra $A$ is defined by the set $\tau(A) = \{ ab : a, b \in A \}$. It is known that $\tau(A)$ is an ideal of $A$, which is a Banach algebra under a suitable norm $\tau_A(\cdot)$. The norm $\tau_A$ is related to the given norm $\| \cdot \|$ on $A$ by $\| a \|^2 = \tau_A(a^*a)$ for each $a \in A$. By [1, Lemma 2.7], if $A$ is proper, then $\tau(A)$ is dense in $A$. The trace functional $tr$ on $\tau(A)$ is defined by $tr(ab) = \langle b, a^* \rangle = \langle a, b^* \rangle = tr(ba)$ for each $a, b \in A$, in particular $\tau(aa^*) = \langle a, a \rangle = \| a \|^2 = \tau_A(a^*a)$ for all $a \in A$.

A nonzero element $e \in A$ is called a projection, if it is self-adjoint and idempotent. In addition, if $eAe = Ce$, then it is called a minimal projection. Two idempotents $e$ and $e'$ are doubly orthogonal if $\langle e, e' \rangle = 0$ and $ee' = e'e = 0$. An idempotent is primitive if it can not be expressed as the sum of two doubly orthogonal idempotent.

**Lemma 1.1.** Suppose that $A$ is a commutative $H^*$-algebra. Then $e$ is a minimal projection if and only if $e$ is a primitive projection.

**Proof.** Suppose that $e$ is a minimal projection. Then $Ae = eAe = Ce$, but $Ce$ is a minimal ideal of $A$ of rank one. So by [1, Lemma 3.4] $e$ is primitive. Conversely, suppose $e$ is a primitive projection in $A$. We will show that $eAe = Ce$ or $Ae^2 = Ce$. Obviously $e = e^2 \in Ae$. Therefore $Ce \subseteq A$. For the other side, on the contrary, suppose that there exists an element $a \in A$ such that $a \notin Ce$, so $a$ and $e$ are independent. Then $Aa$ is a proper ideal of $Ae$, which contradicts minimality of $Ae$ (Note that if $e$ is primitive, then $Ae$ is a minimal ideal). Hence $Ae \subseteq Ce$. \hfill \Box

Each simple $H^*$-algebra (an $H^*$-algebra without nontrivial closed two-sided ideals) contains minimal projections. It is known that all minimal projections in a simple $H^*$-algebra have equal norms [4]. Also note that if $A$ and $B$ are $H^*$-algebras, then $A \oplus B$ is an $H^*$-algebra with $\langle (a_1, b_1), (a_2, b_2) \rangle = \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle$. For more details on $H^*$-algebras, see [5, 10] and references cited therein.

**Definition 1.1.** Let $A$ be a proper $H^*$-algebra. A Hilbert $H^*$-module [4, 7] is a right module $X$ over $A$ with a mapping $[-,-] : X \times X \rightarrow \tau(A)$, which satisfies the following conditions:

(i) $[x|\alpha y] = \alpha [x|y]$,  
(ii) $[x + y|z] = [x|z] + [y|z]$,  
(iii) $[x|ya] = [x|y]a$,  
(iv) $[x|y]^* = [y|x]$,  
(v) For each nonzero element $x$ in $X$ there is a nonzero element $a$ in $A$ such that $[x|x] = a^*a$,  
(vi) $X$ is a Hilbert space with the inner product $(x, y) = tr([x|y])$,  

for each $\alpha \in \mathbb{C}$, $x, y \in X$, $a \in A$.  


The Hilbert $H^*$-module $X$ is full if the ideal $I = [X, X] = \text{span}\{[x|y] : x, y \in X\}$ is dense in $\tau(A)$ under the norm $\tau_A(.)$.

**Lemma 1.2.** Let $X$ be a full Hilbert module over a proper $H^*$-algebra $A$ and $a \in A$. Then $xa = 0$ for all $x \in X$ if and only if $a = 0$.

**Proof.** If $a \in \tau(A)$ and $xa = 0$ for all $x \in X$, then $[xa|ya] = 0$ for all $x, y \in X$. Let $b \in \tau(A)$ be arbitrary. Since $X$ is full, there exists a sequence $\{u_n\}$ in $I$ such that $\lim_{n \to \infty} \tau_A u_n = b$. Each $u_n$ is of the form $\sum_{i=1}^{k_n} \alpha_i [x_i|y_i]$ in which $x_i, y_i \in X$ and $\alpha_i \in \mathbb{C}$.

