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DEPENDENCE RESULT OF THE WEAK SOLUTION OF ROBIN BOUNDARY VALUE PROBLEMS

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Abstract. In this article we establish an approximation result involving the Laplacian with Robin boundary conditions. It informs about the weak solution's dependence from the input function on the boundary.

1. Introduction

Let Ω be a bounded domain with Lipschitz boundary. We consider the problem of the Laplacian with Robin boundary conditions

$$\frac{\partial u}{\partial \nu} + \beta u = 0,$$

where ν is the outward normal vector and β is a measurable positive bounded function on the boundary $\partial\Omega$. This kind of problems was extensively studied by many authors, we refer to [1, 2, 3, 5, 8] and references therein for more details. Such boundary conditions appears in the modelisation of some physical, chimical or biological processes governed by the Laplacian equation.

Such processes are called *Laplacian transport phenomena* or *diffusive transport phenomena*. More in details, the diffusive transport phenomena describes the transport of species between two distinct "regions" separated by an interface. In biology, for example, it describes the process when water and minerals are pumped by roots from earths, or when ions and biological species penetrate through cellular membranes, or also when oxygen molecules diffuse towards and pass through alveolar ducts (see for example [6]).

The physical and biological properties of the interface described above are reproduced by the function β. What motivate this work is the fact that such functions

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can be too complicated to deal with. One can try then to approximate the function β by a sequence of functions $(\beta_n)_{n\geq 0}$ more simple than β. The aim of this article is then to show a dependance result of a sequence of weak solutions $(u_n)_{n\geq 0}$ with a sequence of input functions $(\beta_n)_{n\geq 0}$. The proof is based on a technical Lemma due to Stampaccia [7].

2. Preliminaries and Main Result

We assume that $\Omega \subset \mathbb{R}^d$ $(d \geq 3)$ is a bounded domain with Lipschitz boundary. We denote by σ the restriction to $\partial\Omega$ of the $(d-1)-$dimensional Hausdorff measure.

We know that the following continuous embedding holds,

$$H^1(\Omega) \to L^q(\Omega), \quad q = \frac{2d}{d-2}. \tag{2.1}$$

Moreover each function $u \in H^1(\Omega)$ has a trace which is in $L^s(\partial\Omega)$, where $s = \frac{2(d-1)}{d-2}$; i.e., there is a constant $c > 0$ such that

$$\|u\|_{s,\partial\Omega} \leq c\|u\|_{H^1(\Omega)}, \quad \text{for all } u \in H^1(\Omega). \tag{2.2}$$

Let $\lambda > 0$ be a real number, $f \in L^p(\Omega)$ $(p > d)$ and β be a nonnegative bounded measurable function on $\partial\Omega$. We consider the following Robin boundary value problem

$$\begin{cases} -\Delta u + \lambda u = f, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \beta u = 0, & \text{in } \partial\Omega. \end{cases} \tag{2.3}$$

The form associated with the Laplacian with Robin boundary condition is

$$a_\beta(u,v) = \int_\Omega \nabla u \nabla v dx + \int_{\partial\Omega} \beta uv d\sigma, \quad \text{for all } u, v \in H^1(\Omega).$$

We start by the definition of the weak solution of the problem (2.3).

Definition 2.1. Let $f \in L^p(\Omega)$. For each $\lambda > 0$, a function $u = G_\beta^\lambda f \in H^1(\Omega)$ is called a weak solution of the Robin boundary value Problem (associated with β) if for every $v \in H^1(\Omega)$

$$a_\beta^\lambda(u,v) = \int_\Omega fv dx,$$

where for $u, v \in H^1(\Omega)$

$$a_\beta^\lambda(u,v) = a_\beta(u,v) + \lambda \int_\Omega uv \, dx.$$

It is clear that the closed bilinear form a_β is continuous on $H^1(\Omega)$ and also coercive on $H^1(\Omega)$ in the sense that there exists a constant $c > 0$ such that for all $u \in H^1(\Omega)$

$$a_\beta^\lambda(u,u) \geq \|u\|^2_{H^1(\Omega)}.$$

Let L be the linear functional on $H^1(\Omega)$ defined by: for $v \in H^1(\Omega)$

$$Lv := \int_\Omega fv\, dx.$$

Since $p \geq 2$, the functional L is well defined and continuous on $H^1(\Omega)$. Thus by coerciveness of the bilinear form a_β, the Lax-Milgram Lemma (see [4, Corollaire V.8 p:84]) implies that there exists a unique weak solution $u \in H^1(\Omega)$ of the boundary value problem (2.3).