Hence $a^*ba = \lim_{n \to \infty} \tau_A a^* u_n a = \lim_{n \to \infty} \tau_A a^* \left( \sum_{i=1}^{k_n} \alpha_i [x_i|y_i] \right) a = \lim_{n \to \infty} \tau_A \sum_{i=1}^{k_n} \alpha_i [x_i a|y_i a] = 0$.

Put $b = aa^*$. Therefore $\|a^*a\|^2 = \tau_A(a^*aa^*a) = \text{tr}(a^*aa^*a) = 0$. By [1, Lemma 2.2], $a = 0$.

Suppose that $a \in A - \tau(A)$ and $xa = 0$ for all $x \in X$. Let $b \in A$ be arbitrary. So $ab \in \tau(A)$ and $xab = 0$ for all $x \in X$. Recall that $\|xab\| \leq \|xa\| \|b\|$. By previous argument $ab = 0$ or $aA = \{0\}$. It implies that $a = 0$, since $A$ is proper. $\square$

Let $X$ and $Y$ be Hilbert modules over proper $H^*$-algebras $A$ and $B$, respectively, and $\varphi : \tau(A) \to \tau(B)$ be a norm continuous $*$-homomorphism (morphism). A map $\Phi : X \to Y$ is said to be a $\varphi$-morphism if $[\Phi(x)|\Phi(y)] = \varphi([x|y])$ for all $x, y \in X$. We can extend $\varphi$ to a continuous morphism $\bar{\varphi} : A \to B$. Obviously, $\Phi$ is a $\bar{\varphi}$-morphism, i.e., $[\Phi(x)|\Phi(y)] = \bar{\varphi}([x|y])$ for each $x, y \in X$. From now on we mean by a $\varphi$-morphism, a $\bar{\varphi}$-morphism. It is easy to see that each $\varphi$-morphism is necessarily a linear operator and a module mapping in the sense that $\Phi(xa) = \Phi(x) \varphi(a)$ for all $x \in X, a \in A$.

Let $X$ be a Hilbert $H^*$-module over $A$ and $a \in A$, the left translation $L_a : X \to X$ is defined by $L_a(x) = ax$ for $x \in X$. If $e \in A$ is a projection, then $L_e$ is an orthogonal projection defined on the Hilbert space $(X, (.,.))$. Let us denote $X_e = L_e X$. The subspace $X_e$ is a closed subspace of the Hilbert space $(X, (.,.))$ [4].

**Theorem 1.1.** (see [4, Lemma 2.7]) Let $X$ be a Hilbert $H^*$-module over $A$ and $e$ be a minimal projection in $A$. Then $X_e = \{x \in X : [x|x] = \lambda e, \lambda \geq 0\}$. If $A$ is a simple $H^*$-algebra, then the subspace $X_e$ generates a dense submodule in $X$.

**Remark 1.1.** In the above theorem if $A$ is a commutative, simple and proper $H^*$-algebra, then $X_e = X$. Recall that for each arbitrary minimal projection $e \in A$, we have $A = Ae = eAe = \mathbb{C}e$ [1, Theorem 4.1 and 4.2]. If $x$ is a nonzero element in $X$, then there is a nonzero element $a$ in $A$ such that $[x|x] = a^*a$. Hence $[x|x] \in \tau(A) \subseteq A = \mathbb{C}e$. So there exists a positive number $\lambda$ such that $[x|x] = \lambda e$. Therefore $x \in X_e$.

In this paper, we obtain some conditions under which a commutative diagram of Hilbert $H^*$-modules and morphisms is pullback.
2. Pullback constructions in Hilbert modules over $H^*$-algebras

In this section we introduce a pullback diagram of $H^*$-algebras and investigate some properties of them. For this we need the following definition.

**Definition 2.1.** A commutative diagram of $H^*$-algebras and morphisms

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\varphi_1} & B_1 \\
\downarrow{\psi_1} & & \downarrow{\psi_2} \\
A_2 & \xrightarrow{\varphi_2} & B_2
\end{array}
\]

is pullback if $\ker(\varphi_1) \cap \ker(\psi_1) = \{0\}$ and for any other pair of morphisms $\mu_1 : A \to B_1$ and $\mu_2 : A \to A_2$ from an $H^*$-algebra $A$ that satisfy condition $\psi_2 \mu_1 = \varphi_2 \mu_2$, there is a unique morphism $\mu : A \to A_1$ such that $\mu_1 = \varphi_1 \mu$ and $\mu_2 = \psi_1 \mu$.

It follows that $A_1$ is isomorphic to the restricted direct sum $A_2 \bigoplus B_1 = \{(a_2, b_1) \in A_2 \bigoplus B_1 | \varphi_2(a_2) = \psi_2(b_1)\}$, so that $\varphi_1$ and $\psi_1$ can be identified with projections on first and second coordinates, respectively. In particular, the pullback exists for any triple of $H^*$-algebras $A_2, B_1$ and $B_2$ with linking morphisms $\varphi_2$ and $\psi_2$.

**Theorem 2.1.** Suppose that

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\varphi_1} & B_1 \\
\downarrow{\psi_1} & & \downarrow{\psi_2} \\
A_2 & \xrightarrow{\varphi_2} & B_2
\end{array}
\]

is a commutative diagram of $H^*$-algebras and morphisms. If $\ker \varphi_1 \cap \ker \psi_1 = \{0\}$ and $\psi_1, \psi_2$ are surjective and injective, respectively, then the above diagram is pullback.

**Proof.** It is enough to show that the morphism $\varphi : A_1 \to A_2 \bigoplus B_1$ defined by $\varphi(a_1) = (\psi_1(a_1), \varphi_1(a_1))$ is an isomorphism. Let $(a_1, b_1) \in A_2 \bigoplus B_1$. Then $\psi_2(b_1) = \varphi_2(a_2)$. There exists $a_1 \in A_1$, such that $\psi_1(a_1) = a_2$, since $\psi_1$ is surjective. By the commutativity of the diagram and injectivity of $\psi_2$, we have $b_1 = \psi_2^{-1} \varphi_2(a_2) = \psi_2^{-1} \varphi_2 \psi_1(a_1) = \varphi_1(a_1)$. It proves the surjectivity of $\varphi$.

It is clear that if $\psi_1$ is injective, then so is $\varphi$. For injectivity of $\psi_1$, let $\psi_1(a_1) = 0$. Thus $\varphi_2 \psi_1(a_1) = \varphi_2 \varphi_1(a_1) = 0$ and injectivity of $\psi_2$ implies that $\varphi_1(a_1) = 0$. Hence $a_1 \in \ker \varphi_1 \cap \ker \psi_1 = \{0\}$. \hfill $\square$
Lemma 2.1. Suppose that $\Phi_2 : X_2 \rightarrow Y_2$ and $\Psi_2 : Y_1 \rightarrow Y_2$ are $\varphi_2$, $\psi_2$-morphisms of Hilbert $H^*$-modules, where $\varphi_2 : A_2 \rightarrow B_2$ and $\psi_2 : B_1 \rightarrow B_2$ are morphisms of underlying $H^*$-algebras. Denote by $X_2 \oplus Y_1$ the set $\{(x_2, y_1) \in X_2 \oplus Y_1 : \Phi_2(x_2) = \Psi_2(y_1)\}$, then $X_2 \oplus Y_1$ is a Hilbert module over $H^*$-algebra $A_2 \oplus B_1$ (with operations inherited from the Hilbert $A_2 \oplus B_1$-module $X_2 \oplus Y_1$). If $X_2$ and $Y_1$ are full, then $X_2 \oplus Y_1$ is a full Hilbert module over $A_2 \oplus B_2$.