The following lemma is important in the proof of Theorem 2.2, we can find its proof in [7] Lemma 4.1.

Lemma 2.1. *Let $\varphi = \varphi(t)$ be a nonnegative, nonincreasing function on the half line $t \geq k_0 \geq 0$ such that there are positive constants c, α and $\delta (\delta > 0)$ such that*

$$\varphi(h) \leq c(h-k)^{-\alpha} \varphi(k)^\delta,$$

for all $h > k \geq k_0$. Then we have

$$\varphi(k_0 + d) = 0, \text{ where } d > 0 \text{ satisfies } d^\alpha = c\varphi(k_0)^{\delta-1} 2^{\delta(\delta-1)}.$$

Theorem 2.1. *Let u be a weak solution and assume that $p > d$. Then*

1) *if $\lambda = 0$ and Ω is of finite volume, there exists a strictly positive constants $C_1 = C_1(d, p, |\Omega|)$ such that*

$$|u(x)| \leq C_1 \|f\|_p \quad \text{a.e on } \overline{\Omega},$$

2) *if $\lambda > 0$ and Ω is an arbitrary domain, there exist a strictly positive constant $C_2 = C_2(d, p, \lambda)$ such that*

$$|G_\beta^\lambda f(x)| \leq C_2 \|f\|_p \quad \text{a.e on } \overline{\Omega}.$$

The proof can be found in [8] and is based on the Maza'ya inequality and a standard argument as in Theorem 4.1 of [7].

Our main result is the following Theorem.

Theorem 2.2. *Any sequence $(u_n)_{n\geq 0}$ of weak solutions of the Robin boundary value problem associated to the sequence $(\beta_n)_{\geq 0}$ verify the following inequality:*

$$\|u_n - u_m\|_{\infty, \overline{\Omega}} \leq C \|u_n\|_{\infty, \partial\Omega} \|\beta_n - \beta_m\|_{\infty, \partial\Omega},$$

for all $n, m \in \mathbb{N}$ and where C may depend of λ.

3. Proof of Theorem 2.2

Proof. Let $(u_n)_{n\geq 0}$ be a sequence of weak solutions associated with the sequence $(\beta_n)_{\geq 0}$. Let $k \geq 0$ be a real number and define $u_{n,m} := u_n - u_m$.

Define $v_{n,m} := (|u_{n,m}| - k)^+ \operatorname{sgn}(u_{n,m})$. Then $v_{n,m} \in H^1(\Omega)$ and

$$\nabla v_{n,m} = \begin{cases} \nabla u_{n,m}, & \text{in } \ A_{n,m}(k); \\ 0, & \text{otherwise,} \end{cases}$$

where $A_{n,m}(k) = \{x \in \overline{\Omega} : |u_{n,m}(x)| > k\}$. In the following, we write $u, v, A(k)\dots$ instead of $u_{n,m}, v_{n,m}, A_{n,m}(k)\dots$

It is clear that $a^{\lambda}_{\beta_n}(u_n, v) - a^{\lambda}_{\beta_m}(u_m, v) = 0$. Calculating we obtain:

$$\begin{aligned}
0 &= \int_{\Omega} \nabla(u_n - u_m)\nabla v dx + \int_{\partial\Omega} (\beta_n u_n - \beta_m u_m) v d\sigma + \lambda \int_{\Omega} (u_n - u_m) v dx \\
&= \int_{A(k)} |\nabla v|^2 dx + \int_{\partial\Omega} (\beta_n - \beta_m) u_n + \beta_m (u_n - u_m) v d\sigma + \lambda \int_{\Omega} (u_n - u_m) v dx \\
&= \int_{A(k)} |\nabla v|^2 dx + \int_{\partial\Omega\cap A(k)} (\beta_n - \beta_m) u_n v d\sigma + \int_{\partial\Omega\cap A(k)} \beta_m (u_n - u_m) v d\sigma \\
&\quad + \lambda \int_{A(k)} (u_n - u_m) v dx \\
&= \int_{A(k)} |\nabla v|^2 dx + \int_{\partial\Omega\cap A(k)} (\beta_n - \beta_m) u_n v d\sigma + \int_{\partial\Omega\cap A(k)} \beta_m v^2 d\sigma \\
&\quad + k \int_{\partial\Omega\cap A(k)} \beta_m |v| d\sigma + \lambda \int_{A(k)} v^2 dx + \lambda k \int_{A(k)} |v| dx \\
&= a^{\lambda}_{\beta_m}(v, v) + \int_{\partial\Omega\cap A(k)} (\beta_n - \beta_m) u_n v d\sigma + k \int_{\partial\Omega\cap A(k)} \beta_m |v| d\sigma + \lambda k \int_{A(k)} |v| dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
a^{\lambda}_{\beta_m}(v, v) + \int_{\partial\Omega\cap A(k)} (\beta_n - \beta_m) u_n v d\sigma &= -k \int_{\partial\Omega\cap A(k)} \beta_m |v| d\sigma - \lambda k \int_{A(k)} |v| dx \\
&\le 0.
\end{aligned}$$

Which leads to

$$a^{\lambda}_{\beta_m}(v, v) \le \int_{\partial\Omega\cap A(k)} (\beta_m - \beta_n) u_n v d\sigma.$$

Using the Hölder inequality and (2.2), we obtain the following estimates,

$$\begin{aligned}
a^{\lambda}_{\beta_m}(v, v) &\le \int_{\partial\Omega\cap A(k)} (\beta_m - \beta_n) u_n v d\sigma \\
&\le \|\beta_n - \beta_m\|_{\infty,\partial\Omega} \int_{\partial\Omega\cap A(k)} u_n v d\sigma \\
&\le \|\beta_n - \beta_m\|_{\infty,\partial\Omega} \|u_n\|_{2,\partial\Omega\cap A(k)} \|v\|_{2,\partial\Omega\cap A(k)} \\
&\le \|\beta_n - \beta_m\|_{\infty,\partial\Omega} \|u_n\|_{\infty,\partial\Omega} |\partial\Omega \cap A(k)|^{\frac{1}{2}} |\partial\Omega \cap A(k)|^{\frac{1}{2}-\frac{1}{s}} \|v\|_{s,\partial\Omega} \\
&\le \|\beta_n - \beta_m\|_{\infty,\partial\Omega} \|u_n\|_{\infty,\partial\Omega} |\partial\Omega \cap A(k)|^{1-\frac{1}{s}} \|v\|_{s,\partial\Omega} \\
&\le c\|\beta_n - \beta_m\|_{\infty,\partial\Omega} \|u_n\|_{\infty,\partial\Omega} |\partial\Omega \cap A(k)|^{1-\frac{1}{s}} \|v\|_{H^1(\Omega)}.
\end{aligned}$$

We have then,

$$\begin{aligned}\alpha\|v\|^2_{H^1(\Omega)} &\le a^{\lambda}_{\beta_m}(v,v)\\ &\le c\|\beta_n-\beta_m\|_{\infty,\partial\Omega}\|u_n\|_{\infty,\partial\Omega}|\partial\Omega\cap A(k)|^{1-\frac{1}{s}}\|v\|_{H^1(\Omega)}.\end{aligned}$$

It follows that

$$\|v\|_{H^1(\Omega)} \le c_1\|\beta_n-\beta_m\|_{\infty,\partial\Omega}\|u_n\|_{\infty,\partial\Omega}|\partial\Omega\cap A(k)|^{1-\frac{1}{s}}.$$

Using the inequalities (2.1) and (2.2), we obtain the following estimates,

$$\|v\|_{s,\partial\Omega\cap A(k)} \le c_2\|\beta_n-\beta_m\|_{\infty,\partial\Omega}\|u_n\|_{\infty,\partial\Omega}|\partial\Omega\cap A(k)|^{1-\frac{1}{s}} \tag{3.1}$$

and

$$\|v\|_{q,A(k)} \le c_3\|\beta_n-\beta_m\|_{\infty,\partial\Omega}\|u_n\|_{\infty,\partial\Omega}|\partial\Omega\cap A(k)|^{1-\frac{1}{s}}. \tag{3.2}$$