Proof. Straightforward (see [3, Proposition 2.1]). □

Definition 2.2. A commutative diagram of Hilbert $H^*$-modules and morphisms

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\Phi_1} & Y_1 \\
\downarrow{\psi_1} & & \downarrow{\psi_2} \\
X_2 & \xrightarrow{\Phi_2} & Y_2
\end{array}
\]

is pullback if $\ker(\Phi_1) \cap \ker(\Psi_1) = \{0\}$ and for any other pair of morphisms $\Upsilon_1 : X \rightarrow Y_1$ and $\Upsilon_2 : X \rightarrow X_2$ from a full Hilbert $H^*$-module $X$ such that satisfy the condition $\Psi_2 \Upsilon_1 = \Phi_2 \Upsilon_2$, there exists a unique morphism $\Upsilon : X \rightarrow X_1$ such that $\Upsilon_1 = \Phi_1 \Upsilon$ and $\Upsilon_2 = \Psi_1 \Upsilon$.

It is easily verified that $X_1$ is isomorphic to $X_2 \oplus Y_2$. The following proposition is proved in framework of Hilbert $C^*$-modules. It is easy to show that this proposition holds in the category of Hilbert $H^*$-modules. Density of trace class of a proper $H^*$-algebra in its own is useful in checking commutativity of the diagram of underlying $H^*$-algebras.

Proposition 2.1. (see [3, Proposition 2.3]) Let $X_2$, $Y_1$ and $Y_2$ be Hilbert modules over $H^*$-algebras with linking morphisms $\Phi_2$ and $\Psi_2$. Then

\[
\begin{array}{ccc}
X_2 \oplus Y_1 & \xrightarrow{\Phi_1} & Y_1 \\
\downarrow{\psi_1} & & \downarrow{\psi_2} \\
X_2 & \xrightarrow{\Phi_2} & Y_2
\end{array}
\]

with the projections $\Phi_1(x_2, y_1) = y_1$ and $\Psi_1(x_2, y_1) = x_2$ is a pullback diagram of Hilbert modules over $H^*$-algebras, where $\Phi_1$ is a $\varphi_1$-morphism and $\Psi_1$ is a $\psi_1$-morphism.
of Hilbert modules over $H^*$-algebras and $\varphi_1 : A_2 \oplus_B B_1 \to B_1$ and $\psi_1 : A_2 \oplus_B B_1 \to A_1$ are the corresponding projections.

$$
\begin{array}{c}
A_2 \oplus_B B_1 \xrightarrow{\varphi_1} B_1 \\
\downarrow \psi_1 \quad \downarrow \psi_2 \\
A_2 \xrightarrow{\varphi_2} B_2 
\end{array}
$$

Now we are ready to prove the main theorem of this paper.

**Theorem 2.2.** Suppose that

$$
\begin{array}{c}
X_1 \xrightarrow{\Phi_1} Y_1 \\
\downarrow \psi_1 \quad \downarrow \psi_2 \\
X_2 \xrightarrow{\Phi_2} Y_2 
\end{array}
$$

is a commutative diagram of full Hilbert $H^*$-modules $X_1, X_2$ and $Y_1$ and arbitrary Hilbert $H^*$-module $Y_2$ and continuous morphisms. If this diagram is pullback and $\Psi_2$ is surjective, then the following conditions hold

(i) $\ker \Phi_1 \cap \ker \Psi_1 = \{0\}$,

(ii) $\Psi_1$ is surjective.

Conversely, if (i) and (ii) hold, $\psi_1(\tau(A_1))$ is $\tau_{A_2}$-closed and $\Psi_2$ is injective, then (2.2) is pullback.

**Proof.** Suppose that the above diagram is pullback. By the definition, (i) holds and there exists a unique isomorphism $\Phi : X_1 \to X_2 \oplus_Y Y_1$ defined by $\Phi(x_1) = (\Psi_1(x_1), \Phi_1(x_1)) = (x_2, y_1)$. We will show that the surjectivity of $\Psi_2$ implies surjectivity of $\Psi_1$. Let $x_2 \in X_2$. Then $\Phi_2(x_2) \in Y_2 = \Psi_2(Y_1)$. So $\Phi_2(x_2) = \Psi_2(y_1)$ for some $y_1 \in Y_1$. Thus $x_2, y_1 \in X_2 \oplus_Y Y_1$. Therefore there exists $x_1 \in X_1$ such that $\Phi(x_1) = (\Psi_1(x_1), \Phi_1(x_1)) = (x_2, y_1)$, since $\Phi$ is onto. Hence $\Psi_1$ is surjective.

Conversely, suppose that conditions (i) and (ii) hold, $\psi_1(\tau(A_1))$ is $\tau_{A_2}$-closed and $\Psi_2$ is injective and let (2.1) be the corresponding diagram of underlying $H^*$-algebras. Clearly $\Psi_1, \Psi_2$ are $\psi_1, \psi_2$-morphisms and $\Phi_1, \Phi_2$ are $\varphi_1, \varphi_2$-morphisms of corresponding Hilbert $H^*$-modules. We shall show that the three conditions of Theorem 2.1 hold for the diagram of underlying $H^*$-algebras. The diagram of $H^*$-algebras is commutative, since the diagram of their Hilbert modules is commutative. Note that density of trace class of a proper $H^*$-algebra implies commutativity of the diagram of underlying $H^*$-algebras.

(I) We want to show that $\ker \varphi_1 \cap \ker \psi_1 = \{0\}$. Let $a_1 \in \ker \varphi_1 \cap \ker \psi_1$ and $x_1 \in X_1$ be arbitrary. Then, we have that $[\Phi_1(x_1a_1)] = \varphi_1([x_1a_1]) = 0$. Thus $\|\Phi_1(x_1a_1)\|^2 = \text{tr}([\Phi_1(x_1a_1)]^*) = 0$, so $x_1a_1 \in \ker \Phi_1$. Similarly $x_1a_1 \in \ker \Psi_1$. Hence by (i), $x_1a_1 = 0$, for all $x_1 \in X_1$. Since $X_1$ is full, by Lemma 1.2, $a_1 = 0$. 


(II) Let \( b_1 \in B_1 \) such that \( \psi_2(b_1) = 0 \) and \( y_1 \in Y_1 \) be arbitrary. Then we have that \( \Psi_2(y_1b_1) = \psi_2(b_1) \psi_2(y_1) \). Since \( \psi_2 \) is injective, we have that \( \psi_2(y_1b_1) = 0 \). So \( \|\Psi_2(y_1b_1)\|^2 = \text{tr}(|\Psi_2(y_1b_1)|^2) = 0 \). Then \( y_1b_1 = 0 \) for each \( y_1 \in Y_1 \), since \( \Psi_2 \) is an injection. By the fullness of \( Y_1 \), \( b_1 = 0 \). Then \( \psi_2 \) is injective.

(III) We will show that \( \Psi_1 \) is surjective. First we show that \( \psi_1 \) is injective. If \( a_1 \in \ker \psi_1 \), then commutativity of (2.1), implies that \( \psi_2 \varphi_1(a_1) = \varphi_2 \psi_1(a_1) = 0 \). By (II), \( \psi_2 \) is injective, so \( \varphi_1(a_1) = 0 \) and by (I), \( a_1 = 0 \). Since \( X_2 \) is full, \( \Psi_1 \) is surjective and \( \psi_1(\tau(A_1)) \) is \( \tau_{A_2} \)-closed, we have

\[
\tau(A_2) = \frac{[X_2|X_2]^\tau_{A_2}}{\Psi_1(\tau(A_1))} = \psi_1(\tau(A_1)).
\]

Clearly \( \psi_1(\tau(A_1)) \subseteq \tau(A_2) \). Let \( a_2 \in A_2 \) be arbitrary, since \( A_2 \) is proper, then there exists a sequence \( \{u_n\} \) in \( \tau(A_2) = \psi_1(\tau(A_1)) \) such that \( \lim_{n \to \infty} u_n = a_2 \). Each \( u_n \) is of the form \( \psi_1(a_n b_n) \) in which, \( a_n, b_n \in A_1 \). Since \( \psi_1 : \tau(A_1) \to \tau(A_2) \) is a norm continuous isomorphism and the sequence \( \{\psi_1(a_n b_n)\} \) is Cauchy in \( \tau(A_2) \), then \( \{\psi_1^{-1}(\psi_1(a_n b_n))\} = \{a_n b_n\} \) is Cauchy in \( \tau(A_1) \subseteq A_1 \). Hence this sequence is convergent in \( A_1 \) and \( \lim_{n \to \infty} u_n = \lim_{n \to \infty} \psi_1(a_n b_n) = \psi_1(\lim_{n \to \infty} (a_n b_n)) \in \psi_1(A_1) \), i.e., \( A_2 \subseteq \psi_1(A_1) \).