Let now $h>k\ge 0$. Then $A(h)\subset A(k)$ and on $A(h)$ we have $|v|\ge h-k$. It follows that

$$\begin{aligned}\|v\|_{s,\partial\Omega\cap A(k)} &\ge \|v\|_{s,\partial\Omega\cap A(h)}\\ &\ge \||u|-k\|_{s,\partial\Omega\cap A(h)}\\ &\ge (h-k)|\partial\Omega\cap A(h)|^{\frac{1}{s}}.\end{aligned} \tag{3.3}$$

We deduce from (3.1) that

$$(h-k)|\partial\Omega\cap A(h)|^{\frac{1}{s}} \le c_2\|\beta_n-\beta_m\|_{\infty,\partial\Omega}\|u_n\|_{\infty,\partial\Omega}|\partial\Omega\cap A(k)|^{1-\frac{1}{s}}$$

which reduces to,

$$|\partial\Omega\cap A(h)| \le c_2^s(h-k)^{-s}\|\beta_n-\beta_m\|^s_{\infty,\partial\Omega}\|u_n\|^s_{\infty,\partial\Omega}|\partial\Omega\cap A(k)|^{s-1}.$$

Set $\phi(h)=|\partial\Omega\cap A(h)|$, we obtain,

$$\phi(h)\le C(h-k)^{-s}\phi(k)^{s-1},$$

where $C=c_2^s\|\beta_n-\beta_m\|^s_{\infty,\partial\Omega}\|u_n\|^s_{\infty,\partial\Omega}$.

As $s-1>1$, then the conditions of the Lemma 2.1 are satisfied with $\delta=s-1$ and $k_0=0$, one obtain $\phi(d)=0$ where $d>0$ satisfies $d^s=C\phi(0)^{s-2}2^{(s-1)(s-2)}$, consequently

$$d=c_4\|\beta_n-\beta_m\|_{\infty,\partial\Omega}\|u_n\|_{\infty,\partial\Omega}$$

and

$$\|u_n-u_m\|_{\infty,\partial\Omega} \le c_4\|u_n\|_{\infty,\partial\Omega}\|\beta_n-\beta_m\|_{\infty,\partial\Omega}. \tag{3.4}$$

In the same way as in (3.3), we obtain

$$\|v\|_{q,A(k)} \ge (h-k)|A(k)|^{\frac{1}{q}}.$$

From (3.2), we deduce

$$(h-k)|A(h)|^{\frac{1}{q}} \le c_3\|\beta_n-\beta_m\|_{\infty,\partial\Omega}\|u_n\|_{\infty,\partial\Omega}|\partial\Omega\cap A(k)|^{1-\frac{1}{s}}.$$

We take $k = d$ and $h = \gamma d$ with $\gamma > 1$, we obtain $|A(\gamma d)| = 0$ which leads to

$$\begin{aligned} \|u_n - u_m\|_{\infty,\Omega} &\leq \gamma d \\ &\leq \gamma c_4 \|u_n\|_{\infty,\partial\Omega} \|\beta_n - \beta_m\|_{\infty,\partial\Omega}. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5) we obtain our Theorem. □

Corollary 3.1. *Let $(u_n)_{n\geq 0}$ be a sequence weak solutions associated with the sequence $(\beta_n)_{\geq 0} \in L^\infty(\partial\Omega)$ such that $\inf_n \beta_n > 0$ then if $(u_n)_{n\geq 0}$ is uniformly bounded we have for $p > d$*

$$\|u_n - u_m\|_{\infty,\overline{\Omega}} \leq C\|f\|_p \|\beta_n - \beta_m\|_{\infty,\partial\Omega}, \tag{3.6}$$

for all $n, m \in \mathbb{N}$ and where C may depend of λ.

In the case where the sequence of weak solutions $(u_n)_{n\geq 0}$ is uniformly bounded with respect to n we have the following consequence

Corollary 3.2. *Let $(u_n)_{n\geq 0}$ be a sequence weak solutions associated with the sequence $(\beta_n)_{\geq 0} \in L^\infty(\partial\Omega)$ such that $\inf_n \beta_n > 0$ and $\lim_n \beta_n(x) = \beta(x)$ a.e $x \in \partial\Omega$ then if $(u_n)_{n\geq 0}$ is uniformly bounded we have $\lim_n u_n(x) = u(x)$ a.e $x \in \overline{\Omega}$, where u is the weak solution associated with β.*

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