Then \( \psi_1 \) is surjective, and by Theorem 2.1, diagram (2.1) is pullback. Therefore \( \varphi : A_1 \to A_2 \ominus B_1 \) is defined by \( \varphi(a_1) = (\psi_1(a_1), \varphi_1(a_1)) \), is an isomorphism.

Define \( \Phi : X_1 \to X_2 \oplus Y_2 Y_1 \) by \( \Phi(x_1) = (\Psi_1(x_1), \Phi_1(x_1)) \) and show that \( \Phi \) is an isomorphism of Hilbert \( H^* \)-modules. Let \( (x_2, y_1) \in X_2 \oplus Y_2 Y_1 \). By the surjectivity of \( \Psi_1 \), \( x_2 = \Psi_1(x_1) \) for some \( x_1 \in X_1 \). By the commutativity of the diagram (2), \( \Psi_2 \Phi_1(x_1) = \Phi_2 \Psi_1(x_1) = \Phi_2(x_2) = \Psi_2(y_1) \). Since \( \Psi_2 \) is injective, we have \( \Phi_1(x_1) = y_1 \). So \( \Phi \) is a surjection. Also (i) implies that \( \Phi \) is an injection. On the other hand

\[
[\Phi(x_1)|\Phi(x_1)] = [(\Psi_1(x_1), \Phi_1(x_1))|\Phi_1(x_1)]
\]

\[
= ([\Psi_1(x_1)|\Psi_1(x_1)], [\Phi_1(x_1)|\Phi_1(x_1)])
\]

\[
= (\psi_1([x_1|x_1]), \varphi_1([x_1|x_1])) = \varphi([x_1|x_1]).
\]

So \( \Phi \) is a \( \varphi \)-morphism. Hence \( X_1 \cong X_2 \oplus Y_2 Y_1 \). By Proposition 2.1, diagram (2) is a pullback diagram of Hilbert \( H^* \)-modules. \( \square \)

Recall that a Hilbert \( H^* \)-module \( X \) over \( A \) is faithful if \( \{a \in A : Xa = \{0\}\} = \{0\} \). By [4, Remark 1.6], for each faithful Hilbert \( H^* \)-module \( X \) over a proper \( H^* \)-algebra \( A \) there exists a family \( \{X_i\}_{i \in I} \) of Hilbert \( H^* \)-modules, where each \( X_i \) is a Hilbert \( H^* \)-module over a simple \( H^* \)-algebra \( A_i \), such that \( X \) is equal to the mixed product of the family \( \{X_i\}_{i \in I} \),

\[
X = \bigotimes_{i \in I} X_i = \left\{ \{x_i\} \in \prod_{i \in I} X_i : \sum_{i \in I} \|x_i\|^2 < \infty \right\}.
\]
Example 2.1. The Hilbert space $l^2 = \left\{ (a_n) : a_n \in \mathbb{C}, \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$ is a commutative and proper $H^*$-algebra, where for each $(a_n)$ and $(b_n)$ in $l^2$, $(a_n)(b_n) = (a_n b_n)$ and $(a_n)^* = (\overline{a_n})$ [1, Example 3]. If $e_1 = (1, 0, 0, 0, \ldots), e_2 = (0, 1, 0, 0, \ldots), \ldots$, then $\{e_i\}_{i \in \mathbb{N}}$ is a maximal family of doubly orthogonal primitive elements of $l^2$. Put $I = \mathbb{C} e_i$ for each $i \in \mathbb{N}$. Then $I$ is a simple and proper $H^*$-algebra. It is easy to verify that $l^2$ is a faithful Hilbert module over itself, under the inner product $[(a_j)](b_j) = (a_j \overline{b_j}) \in \tau(l^2)$. Since $l^2$ is also a proper $H^*$-algebra, then there exists the family $\{I_i\}_{i \in \mathbb{N}}$ of Hilbert $H^*$-modules, where each $I_i$ is a Hilbert $H^*$-module over itself as a simple $H^*$-algebra. Hence $l^2 = \bigotimes_{i \in \mathbb{N}} I_i$.

Let $A_1$ and $A_2$ be simple and proper $H^*$-algebras and $\varphi$ be a surjective morphism from $A_1$ into $A_2$. If $e_1$ is a minimal projection in $A_1$, then $\varphi(e_1)$ is a minimal projection in $A_2$, since

(i) $\varphi(e_1) = \varphi(e_1^2) = (\varphi(e_1))^2$

(ii) $\varphi(e_1) = \varphi(e_1^*) = (\varphi(e_1))^*$

(iii) $\varphi(e_1) A_2 \varphi(e_1) = \varphi(e_1) \varphi(A_1) \varphi(e_1) = \varphi(e_1 A_1 e_1) = \varphi(\mathbb{C} e_1) = \mathbb{C} \varphi(e_1)$.

If $A$ and $B$ are commutative simple and proper $H^*$-algebras and $\varphi : A \rightarrow B$ is a nonzero morphism and $e, e'$ are minimal projections in $A$ and $B$, respectively, then for some complex number $\lambda$, $\varphi(\lambda e) = e'$. It implies that every nonzero morphism $\varphi$ is a surjection. One can easily concludes that $\varphi$ is an injection, too. Let (2.2) be a commutative diagram of Hilbert modules over commutative simple and proper $H^*$-algebras and morphisms and let (2.1) be its underlying diagram of $H^*$-algebras and morphisms. Then for an arbitrary minimal projection $e_1$ in $A_1$, there exist minimal projections $e_1' = \varphi_1(e_1)$ in $B_1$, $e_2 = \psi_1(e_1)$ in $A_2$ and $e_2' = \psi_2(e_1')$ in $B_2$. Obviously, by the commutativity of diagram, $\varphi_2 \psi_1(e_1) = \psi_2 \varphi_1(e_1)$. So $\varphi_2(e_2) = e_2'$.

Suppose that $X_{1,e_1} = \{x_1 \in X_1 : [x_1|x_1] = \lambda e_1, \lambda \geq 0\}$ and $Y_{1,e_1'}, X_{2,e_2}, Y_{2,e_2'}$ are defined similarly. If $x_1 \in X_{1,e_1}$, then $[\Phi_1(x_1)|\Phi_1(x_1)] = \varphi_1([x_1|x_1]) = \varphi_1(\lambda e_1) = \lambda \varphi_1(e_1) = \lambda e_1'$ for some $\lambda \geq 0$. Therefore the $\varphi_1$-morphism $\Phi_1 : X_{1,e_1} \rightarrow Y_{1,e_1'}$ is well-defined.

Recall that if $\{e_i\}_{i \in I}$ is a maximal family of doubly orthogonal minimal projections in commutative proper $H^*$-algebra $A$, then $A$ is the direct sum of the minimal left ideals $A e_i$ or the minimal right ideals $e_i A$ [1, Theorem 4.1]. Also by [6, Lemma 34.14], we know that every minimal ideal in $A$ is of the form $A e$ or $e A$ for some minimal projection $e$.

Corollary 2.1. Let (2.2) be a commutative diagram of faithful Hilbert modules over commutative proper $H^*$-algebras and morphisms. If their underlying $H^*$-algebras have the same cardinal of doubly orthogonal minimal projections and $\Psi_1$ is surjective, then (2.2) is pullback.
Proof. Suppose that \( \{ e_{1,i} \}_{i \in I}, \{ e_{2,i} \}_{i \in I} = \{ \varphi_1(e_{1,i}) \}_{i \in I}, \{ e_{1,i}' \}_{i \in I} = \{ \varphi_1(e_{1,i}) \}_{i \in I} \) and \( \{ e_{2,i} \}_{i \in I} = \{ \varphi_2(e_{1,i}) \}_{i \in I} \) are the maximal family of doubly orthogonal minimal projections of \( A_1, A_2, B_1 \) and \( B_2 \), respectively. Note that these \( H^* \)-algebras have the same cardinal of doubly orthogonal minimal projections. Put \( A_{1,i} (= A_i e_{1,i}) \) for each \( i \in I \). Then \( \{ A_{1,i} \}_{i \in I} \) is the family of minimal closed ideals of \( A_1 \). Also there exists a suitable family \( \{ X_{1,i} \}_{i \in I} \), of faithful Hilbert modules over simple \( H^* \)-algebras \( A_{1,i} \), such that \( X_1 \) equals the mixed products of the family \( \{ X_{1,i} \}_{i \in I} \) \([7, \text{Theorem } 2.3]\).

Similarly we can assume that \( X_2, Y_1, Y_2 \) are the mixed products of the family \( \{ X_{2,i} \}_{i \in I}, \{ Y_{1,i} \}_{i \in I}, \{ Y_{2,i} \}_{i \in I} \), respectively.

Now by Theorem 1.1, we can replace the above families of Hilbert modules over the simple and proper \( H^* \)-algebras \( \{ A_{1,i} \}_{i \in I}, \{ B_{1,i} \}_{i \in I}, \{ A_{2,i} \}_{i \in I}, \{ B_{2,i} \}_{i \in I} \), by \( \{ X_{2,i} \}_{i \in I}, \{ X_{1,i} \}_{i \in I}, \{ Y_{2,i} \}_{i \in I}, \{ Y_{1,i} \}_{i \in I} \) and \( \{ Y_{2,i} \}_{i \in I} \), respectively. By the assumption \( \Psi_1 \) is surjective, then \( \Psi_{1,i} : X_{1,i} \rightarrow X_{2,i} \) is surjective, where \( \Psi_{1,i} = \Psi_1 |_{X_{1,i}} \), for each \( i \in I \).

Since for an arbitrary element \( x_2 \in X_{2,i} \), and surjectivity of \( \Psi_1 \), there exists \( x_1 \in X_1 \) such that \( [x_2]_{x_2} = [\Psi_1(x_1)]_{\Psi_1(x_1)} = \psi_1([x_1]_1) \). Furthermore for some positive number \( \lambda \), we have \([x_1]_{x_1} = \lambda e_{2,i} = \lambda \psi_1(e_{1,i}) = \psi_1(\lambda e_{1,i}) \). Since \( \psi_1 \) is an isomorphism, then \([x_1]_{x_1} = \lambda e_{1,i} \). So \( x_1 \in X_{1,i} \), and \( \Psi_{1,i} \) is surjective. Now we are going to show the injectivity of \( \Psi_{2,i} \). Let \( y_1 \in Y_{1,i} \) and \( \Psi_{2,i}(y_1) = 0 \). Then \([y_1]_{y_1} = \lambda e_{2,i} \) for some positive number \( \lambda \) and \( [\Psi_{2,i}(y_1)]_{\Psi_{2,i}(y_1)} = \psi_{2,i}(\lambda e_{1,i}) = \lambda e_{2,i} = 0 \). Hence, \( \lambda = 0 \), so \([y_1]_{y_1} = 0 \). By the definition of Hilbert \( H^* \)-module, \( y_1 = 0 \). We can prove the injectivity of other morphisms in a similar fashion. This implies that \( \ker(\Phi_{1,i}) \cap \ker(\Psi_{1,i}) = \{0\} \). Finally one can easily verify the fullness of each of Hilbert \( H^* \)-modules in the following diagram. Hence by Theorem 2.2, the following diagram is pullback for each \( i \in I \).

\[
\begin{array}{ccc}
X_{1,i} & \xrightarrow{\Phi_{1,i}} & Y_{1,i}' \\
\downarrow \Psi_{1,i} & & \downarrow \Psi_{2,i} \\
X_{2,i} & \xrightarrow{\Phi_{2,i}} & Y_{2,i}'
\end{array}
\]

In particular, the following diagram or \((2.2)\) is pullback. (see \([9, \text{Proposition } 4.8]\))

\[
\begin{array}{ccc}
\bigotimes X_{1,i} & \oplus \Phi_{1,i} & \bigotimes Y_{1,i}' \\
\downarrow \Psi_{1,i} & & \downarrow \Psi_{2,i} \\
\bigotimes X_{2,i} & \oplus \Phi_{2,i} & \bigotimes Y_{2,i}'
\end{array}
\]

\[
\square
\]

REFERENCES


---

1Department of Mathematics,
Mashhad Branch, Islamic Azad University,
Mashhad, Iran

E-mail address: khanehgir@mshdiau.ac.ir
E-mail address: amyari@mshdiau.ac.ir, maryam_amyari@yahoo.com
E-mail address: MMkh926@gmail.com